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Tensor product dual frames

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Abstract

To construct dual frames with good structure for a given frame is a fundamental problem in the theory of frames. The tensor product duals of tensor product frames can provide a rank-one decomposition of bounded antilinear operators between two Hilbert spaces. This paper addresses tensor product dual frames. We derive a necessary and sufficient condition for two tensor product Bessel sequences to be a pair of dual frames; obtain explicit expressions of all dual frames and tensor product dual frames of tensor product frames; and demonstrate the existence of non-tensor product dual frames of tensor product frames.

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1 Introduction

Let $\mathcal{H}$ be a separable Hilbert space, and $\{x_i\}_{i\in I}$ be a countable sequence in $\mathcal{H}$. It is called a *frame* for $\mathcal{H}$ if there exist constants $0 < C_1 \le C_2 < \infty$ such that

$$C_1\|x\|^2 \le \sum_{i\in I}\left|\langle x, x_i\rangle\right|^2 \le C_2\|x\|^2 \quad \text{for } x \in \mathcal{H}, \tag{1.1}$$

where C_1, C_2 are called *frame bounds*. It is called a *Bessel sequence* in $\mathcal{H}$ if the right-hand side inequality of (1.1) holds. And it is called a *Riesz basis* if it is a frame for $\mathcal{H}$, while it ceases to be a frame for $\mathcal{H}$ whenever an arbitrary element is removed. Given a frame $\{x_i\}_{i\in I}$ for $\mathcal{H}$, a sequence $\{y_i\}_{i\in I}$ in $\mathcal{H}$ is called a *dual frame* of $\{x_i\}_{i\in I}$ if it is a frame for $\mathcal{H}$ such that

$$x = \sum_{i\in I}\langle x, x_i\rangle y_i \quad \text{for } x \in \mathcal{H}. \tag{1.2}$$

It is easy to check that if $\{y_i\}_{i\in I}$ is a dual frame of $\{x_i\}_{i\in I}$, then $\{x_i\}_{i\in I}$ is also a dual frame of $\{y_i\}_{i\in I}$. So we say $(\{x_i\}_{i\in I}, \{y_i\}_{i\in I})$ is a pair of dual frames in this case. It is well known that $(\{x_i\}_{i\in I}, \{y_i\}_{i\in I})$ is a pair of dual frames for $\mathcal{H}$ if and only if $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ are Bessel sequences in $\mathcal{H}$ satisfying (1.2), and that a Bessel sequence (frame, Riesz sequence) is exactly the image of an orthonormal basis for $\mathcal{H}$ under a linear bounded operator (bounded surjection, bounded bijection) on $\mathcal{H}$. For basics on frames, see, e.g., [3, 18].

Throughout this paper, we denote by $\mathcal{L}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$. We define the operation "♠" as follows. For $h_1, h_2 \in \mathcal{H}$, we define the operator $h_1♠h_2$ on

$\mathcal{H}$ by

$$(h_1♠h_2)(x) = \langle x, h_2\rangle h_1 \quad \text{for } x \in \mathcal{H}. \tag{1.3}$$

It is well known that an arbitrary rank-one linear operator is always of this form. Herein, we use ♠ instead of ⊗ since ⊗ denotes rank-one antilinear operator throughout this paper.

Now we turn to the tensor product of Hilbert spaces. There are several ways of defining the tensor product of Hilbert spaces. Folland in [6], Kadison and Ringrose in [9] represented the tensor product of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ as a certain linear space of operators. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex separable Hilbert spaces. An operator T from $\mathcal{H}_2$ to $\mathcal{H}_1$ is said to be antilinear if

$$T(\alpha x + \beta y) = \overline{\alpha} Tx + \overline{\beta} Ty \tag{1.4}$$

for $x, y \in \mathcal{H}_2$ and $\alpha, \beta \in \mathbb{C}$. We consider the set of all bounded antilinear operators from $\mathcal{H}_2$ to $\mathcal{H}_1$. The norm of an antilinear operator $T : \mathcal{H}_2 \to \mathcal{H}_1$ is defined as in the linear case:

$$\|T\| = \sup_{\|g\|=1} \|Tg\|. \tag{1.5}$$

By a standard argument as in [10], given a bounded antilinear operator $T : \mathcal{H}_2 \to \mathcal{H}_1$, all series $\sum_{j\in J} \|Tu_j\|^2$ associated with an orthonormal basis $\{u_j\}_{j\in J}$ for $\mathcal{H}_2$ take the same value (not necessarily finite).

Definition 1.1 Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Then the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ is the set of all bounded antilinear maps $T : \mathcal{H}_2 \to \mathcal{H}_1$ such that $\sum_{j\in J} \|Tu_j\|^2 < \infty$ for some, and hence every orthonormal basis $\{u_j\}_{j\in J}$ of $\mathcal{H}_2$. Moreover, for every $T \in \mathcal{H}_1 \otimes \mathcal{H}_2$, we set

$$|||T|||^2 = \sum_{j\in J} \|Tu_j\|^2.$$

Recall from [6, Theorem 7.12], that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a Hilbert space with the norm $|||\cdot|||$ and the associated inner product

$$\langle Q, T\rangle = \sum_{j\in J} \langle Qu_j, Tu_j\rangle \quad \text{for } Q, T \in \mathcal{H}_1 \otimes \mathcal{H}_2. \tag{1.6}$$

Let $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$, we define their *tensor product* $f \otimes g$ by

$$(f \otimes g)(g') = \langle g, g'\rangle f \quad \text{for } g' \in \mathcal{H}_2. \tag{1.7}$$

Obviously, $f \otimes g$ belongs to $\mathcal{H}_1 \otimes \mathcal{H}_2$. By Lemma 2.2 below, a bounded antilinear operator T from $\mathcal{H}_2$ to $\mathcal{H}_1$ is rank-one if and only if $T = f \otimes g$ for some $0 \neq f \in \mathcal{H}_1$ and $0 \neq g \in \mathcal{H}_2$. Also, by [6], for any $f, f' \in \mathcal{H}_1$ and $g, g' \in \mathcal{H}_2$,

$$\langle f \otimes g, f' \otimes g'\rangle = \langle f, f'\rangle\langle g, g'\rangle. \tag{1.8}$$

Moreover, by Proposition 7.14 in [6], the tensor product of two orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. Throughout this paper, we always use $\{e_i\}_{i\in I}$ and $\{u_j\}_{j\in J}$ to denote orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

We now consider the tensor product of operators. Given $T \in \mathcal{L}(\mathcal{H}_1)$ and $Q \in \mathcal{L}(\mathcal{H}_2)$, define the tensor product $T \otimes Q$ of T and Q by

$$(T \otimes Q)B = TBQ^* \quad \text{for } B \in \mathcal{H}_1 \otimes \mathcal{H}_2.$$

By a standard argument, we have that $T \otimes Q \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. The following proposition is repeated from [6, 7, 12].

Proposition 1.1 *Suppose $T, T' \in \mathcal{L}(\mathcal{H}_1)$, $Q, Q' \in \mathcal{L}(\mathcal{H}_2)$, then*

(a) $T \otimes Q \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ *and* $\||T \otimes Q|\| = \|T\|\|Q\|$.
(b) $(T \otimes Q)(f \otimes g) = Tf \otimes Qg$ *for all* $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$.
(c) $(T \otimes Q)(T' \otimes Q') = (TT') \otimes (QQ')$.
(d) *If T and Q are invertible operators, then $T \otimes Q$ is an invertible operator and* $(T \otimes Q)^{-1} = T^{-1} \otimes Q^{-1}$.
(e) $(T \otimes Q)^* = T^* \otimes Q^*$.
(f) *Let $f, f' \in \mathcal{H}_1\backslash\{0\}$ and $g, g' \in \mathcal{H}_2\backslash\{0\}$. If $f \otimes g = f' \otimes g'$, then there exist constants a and b with $ab = 1$ such that $f' = af$ and $g' = bg$.*

In [5, 8], it is proven that the tensor product of a sequence with itself is a frame if this sequence is a frame. In [10], it is proven that the tensor product of two frames must be a new frame for the corresponding tensor product space. In 2008, Bourouihiya in [1] and Upender in [17] obtained the following proposition which is generalized to the G-frame setting in [16]. For basics on G-frames, see, e.g., [13–15] and the references therein.

Proposition 1.2 *The sequence $\{f_i \otimes g_j : i \in I, j \in J\}$ is a Bessel sequence (frame, Riesz basis) for $\mathcal{H}_1 \otimes \mathcal{H}_2$ if and only if $\{f_i\}_{i\in I}$ and $\{g_j\}_{j\in J}$ are Bessel sequences (frames, Riesz bases) for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.*

A fundamental problem in the frame theory is to find dual frames with good properties. A pair of dual frames gives an expansion of the elements in the space which is like atomic decomposition in harmonic analysis. Unfortunately, for a given tensor product frame, little has been known to us except for its canonical dual in [10]. This study aims to investigate dual frames of a general tensor product frame $\{f_i \otimes g_j\}_{(i,j)\in I\times J}$. Particularly, we are interested in dual frames with tensor product structure.

From the viewpoint of operators, [4, Theorem 3], demonstrates that, on an arbitrary ellipsoidal surface Δ in a separable Hilbert space $\mathcal{H}$, there exists a sequence $\{x_i\}_{i\in I}$ such that some multiple of the identity operator $\lambda I_{\mathcal{H}}$ has the following rank-one decomposition:

$$\lambda I_{\mathcal{H}} = \sum_{i\in I} x_i \spadesuit x_i,$$

where $I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$. For the case of $\dim \mathcal{H} < \infty$, [2] characterizes sequences $\{a_i\}_{i=1}^M$ such that

$$\lambda I_{\mathcal{H}} = \sum_{i=1}^{M} x_i \spadesuit x_i$$

for some constant λ and $\{x_i\}_{i=1}^M$, where $\|x_i\| = a_i$ for $1 \le i \le M$. For the case of $\dim \mathcal{H} = \infty$, [11, Proposition 7], presents another sufficient condition on $\{a_i\}_{i=1}^\infty$ satisfying the following identity:

$$I_{\mathcal{H}} = \sum_{i=1}^{\infty} x_i \spadesuit x_i$$

with $\|x_i\| = \sqrt{a_i}$. Furthermore, the decomposition of more general operators was studied in [11]. For an arbitrary positive operator B with finite rank, [11, Theorem 2], characterizes a class of sequences $\{a_i\}_{i=1}^M$ for which there exists $\{x_i\}_{i=1}^M$ with $\|x_i\| = \sqrt{a_i}$ such that

$$B = \sum_{i=1}^{M} x_i \spadesuit x_i.$$

And for an arbitrary positive non-compact operator B, [11, Theorem 6], gives a sufficient condition on $\{a_i\}_{i=1}^\infty$ for which there exists $\{x_i\}_{i=1}^\infty$ with $\|x_i\| = \sqrt{a_i}$ such that

$$B = \sum_{i=1}^{\infty} x_i \spadesuit x_i.$$

In summary, these results focus on decomposing an bounded linear operator on $\mathcal{H}$ into a sum of rank-one linear operators. Actually, a pair of tensor product dual frames also gives some rank-one decomposition of operators. Recall that every $T \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is a bounded antilinear operator from $\mathcal{H}_2$ to $\mathcal{H}_1$ and that, if $\{f_i \otimes g_j\}_{(i,j)\in I\times J}$ and $\{\widetilde{f}_i \otimes \widetilde{g}_j\}_{(i,j)\in I\times J}$ form a pair of dual frames for $\mathcal{H}_1 \otimes \mathcal{H}_2$, then

$$T = \sum_{(i,j)\in I\times J} a_{i,j} \widetilde{f}_i \otimes \widetilde{g}_j \tag{1.9}$$

for $T \in \mathcal{H}_1 \otimes \mathcal{H}_2$, where $a_{i,j} = \langle T, f_i \otimes g_j\rangle$. Therefore, the $\spadesuit$-based decomposition presents a rank-one linear operator decomposition of bounded linear operators on $\mathcal{H}$, while the $\otimes$-based (1.9) presents a rank-one antilinear operator decomposition of bounded antilinear operators from $\mathcal{H}_2$ to $\mathcal{H}_1$.

The rest of this paper is organized as follows. In Sect. 2, we characterize tensor product dual frames, give an explicit expression of all dual frames of tensor product frames, and prove the existence of non-tensor product dual frames. In Sect. 3, we present an explicit expression of all tensor product dual frames of tensor product frames. Some examples are also provided in Sect. 2 to illustrate the generality of the theory.

2 Dual characterization

This section is devoted to dual characterizations. We will derive a necessary and sufficient condition for two tensor product Bessel sequences to be a pair of dual frames; obtain an explicit expression of all duals of tensor product frames; and prove the existence of non-tensor product duals of tensor product frames. For this purpose, we first introduce some notations.

Let $\mathbf{f} = \{f_i\}_{i\in I}$ and $\mathbf{g} = \{g_j\}_{j\in J}$ be sequences in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We write

$$\mathbf{f} \otimes \mathbf{g} = \{f_i \otimes g_j : (i,j) \in I \times J\}.$$

Suppose that $\mathbf{f}$ and $\mathbf{g}$ are Bessel sequences in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We denote by $T_{\mathbf{f}}$, $T_{\mathbf{g}}$, and $T_{\mathbf{f}\otimes\mathbf{g}}$ the synthesis operators corresponding to $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{f} \otimes \mathbf{g}$, that is,

$$T_{\mathbf{f}}c = \sum_{i\in I} c(i)f_i \quad \text{for } c \in l^2(I),$$
$$T_{\mathbf{g}}c = \sum_{j\in J} c(j)g_j \quad \text{for } c \in l^2(J)$$

and

$$T_{\mathbf{f}\otimes\mathbf{g}}c = \sum_{(i,j)\in I\times J} c(i,j)f_i \otimes g_j \quad \text{for } c \in l^2(I \times J).$$

And we denote by $S_{\mathbf{f}}$, $S_{\mathbf{g}}$, $S_{\mathbf{f}\otimes\mathbf{g}}$ the frame operators corresponding to $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{f} \otimes \mathbf{g}$, accordingly, i.e.,

$$S_{\mathbf{f}} = T_{\mathbf{f}}T_{\mathbf{f}}^*, \qquad S_{\mathbf{g}} = T_{\mathbf{g}}T_{\mathbf{g}}^*, \qquad S_{\mathbf{f}\otimes\mathbf{g}} = T_{\mathbf{f}\otimes\mathbf{g}}T_{\mathbf{f}\otimes\mathbf{g}}^*.$$

In particular, for the finite dimensional case, we also introduce the following notations. Let $\{e_i\}_{i=1}^m$ and $\{u_j\}_{j=1}^n$ be orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then each element of $\mathcal{H}_1$ and $\mathcal{H}_2$ can be uniquely represented in terms of $\{e_i\}_{i=1}^m$ and $\{u_j\}_{j=1}^n$, respectively. For $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$, we define $c(f) \in \mathbb{C}^m$ and $c(g) \in \mathbb{C}^n$ by

$$f = \begin{pmatrix} e_1, & e_2, & \cdots, & e_m \end{pmatrix} c(f) \tag{2.1}$$

and

$$g = \begin{pmatrix} u_1, & u_2, & \cdots, & u_n \end{pmatrix} c(g) \tag{2.2}$$

with

$$c(f) = \begin{pmatrix} c_1(f) \\ c_2(f) \\ \vdots \\ c_m(f) \end{pmatrix} \quad \text{and} \quad c(g) = \begin{pmatrix} c_1(g) \\ c_2(g) \\ \vdots \\ c_n(g) \end{pmatrix}. \tag{2.3}$$

Given sequences $\mathbf{f} = \{f_i\}_{i=1}^{M}$ in $\mathcal{H}_1$ and $\mathbf{g} = \{g_j\}_{j=1}^{N}$ in $\mathcal{H}_2$, we denote by $M_\mathbf{f}$ and $M_\mathbf{g}$ the following $m \times M$ and $n \times N$ matrices:

$$M_\mathbf{f} = \begin{pmatrix} c(f_1), & c(f_2), & \cdots, & c(f_M) \end{pmatrix} \tag{2.4}$$

and

$$M_\mathbf{g} = \begin{pmatrix} c(g_1), & c(g_2), & \cdots, & c(g_N) \end{pmatrix}. \tag{2.5}$$

The following lemma demonstrates that the synthesis (analysis) operator associated with the tensor product of two Bessel sequences is exactly the tensor product of their respective synthesis (analysis) operators.

Lemma 2.1 *Let* $\mathbf{f} = \{f_i\}_{i\in I}$ *and* $\mathbf{g} = \{g_j\}_{j\in J}$ *be Bessel sequences in* $\mathcal{H}_1$ *and* $\mathcal{H}_2$, *respectively. Then*

$$T_{\mathbf{f}\otimes\mathbf{g}} = T_\mathbf{f} \otimes T_\mathbf{g} \quad \textit{and} \quad T^*_{\mathbf{f}\otimes\mathbf{g}} = T^*_\mathbf{f} \otimes T^*_\mathbf{g}.$$

Proof Since $\mathbf{f}$ and $\mathbf{g}$ are Bessel sequences in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, $T_\mathbf{f}$ and $T_\mathbf{g}$ are all linear bounded operators. For arbitrary $c \in l^2(I)$ and $d \in l^2(J)$, by Proposition 1.1(b), we see that

$$\begin{aligned} T_{\mathbf{f}\otimes\mathbf{g}} &= \sum_{(i,j)\in I\times J} c(i)d(j)f_i \otimes g_j \\ &= \sum_{(i,j)\in I\times J} \bigl(c(i)f_i\bigr) \otimes \bigl(d(j)g_j\bigr) \\ &= (T_\mathbf{f} c) \otimes (T_\mathbf{g} d) \\ &= (T_\mathbf{f} \otimes T_\mathbf{g})(c \otimes d). \end{aligned}$$

Also observe that $\{c \otimes d : c \in l^2(I), d \in l^2(J)\}$ is dense in $l^2(I \times J)$. It follows that

$$T_{\mathbf{f}\otimes\mathbf{g}} = T_\mathbf{f} \otimes T_\mathbf{g},$$

whence

$$T^*_{\mathbf{f}\otimes\mathbf{g}} = T^*_\mathbf{f} \otimes T^*_\mathbf{g}.$$ □

Similarly to the case of "♠", the following lemma shows that the tensor products take over all rank-one antilinear operators from $\mathcal{H}_2$ to $\mathcal{H}_1$.

Lemma 2.2 *For* $T \in \mathcal{H}_1 \otimes \mathcal{H}_2$, $\dim(\mathrm{range}(T)) \le 1$ *if and only if*

$$T = f \otimes g$$

for some $f \in \mathcal{H}_1$ *and* $g \in \mathcal{H}_2$.

Proof The sufficiency is obvious. Next we prove the necessity. Suppose $\dim(\mathrm{range}(T)) \le 1$. If $\dim(\mathrm{range}(T)) = 0$, then $T = 0 \otimes g$ for an arbitrary $g \in \mathcal{H}_2$. Next we suppose $\dim(\mathrm{range}(T)) = 1$. Take $0 \neq f \in \mathrm{range}(T)$. Then, to every $h \in \mathcal{H}_2$, there corresponds a unique number C_h such that

$$Th = C_h f.$$

Define $\Lambda : \mathcal{H}_2 \to \mathbb{C}$ by

$$\Lambda h = \overline{C_h} \quad \text{for } h \in \mathcal{H}_2.$$

Then

$$\overline{(\alpha_1 \Lambda h_1 + \alpha_2 \Lambda h_2)} f = T(\alpha_1 h_1 + \alpha_2 h_2) = \overline{\Lambda(\alpha_1 h_1 + \alpha_2 h_2)} f,$$

equivalently,

$$\Lambda(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 \Lambda h_1 + \alpha_2 \Lambda h_2 \tag{2.6}$$

for $\alpha_1, \alpha_2 \in \mathbb{C}$ and $h_1, h_2 \in \mathcal{H}_2$. Also observing that

$$|\Lambda h| \|f\| = \|Th\| \le \|T\| \|h\|,$$

we see that

$$|\Lambda h| \le \frac{\|T\|}{\|f\|} \|h\|.$$

This together with (2.6) leads to Λ is a linear bounded functional on $\mathcal{H}_2$. So there exists unique $g \in \mathcal{H}_2$ such that

$$\Lambda h = \langle h, g \rangle \quad \text{for } h \in \mathcal{H}_2.$$

It follows that

$$Th = \overline{\Lambda h} f = \langle g, h \rangle f = f \otimes g(h)$$

for $h \in \mathcal{H}_2$, and thus $T = f \otimes g$. □

The following lemma gives an explicit expression of the tensor product of two finite-dimensional spaces. It will be used in Example 2.1 below.

Lemma 2.3 *Let $\{e_i\}_{i=1}^m$ and $\{u_j\}_{j=1}^n$ be orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then*

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \Big\{ \mathcal{A} : \mathcal{A}\begin{pmatrix} u_1, & u_2, & \cdots, & u_n \end{pmatrix} z = \begin{pmatrix} e_1, & e_2, & \cdots, & e_m \end{pmatrix} A\overline{z} \\ \text{for } z \in \mathbb{C}^n, A \text{ is an } m \times n \text{ matrix} \Big\}. \tag{2.7}$$

Proof By a simple computation, we have

$$\begin{aligned}(e_i \otimes u_j)(h) &= \overline{c_j(h)} e_i \\ &= \begin{pmatrix} e_1, & e_2, & \cdots, & e_m \end{pmatrix} E_{i,j} \overline{c(h)} \end{aligned} \tag{2.8}$$

for $1 \le i \le m$, $1 \le j \le n$, $h \in \mathcal{H}_2$, where $E_{i,j}$ denotes the $m \times n$ matrix with (i,j)-entry being 1 and others being zero, $\overline{c(h)}$ denotes the conjugate of $c(h)$. Since $\{e_i \otimes u_j : 1 \le i \le m, 1 \le j \le n\}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$, observe that

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \operatorname{span}\{e_i \otimes u_j : 1 \le i \le m, 1 \le j \le n\}.$$

This leads to (2.7) by (2.8). The proof is completed. □

Next we turn to the main results of this section. The first theorem gives a necessary and sufficient condition for two tensor product Bessel sequences to form a pair of dual frames.

Theorem 2.1 *Assume that* $\mathbf{f} = \{f_i\}_{i \in I}$ *and* $\widetilde{\mathbf{f}} = \{\widetilde{f}_i\}_{i \in I}$ *are Bessel sequences in* $\mathcal{H}_1$, *and* $\mathbf{g} = \{g_j\}_{j \in J}$ *and* $\widetilde{\mathbf{g}} = \{\widetilde{g}_j\}_{j \in J}$ *are Bessel sequences in* $\mathcal{H}_2$. *Then* $\mathbf{f} \otimes \mathbf{g}$ *and* $\widetilde{\mathbf{f}} \otimes \widetilde{\mathbf{g}}$ *form a pair of dual frames in* $\mathcal{H}_1 \otimes \mathcal{H}_2$ *if and only if there exist constants* a *and* b *with* $ab = 1$ *such that*

$$T_{\mathbf{f}} T_{\widetilde{\mathbf{f}}}^* = a I_{\mathcal{H}_1} \quad \text{and} \quad T_{\mathbf{g}} T_{\widetilde{\mathbf{g}}}^* = b I_{\mathcal{H}_2}. \tag{2.9}$$

Proof Since $\mathbf{f}$ and $\widetilde{\mathbf{f}}$ are Bessel sequences in $\mathcal{H}_1$ and $\mathbf{g}$ and $\widetilde{\mathbf{g}}$ are Bessel sequences in $\mathcal{H}_2$, by the definition of dual frame, $\mathbf{f} \otimes \mathbf{g}$ and $\widetilde{\mathbf{f}} \otimes \widetilde{\mathbf{g}}$ are a pair of dual frames in $\mathcal{H}_1 \otimes \mathcal{H}_2$ if and only if

$$T_{\mathbf{f} \otimes \mathbf{g}} T_{\widetilde{\mathbf{f}} \otimes \widetilde{\mathbf{g}}}^* = I_{\mathcal{H}_1 \otimes \mathcal{H}_2}.$$

This is in turn equivalent to

$$T_{\mathbf{f} \otimes \mathbf{g}} T_{\widetilde{\mathbf{f}} \otimes \widetilde{\mathbf{g}}}^* (e_i \otimes u_j) = e_i \otimes u_j \quad \text{for } (i,j) \in I \times J \tag{2.10}$$

due to the fact that $\{e_i \otimes u_j : (i,j) \in I \times J\}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. Therefore, to finish the proof, we only need to demonstrate that (2.10) is equivalent to (2.9). Next we do this. By Lemma 2.1 and Proposition 1.1(c), we have that

$$\begin{aligned} T_{\mathbf{f} \otimes \mathbf{g}} T_{\widetilde{\mathbf{f}} \otimes \widetilde{\mathbf{g}}}^* &= (T_{\mathbf{f}} \otimes T_{\mathbf{g}}) \bigl(T_{\widetilde{\mathbf{f}}}^* \otimes T_{\widetilde{\mathbf{g}}}^*\bigr) \\ &= \bigl(T_{\mathbf{f}} T_{\widetilde{\mathbf{f}}}^*\bigr) \otimes \bigl(T_{\mathbf{g}} T_{\widetilde{\mathbf{g}}}^*\bigr). \end{aligned}$$

It follows that

$$T_{\mathbf{f} \otimes \mathbf{g}} T_{\widetilde{\mathbf{f}} \otimes \widetilde{\mathbf{g}}}^* (e_i \otimes u_j) = \bigl(T_{\mathbf{f}} T_{\widetilde{\mathbf{f}}}^* e_i\bigr) \otimes \bigl(T_{\mathbf{g}} T_{\widetilde{\mathbf{g}}}^* u_j\bigr) \quad \text{for } (i,j) \in I \times J$$

by Proposition 1.1(b). So (2.10) can be rewritten as

$$\bigl(T_{\mathbf{f}} T_{\widetilde{\mathbf{f}}}^* e_i\bigr) \otimes \bigl(T_{\mathbf{g}} T_{\widetilde{\mathbf{g}}}^* u_j\bigr) = e_i \otimes u_j \quad \text{for } (i,j) \in I \times J, \tag{2.11}$$

equivalently, for each $(i,j) \in I \times J$,

$$T_{\mathbf{f}} T_{\widetilde{\mathbf{f}}}^{*} e_i = a_i e_i \quad \text{and} \quad T_{\mathbf{g}} T_{\widetilde{\mathbf{g}}}^{*} u_j = b_j u_j \quad \text{with } a_i b_j = 1 \tag{2.12}$$

by Proposition 1.1(f), where a_i, b_j are two constants. Fix $i_0 \in I$, then $b_j = \frac{1}{a_{i_0}}$ for every $j \in J$. It follows that all b_j take the same value b. Again from $a_i b = 1$ for every $i \in I$, we see that all a_i take the same value a. This implies that (2.12) is equivalent to

$$T_{\mathbf{f}} T_{\widetilde{\mathbf{f}}}^{*} = a I_{\mathcal{H}_1} \quad \text{and} \quad T_{\mathbf{g}} T_{\widetilde{\mathbf{g}}}^{*} = b I_{\mathcal{H}_2} \quad \text{with } ab = 1$$

due to the fact that $\{e_i\}_{i\in I}$ and $\{u_j\}_{j\in J}$ are orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The proof is completed. □

Observe that, given Bessel sequences $\mathbf{f} = \{f_i\}_{i\in I}$, $\widetilde{\mathbf{f}} = \{\widetilde{f}_i\}_{i\in I}$ in $\mathcal{H}_1$ and $\mathbf{g} = \{g_j\}_{j\in J}$, $\widetilde{\mathbf{g}} = \{\widetilde{g}_j\}_{j\in J}$ in $\mathcal{H}_2$, $\mathbf{f}$ and $\widetilde{\mathbf{f}}$ ($\mathbf{g}$ and $\widetilde{\mathbf{g}}$) form a pair of dual frames in $\mathcal{H}_1$ ($\mathcal{H}_2$) if and only if $T_{\mathbf{f}} T_{\widetilde{\mathbf{f}}}^{*} = I_{\mathcal{H}_1}$ ($T_{\mathbf{g}} T_{\widetilde{\mathbf{g}}}^{*} = I_{\mathcal{H}_2}$). As a special case of Theorem 2.1, we have the following corollary. It shows that the dual property is preserved under the tensor product operation.

Corollary 2.1 *Let $\mathbf{f} = \{f_i\}_{i\in I}$ and $\mathbf{g} = \{g_j\}_{j\in J}$ be frames for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume that $\widetilde{\mathbf{f}} = \{\widetilde{f}_i\}_{i\in I}$ and $\widetilde{\mathbf{g}} = \{\widetilde{g}_j\}_{j\in J}$ are dual frames of $\mathbf{f}$ and $\mathbf{g}$, respectively. Then $\widetilde{\mathbf{f}} \otimes \widetilde{\mathbf{g}}$ is a dual frame of $\mathbf{f} \otimes \mathbf{g}$.*

Corollary 2.1 and Proposition 1.2 show that a tensor product frame always admits a tensor product dual. It is natural to ask whether all duals of a tensor product frame are of tensor product. The following theorem characterizes all duals of a general tensor product frame. Then using it we derive Theorem 2.3 which shows the existence of non-tensor product duals of tensor product frames.

Theorem 2.2 *Let $\mathbf{f} \otimes \mathbf{g}$ be a frame for $\mathcal{H}_1 \otimes \mathcal{H}_2$ with $\mathbf{f} = \{f_i\}_{i\in I}$ and $\mathbf{g} = \{g_j\}_{j\in J}$. Then the dual frames of $\mathbf{f} \otimes \mathbf{g}$ are precisely the families*

$$\left\{ S_{\mathbf{f}}^{-1} f_i \otimes S_{\mathbf{g}}^{-1} g_j + w_{i,j} - \sum_{(i',j')\in I\times J} \langle S_{\mathbf{f}}^{-1} f_i, f_{i'} \rangle \langle S_{\mathbf{g}}^{-1} g_j, g_{j'} \rangle w_{i',j'} \right\}_{(i,j)\in I\times J}, \tag{2.13}$$

where $\{w_{i,j}\}_{(i,j)\in I\times J}$ is a Bessel sequence in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Proof By Proposition 1.2, $\mathbf{f}$ and $\mathbf{g}$ are frames for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. So $S_{\mathbf{f}}$ and $S_{\mathbf{g}}$ are well-defined, bounded, and invertible. By Theorem 6.3.7 in [3], the dual frames are precisely the families of the form

$$\left\{ S_{\mathbf{f}\otimes\mathbf{g}}^{-1} (f_i \otimes g_j) + w_{i,j} - \sum_{(i',j')\in I\times J} \langle S_{\mathbf{f}\otimes\mathbf{g}}^{-1} (f_i \otimes g_j), f_{i'} \otimes g_{j'} \rangle w_{i',j'} \right\}_{(i,j)\in I\times J}, \tag{2.14}$$

where $\{w_{i,j}\}_{(i,j)\in I\times J}$ is a Bessel sequence in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Also, by Proposition 3.2 of [10], $S_{\mathbf{f}\otimes\mathbf{g}}^{-1} = S_{\mathbf{f}}^{-1} \otimes S_{\mathbf{g}}^{-1}$. This implies that

$$S_{\mathbf{f}\otimes\mathbf{g}}^{-1} (f_i \otimes g_j) = S_{\mathbf{f}}^{-1} f_i \otimes S_{\mathbf{g}}^{-1} g_j,$$

$$\langle S_{\mathbf{f}\otimes\mathbf{g}}^{-1}(f_i \otimes g_j), f_{i'} \otimes g_{j'}\rangle = \langle S_{\mathbf{f}}^{-1}f_i, f_{i'}\rangle\langle S_{\mathbf{g}}^{-1}g_j, g_{j'}\rangle$$

for $(i,j),(i',j') \in I \times J$ by Proposition 1.1(b) and (1.8). Therefore, (2.14) can be written as (2.13). The proof is completed. □

Lemma 2.2 shows that every element of $\mathcal{H}_1 \otimes \mathcal{H}_2$ is of tensor product whenever one of $\dim(\mathcal{H}_1)$ and $\dim(\mathcal{H}_2)$ equals 1. So a tensor product frame in $\mathcal{H}_1 \otimes \mathcal{H}_2$ admits a non-tensor product dual only if both $\dim(\mathcal{H}_1)$ and $\dim(\mathcal{H}_2)$ are greater than 1. Next, with the help of Theorem 2.2, we will prove that this condition is also sufficient for a redundant tensor product frame to admit a non-tensor product dual.

Theorem 2.3 *Let* $\dim(\mathcal{H}_1), \dim(\mathcal{H}_2) > 1$, *and* $\mathbf{f}\otimes\mathbf{g}$ *be a redundant frame for* $\mathcal{H}_1 \otimes \mathcal{H}_2$ *with* $\mathbf{f} = \{f_i\}_{i\in I}$ *and* $\mathbf{g} = \{g_j\}_{j\in J}$. *Then* $\mathbf{f}\otimes\mathbf{g}$ *admits at least one non-tensor product dual.*

Proof By Theorem 2.2 and Lemma 2.2, to finish the proof, we only need to find a Bessel sequence $\{w_{i,j}\}_{(i,j)\in I\times J}$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that

$$\dim\big(\mathrm{range}(T_{i,j})\big) \geq 2 \quad \text{for some } (i,j) \in I \times J, \tag{2.15}$$

where

$$T_{i,j} = S_{\mathbf{f}}^{-1}f_i \otimes S_{\mathbf{g}}^{-1}g_j + w_{i,j} - \sum_{(i',j')\in I\times J} \langle S_{\mathbf{f}}^{-1}f_i, f_{i'}\rangle\langle S_{\mathbf{g}}^{-1}g_j, g_{j'}\rangle w_{i',j'}.$$

Since $\mathbf{f}\otimes\mathbf{g}$ is not a Riesz basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$, there exist $(i_1,j_1),(i_2,j_2) \in I \times J$ such that

$$\langle S_{\mathbf{f}\otimes\mathbf{g}}^{-1}(f_{i_1} \otimes g_{j_1}), f_{i_2} \otimes g_{j_2}\rangle \neq \delta_{i_1,i_2}\delta_{j_1,j_2} \tag{2.16}$$

by [3, Theorem 7.1]. By [10, Proposition 3.2], Proposition 1.1(b), and (1.8), we have

$$\langle S_{\mathbf{f}\otimes\mathbf{g}}^{-1}(f_{i_1} \otimes g_{j_1}), f_{i_2} \otimes g_{j_2}\rangle = \langle S_{\mathbf{f}}^{-1}f_{i_1}, f_{i_2}\rangle\langle S_{\mathbf{g}}^{-1}g_{j_1}, g_{j_2}\rangle.$$

Hence, (2.16) is equivalent to

$$\langle S_{\mathbf{f}}^{-1}f_{i_1}, f_{i_2}\rangle\langle S_{\mathbf{g}}^{-1}g_{j_1}, g_{j_2}\rangle \neq \delta_{i_1,i_2}\delta_{j_1,j_2}. \tag{2.17}$$

By a simple argument, we have that (2.17) holds if and only if one of the following three conditions happens:

Condition 1 $\langle S_{\mathbf{f}}^{-1}f_{i_1}, f_{i_1}\rangle\langle S_{\mathbf{g}}^{-1}g_{j_1}, g_{j_1}\rangle \neq 1$, $S_{\mathbf{f}}^{-1}f_{i_1} \otimes S_{\mathbf{g}}^{-1}g_{j_1} \neq 0$.

Condition 2 $\langle S_{\mathbf{f}}^{-1}f_{i_1}, f_{i_1}\rangle\langle S_{\mathbf{g}}^{-1}g_{j_1}, g_{j_1}\rangle \neq 1$, $S_{\mathbf{f}}^{-1}f_{i_1} \otimes S_{\mathbf{g}}^{-1}g_{j_1} = 0$.

Condition 3 $(i_1,j_1) \neq (i_2,j_2)$, $\langle S_{\mathbf{f}}^{-1}f_{i_1}, f_{i_1}\rangle\langle S_{\mathbf{g}}^{-1}g_{j_1}, g_{j_1}\rangle = 1$, $\langle S_{\mathbf{f}}^{-1}f_{i_1}, f_{i_2}\rangle\langle S_{\mathbf{g}}^{-1}g_{j_1}, g_{j_2}\rangle \neq 0$.

Next we prove that, whenever one of the above conditions is satisfied, we may construct a Bessel sequence $\{w_{i,j}\}_{(i,j)\in I\times J}$ such that $\mathrm{range}(T_{i_1,j_1})$ contains two orthogonal nonzero vectors. This leads to (2.15) with (i_1,j_1).

For Condition 1, choose unit vectors $u \in \mathcal{H}_1$ and $v \in \mathcal{H}_2$ such that

$$\langle u, S_{\mathbf{f}}^{-1}f_{i_1}\rangle = \langle v, S_{\mathbf{g}}^{-1}g_{j_1}\rangle = 0. \tag{2.18}$$

This can be done since $\dim\mathcal{H}_1, \dim\mathcal{H}_2 > 1$. Define $\{w_{i,j}\}_{(i,j)\in I\times J}$ by

$$w_{i,j} = \begin{cases} \frac{u\otimes v}{1-\langle S_{\mathbf{f}}^{-1}f_{i_1},f_{i_1}\rangle\langle S_{\mathbf{g}}^{-1}g_{j_1},g_{j_1}\rangle} & \text{if } (i,j) = (i_1,j_1); \\ 0 & \text{otherwise.} \end{cases}$$

Then by a direct computation, we have

$$T_{i_1,j_1} = S_{\mathbf{f}}^{-1}f_{i_1} \otimes S_{\mathbf{g}}^{-1}g_{j_1} + u \otimes v.$$

It follows from (2.18) that

$$T_{i_1,j_1}\big(S_{\mathbf{g}}^{-1}g_{j_1}\big) = \big\|S_{\mathbf{g}}^{-1}g_{j_1}\big\|^2 S_{\mathbf{f}}^{-1}f_{i_1}, \qquad T_{i_1,j_1}(v) = u,$$

which are two orthogonal nonzero vectors.

For Condition 2, choose unit vectors $u_1, u_2 \in \mathcal{H}_1$ and $v_1, v_2 \in \mathcal{H}_2$ such that

$$\langle u_1, u_2\rangle = \langle v_1, v_2\rangle = 0. \tag{2.19}$$

Define $\{w_{i,j}\}_{(i,j)\in I\times J}$ by

$$w_{i,j} = \begin{cases} u_1 \otimes v_1 + u_2 \otimes v_2 & \text{if } (i,j) = (i_1,j_1); \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$T_{i_1,j_1} = w_{i_1,j_1}.$$

It follows from (2.19) that

$$T_{i_1,j_1}(v_1) = u_1, \qquad T_{i_1,j_1}(v_2) = u_2,$$

which are two orthogonal unit vectors.

For Condition 3, choose unit vectors $u \in \mathcal{H}_1$ and $v \in \mathcal{H}_2$ such that

$$\big\langle u, S_{\mathbf{f}}^{-1}f_{i_1}\big\rangle = \big\langle v, S_{\mathbf{g}}^{-1}g_{j_1}\big\rangle = 0. \tag{2.20}$$

Define $\{w_{i,j}\}_{(i,j)\in I\times J}$ by

$$w_{i,j} = \begin{cases} -\frac{u\otimes v}{\langle S_{\mathbf{f}}^{-1}f_{i_1},f_{i_2}\rangle\langle S_{\mathbf{g}}^{-1}g_{j_1},g_{j_2}\rangle} & \text{if } (i,j) = (i_2,j_2); \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$T_{i_1,j_1} = S_{\mathbf{f}}^{-1}f_{i_1} \otimes S_{\mathbf{g}}^{-1}g_{j_1} + u \otimes v.$$

It follows from (2.20) that

$$T_{i_1,j_1}\left(S_{\mathbf{g}}^{-1}g_{j_1}\right) = \left\|S_{\mathbf{g}}^{-1}g_{j_1}\right\|^2 S_{\mathbf{f}}^{-1}f_{i_1}, \qquad T_{i_1,j_1}(v) = u,$$

which are two orthogonal nonzero vectors. The proof is completed. □

Next we give an example of Theorem 2.3 in the finite dimensional case.

Example 2.1 Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two finite dimensional Hilbert spaces, and $\{e_i\}_{i=1}^m$ and $\{u_j\}_{j=1}^n$ be orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume that $\mathbf{f}\otimes\mathbf{g}$ is a redundant frame for $\mathcal{H}_1\otimes\mathcal{H}_2$ with $\mathbf{f} = \{f_i\}_{i=1}^M$ and $\mathbf{g} = \{g_j\}_{j=1}^N$. Then, by a simple calculation, for any $f\in\mathcal{H}_1$, $g\in\mathcal{H}_2$, we have that

$$\begin{aligned}
&c(S_{\mathbf{f}}f) = M_{\mathbf{f}}M_{\mathbf{f}}^*c(f), \qquad c(S_{\mathbf{g}}g) = M_{\mathbf{g}}M_{\mathbf{g}}^*c(g),\\
&c\left(S_{\mathbf{f}}^{-1}f\right) = \left(M_{\mathbf{f}}M_{\mathbf{f}}^*\right)^{-1}c(f), \qquad c\left(S_{\mathbf{g}}^{-1}g\right) = \left(M_{\mathbf{g}}M_{\mathbf{g}}^*\right)^{-1}c(g),
\end{aligned}$$

and

$$\begin{aligned}
&\left(S_{\mathbf{f}}^{-1}f_i\otimes S_{\mathbf{g}}^{-1}g_j\right)g\\
&\quad = \left(S_{\mathbf{f}}^{-1}f_i\otimes S_{\mathbf{g}}^{-1}g_j\right)\begin{pmatrix}u_1, & \cdots, & u_n\end{pmatrix}c(g)\\
&\quad = \begin{pmatrix}e_1, & \cdots, & e_m\end{pmatrix}\left(c\left(S_{\mathbf{f}}^{-1}f_i\right)\otimes c\left(S_{\mathbf{g}}^{-1}g_j\right)\right)c(g)\\
&\quad = \begin{pmatrix}e_1, & \cdots, & e_m\end{pmatrix}\begin{pmatrix}c_1(S_{\mathbf{g}}^{-1}g_j)c(S_{\mathbf{f}}^{-1}f_i), & \cdots, & c_n(S_{\mathbf{g}}^{-1}g_j)c(S_{\mathbf{f}}^{-1}f_i)\end{pmatrix}\overline{c(g)}\\
&\quad = \begin{pmatrix}e_1, & \cdots, & e_m\end{pmatrix}\Lambda_{i,j}\overline{c(g)},
\end{aligned} \tag{2.21}$$

for $(i,j)\in\{1,2,\ldots,M\}\times\{1,2,\ldots,N\}$. By Lemma 2.3, an arbitrary sequence $\{w_{i,j}\}_{i=1,j=1}^{M,N}$ in $\mathcal{H}_1\otimes\mathcal{H}_2$ has the form

$$w_{i,j}g = w_{i,j}\begin{pmatrix}u_1 & \cdots & u_n\end{pmatrix}c(g) = \begin{pmatrix}e_1 & \cdots & e_m\end{pmatrix}\Omega_{i,j}\overline{c(g)} \tag{2.22}$$

for $(i,j)\in\{1,2,\ldots,M\}\times\{1,2,\ldots,N\}$ and $g\in\mathcal{H}_2$, where $\Omega_{i,j}$ is an $m\times n$ matrix for every $(i,j)\in\{1,2,\ldots,M\}\times\{1,2,\ldots,N\}$. It follows by Theorem 2.2 that all dual frames of $\mathbf{f}\otimes\mathbf{g}$ have the form

$$\begin{aligned}
\Bigg\{T_{i,j} : T_{i,j}g = \begin{pmatrix}e_1, & \cdots, & e_m\end{pmatrix}\Bigg[\Lambda_{i,j} + \Omega_{i,j}\\
- \sum_{i'=1}^{M}\sum_{j'=1}^{N}\left\langle c\left(S_{\mathbf{f}}^{-1}f_i\right), c(f_{i'})\right\rangle\left\langle c\left(S_{\mathbf{g}}^{-1}g_j\right), c(g_{j'})\right\rangle\Omega_{i',j'}\Bigg]\overline{c(g)} \text{ for } g\in\mathcal{H}_2\Bigg\}_{i=1,j=1}^{M,N},
\end{aligned} \tag{2.23}$$

where $\Omega_{i,j}$ is an $m\times n$ matrix for each $(i,j)\in\{1,2,\ldots,M\}\times\{1,2,\ldots,N\}$.

Write

$$\Upsilon_{i,j} = \Lambda_{i,j} + \Omega_{i,j} - \sum_{i'=1}^{M}\sum_{j'=1}^{N}\left\langle c\left(S_{\mathbf{f}}^{-1}f_i\right), c(f_{i'})\right\rangle\left\langle c\left(S_{\mathbf{g}}^{-1}g_j\right), c(g_{j'})\right\rangle\Omega_{i',j'}$$

for $(i,j) \in \{1,2,\dots,M\} \times \{1,2,\dots,N\}$. Obviously, (2.23) is determined by $\{\Omega_{i,j}\}_{i=1,j=1}^{M,N}$. Hence, by Lemma 2.2, constructing a non-tensor product dual frame of $\mathbf{f} \otimes \mathbf{g}$ reduces to constructing $\{\Omega_{i,j}\}_{i=1,j=1}^{M,N}$ satisfying

$$\operatorname{rank}(\Upsilon_{i,j}) \geq 2 \quad \text{for some } (i,j) \in \{1,2,\dots,M\} \times \{1,2,\dots,N\}. \tag{2.24}$$

Since $\mathbf{f} \otimes \mathbf{g}$ is not a Riesz basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$, by [3, Theorem 7.1], we have that

$$\begin{aligned}\big\langle c\big(S_{\mathbf{f}}^{-1} f_{i_1}\big), c(f_{i_2})\big\rangle\big\langle c\big(S_{\mathbf{g}}^{-1} g_{j_1}\big), c(g_{j_2})\big\rangle &= \big\langle S_{\mathbf{f}}^{-1} f_{i_1}, f_{i_2}\big\rangle\big\langle S_{\mathbf{g}}^{-1} g_{j_1}, g_{j_2}\big\rangle \\ &= \big\langle S_{\mathbf{f}\otimes\mathbf{g}}^{-1}(f_{i_1} \otimes g_{j_1}), f_{i_2} \otimes g_{j_2}\big\rangle \\ &\neq \delta_{i_1,i_2}\delta_{j_1,j_2}\end{aligned} \tag{2.25}$$

for some $(i_1,j_1),(i_2,j_2) \in \{1,2,\dots,M\} \times \{1,2,\dots,N\}$. According to the arguments in the proof of Theorem 2.3, we divide the following three cases to construct $\{\Omega_{i,j}\}_{i=1,j=1}^{M,N}$ satisfying (2.24).

Case 1. $\langle c(S_{\mathbf{f}}^{-1} f_{i_1}), c(f_{i_1})\rangle\langle c(S_{\mathbf{g}}^{-1} g_{j_1}), c(g_{j_1})\rangle \neq 1$, $c(S_{\mathbf{f}}^{-1} f_{i_1}) \otimes c(S_{\mathbf{g}}^{-1} g_{j_1}) \neq 0$.

Choose two linearly-independent nonzero vectors $d_1, d_2 \in \mathbb{C}^m$. Define $\{\Omega_{i,j}\}_{i=1,j=1}^{M,N}$ by $\Omega_{i,j} =$

$$\begin{cases} \frac{(d_1 - c_1(S_{\mathbf{g}}^{-1} g_{j_1}) c(S_{\mathbf{f}}^{-1} f_{i_1}), d_2 - c_2(S_{\mathbf{g}}^{-1} g_{j_1}) c(S_{\mathbf{f}}^{-1} f_{i_1}), 0, \dots, 0)}{1 - \langle c(S_{\mathbf{f}}^{-1} f_{i_1}), c(f_{i_1})\rangle\langle c(S_{\mathbf{g}}^{-1} g_{j_1}), c(g_{j_1})\rangle} & \text{if } (i,j) = (i_1,j_1); \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\Upsilon_{i_1,j_1} = \begin{pmatrix} d_1, & d_2, & c_3(S_{\mathbf{g}}^{-1} g_{j_1}) c(S_{\mathbf{f}}^{-1} f_{i_1}), & \cdots, & c_n(S_{\mathbf{g}}^{-1} g_{j_1}) c(S_{\mathbf{f}}^{-1} f_{i_1}) \end{pmatrix}$$

by a simple computation. This implies that (2.24) holds for (i_1,j_1).

Case 2. $\langle c(S_{\mathbf{f}}^{-1} f_{i_1}), c(f_{i_1})\rangle\langle c(S_{\mathbf{g}}^{-1} g_{j_1}), c(g_{j_1})\rangle \neq 1$, $c(S_{\mathbf{f}}^{-1} f_{i_1}) \otimes c(S_{\mathbf{g}}^{-1} g_{j_1}) = 0$.

Choose two linearly-independent nonzero vectors $d_1, d_2 \in \mathbb{C}^m$. Define $\{\Omega_{i,j}\}_{i=1,j=1}^{M,N}$ by

$$\Omega_{i,j} = \begin{cases} \begin{pmatrix} d_1, & d_2, & 0, & \cdots, & 0 \end{pmatrix} & \text{if } (i,j) = (i_1,j_1); \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\Upsilon_{i_1,j_1} = \Omega_{i_1,j_1}.$$

This implies that (2.24) holds for (i_1,j_1).

Case 3. $(i_1,j_1) \neq (i_2,j_2)$, $\langle c(S_{\mathbf{f}}^{-1} f_{i_1}), c(f_{i_1})\rangle\langle c(S_{\mathbf{g}}^{-1} g_{j_1}), c(g_{j_1})\rangle = 1$, and $\langle c(S_{\mathbf{f}}^{-1} f_{i_1}), c(f_{i_2})\rangle \times \langle c(S_{\mathbf{g}}^{-1} g_{j_1}), c(g_{j_2})\rangle \neq 0$.

Choose two linearly-independent nonzero vectors $d_1, d_2 \in \mathbb{C}^m$. Define $\{\Omega_{i,j}\}_{i=1,j=1}^{M,N}$ by $\Omega_{i,j} =$

$$\begin{cases} \frac{(c_1(S_{\mathbf{g}}^{-1} g_{j_1}) c(S_{\mathbf{f}}^{-1} f_{i_1}) - d_1, c_2(S_{\mathbf{g}}^{-1} g_{j_1}) c(S_{\mathbf{f}}^{-1} f_{i_1}) - d_2, 0, \dots, 0)}{\langle c(S_{\mathbf{f}}^{-1} f_{i_1}), c(f_{i_2})\rangle\langle c(S_{\mathbf{g}}^{-1} g_{j_1}), c(g_{j_2})\rangle} & \text{if } (i,j) = (i_2,j_2); \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\Upsilon_{i_1,j_1} = \begin{pmatrix} d_1, & d_2, & c_3(S_{\mathbf{g}}^{-1}g_{j_1})c(S_{\mathbf{f}}^{-1}f_{i_1}), & \cdots, & c_n(S_{\mathbf{g}}^{-1}g_{j_1})c(S_{\mathbf{f}}^{-1}f_{i_1}) \end{pmatrix}$$

by a simple computation. This implies that (2.24) holds for (i_1, j_1). The proof is completed.

3 Tensor product dual expression

This section is devoted to expressing tensor product duals of tensor product frames. For this purpose, we first give the following lemma.

Lemma 3.1 *Let $\{e_i\}_{i=1}^m$ and $\{u_j\}_{j=1}^n$ be orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, $\mathbf{f} = \{f_i\}_{i=1}^M$, $\widetilde{\mathbf{f}} = \{\widetilde{f}_i\}_{i=1}^M$ be sequences in $\mathcal{H}_1$, and $\mathbf{g} = \{g_j\}_{j=1}^N$, $\widetilde{\mathbf{g}} = \{\widetilde{g}_j\}_{j=1}^N$ be sequences in $\mathcal{H}_2$. Then, for any $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$,*

$$T_{\mathbf{f}}T_{\widetilde{\mathbf{f}}}^*f = \begin{pmatrix} e_1, & e_2, & \cdots, & e_m \end{pmatrix} M_{\mathbf{f}}M_{\widetilde{\mathbf{f}}}^*c(f) \tag{3.1}$$

and

$$T_{\mathbf{g}}T_{\widetilde{\mathbf{g}}}^*g = \begin{pmatrix} u_1, & u_2, & \cdots, & u_n \end{pmatrix} M_{\mathbf{g}}M_{\widetilde{\mathbf{g}}}^*c(g), \tag{3.2}$$

where $T_{\mathbf{f}}$, $T_{\widetilde{\mathbf{f}}}$, $T_{\mathbf{g}}$, $T_{\widetilde{\mathbf{g}}}$, $M_{\mathbf{f}}$, $M_{\widetilde{\mathbf{f}}}$, $M_{\mathbf{g}}$, and $M_{\widetilde{\mathbf{g}}}$ are defined as at the beginning of Sect. 2.

Proof We only prove (3.1). (3.2) can be proved similarly. Arbitrarily fix $1 \le i \le m$. Since $T_{\widetilde{\mathbf{f}}}^*e_i = \{\overline{c_i(\widetilde{f}_l)}\}_{l=1}^M$, we have

$$\begin{aligned} T_{\mathbf{f}}T_{\widetilde{\mathbf{f}}}^*e_i &= \begin{pmatrix} f_1, & f_2 & \cdots, & f_m \end{pmatrix} \begin{pmatrix} \overline{c_i(\widetilde{f}_1)} \\ \overline{c_i(\widetilde{f}_2)} \\ \vdots \\ \overline{c_i(\widetilde{f}_M)} \end{pmatrix} \\ &= \begin{pmatrix} e_1, & e_2, & \cdots, & e_m \end{pmatrix} M_{\mathbf{f}} \begin{pmatrix} \overline{c_i(\widetilde{f}_1)} \\ \overline{c_i(\widetilde{f}_2)} \\ \vdots \\ \overline{c_i(\widetilde{f}_M)} \end{pmatrix}. \end{aligned}$$

It follows that

$$\begin{aligned} T_{\mathbf{f}}T_{\widetilde{\mathbf{f}}}^*f &= T_{\mathbf{f}}T_{\widetilde{\mathbf{f}}}^* \begin{pmatrix} e_1, & e_2, & \cdots, & e_m \end{pmatrix} c(f) \\ &= \begin{pmatrix} e_1, & e_2, & \cdots, & e_m \end{pmatrix} M_{\mathbf{f}}M_{\widetilde{\mathbf{f}}}^*c(f) \end{aligned}$$

for $f \in \mathcal{H}_1$. The proof is completed. □

Theorem 3.1 *Let $\{e_i\}_{i\in I}$ and $\{u_j\}_{j\in J}$ be orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Given sequences $\mathbf{f} = \{f_i\}_{i\in I}$, $\widetilde{\mathbf{f}} = \{\widetilde{f}_i\}_{i\in I}$ in $\mathcal{H}_1$ and $\mathbf{g} = \{g_j\}_{j\in J}$, $\widetilde{\mathbf{g}} = \{\widetilde{g}_j\}_{j\in J}$ in $\mathcal{H}_2$, $\mathbf{f} \otimes \mathbf{g}$ and $\widetilde{\mathbf{f}} \otimes \widetilde{\mathbf{g}}$*

form a pair of dual frames in $\mathcal{H}_1 \otimes \mathcal{H}_2$ *if and only if there exist constants* a, b *with* $ab = 1$, *linear bounded operators* $U, \widetilde{U}$ *on* $\mathcal{H}_1$ *and* $V, \widetilde{V}$ *on* $\mathcal{H}_2$ *such that*

$$f_i = Ue_i, \qquad \widetilde{f}_i = \widetilde{U}e_i \quad \textit{for } i \in I, \tag{3.3}$$

$$g_j = Vu_j, \qquad \widetilde{g}_j = \widetilde{V}u_j \quad \textit{for } j \in J \tag{3.4}$$

and

$$U\widetilde{U}^* = aI_{\mathcal{H}_1}, \qquad V\widetilde{V}^* = bI_{\mathcal{H}_2}. \tag{3.5}$$

Proof Observe that, if $\dim \mathcal{H}_1 < \infty (\dim \mathcal{H}_2 < \infty)$, then $\mathbf{f} \otimes \mathbf{g}$ and $\widetilde{\mathbf{f}} \otimes \widetilde{\mathbf{g}}$ forming a pair of dual frames in $\mathcal{H}_1 \otimes \mathcal{H}_2$ implies that $\mathbf{f}$ and $\widetilde{\mathbf{f}}$ ($\mathbf{g}$ and $\widetilde{\mathbf{g}}$) are Riesz bases for $\mathcal{H}_1$ ($\mathcal{H}_2$) due to $\mathrm{card}(I) = \dim \mathcal{H}_1$ ($\mathrm{card}(J) = \dim \mathcal{H}_2$). This implies that there exist linear bounded and invertible operators U and $\widetilde{U}$ (V and $\widetilde{V}$) such that (3.3) ((3.4)) holds. For the case that $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = \infty$, observe that a frame must be an image of an orthonormal basis under a linear bounded surjection, and that the bounded operators U, $\widetilde{U}$, V, and $\widetilde{V}$ satisfying (3.5) are surjections. We may as well assume that $\mathbf{f}$, $\widetilde{\mathbf{f}}$, $\mathbf{g}$, and $\widetilde{\mathbf{g}}$ have the following form:

$$f_i = Ue_i, \qquad \widetilde{f}_i = \widetilde{U}e_i \quad \text{for } i \in I,$$
$$g_j = Vu_j, \qquad \widetilde{g}_j = \widetilde{V}u_j \quad \text{for } j \in J,$$

where U and $\widetilde{U}$ are linear bounded surjections on $\mathcal{H}_1$, and V and $\widetilde{V}$ are linear bounded surjections on $\mathcal{H}_2$. Observe that $T_{\mathbf{f}}$, $T_{\widetilde{\mathbf{f}}}$, $T_{\mathbf{g}}$, and $T_{\widetilde{\mathbf{g}}}$ are all linear bounded operators. By Theorem 2.1, $\mathbf{f} \otimes \mathbf{g}$ and $\widetilde{\mathbf{f}} \otimes \widetilde{\mathbf{g}}$ are a pair of dual frames in $\mathcal{H}_1 \otimes \mathcal{H}_2$ if and only if there exist constants a and b with $ab = 1$ such that

$$T_{\mathbf{f}} T_{\widetilde{\mathbf{f}}}^* = aI_{\mathcal{H}_1} \quad \text{and} \quad T_{\mathbf{g}} T_{\widetilde{\mathbf{g}}}^* = bI_{\mathcal{H}_2}. \tag{3.6}$$

By (3.3) and (3.4), we have

$$\begin{aligned} T_{\mathbf{f}} T_{\widetilde{\mathbf{f}}}^* e_i &= \sum_{k \in I} \langle e_i, \widetilde{U}e_k \rangle Ue_k \\ &= U\left(\sum_{k \in I} \langle \widetilde{U}^* e_i, e_k \rangle e_k \right) \\ &= U\widetilde{U}^* e_i \end{aligned}$$

for $i \in I$. Similarly,

$$T_{\mathbf{g}} T_{\widetilde{\mathbf{g}}}^* u_j = V\widetilde{V}^* u_j \quad \text{for } j \in J.$$

Therefore, (3.6) is equivalent to

$$U\widetilde{U}^* e_i = ae_i \quad \text{and} \quad V\widetilde{V}^* u_j = bu_j \tag{3.7}$$

for $i \in I$ and $j \in J$. This implies that (3.7) is equivalent to

$$U\widetilde{U}^* = aI_{\mathcal{H}_1} \quad \text{and} \quad V\widetilde{V}^* = bI_{\mathcal{H}_2}$$

due to the fact that $\{e_i\}_{i\in I}$ and $\{u_j\}_{j\in J}$ are orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The proof is completed. □

Remark 3.1 In Theorem 3.1, if $\dim \mathcal{H}_1, \dim \mathcal{H}_2 < \infty$ in addition, then by the beginning argument in the proof of the theorem, $\mathbf{f}$ and $\mathbf{g}$ are Riesz bases for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively whenever $\mathbf{f} \otimes \mathbf{g}$ is a frame for $\mathcal{H}_1 \otimes \mathcal{H}_2$. Therefore, Theorem 3.1 cannot be applied to the case that $\mathbf{f} \otimes \mathbf{g}$ is a redundant frame for $\mathcal{H}_1 \otimes \mathcal{H}_2$ with $\dim \mathcal{H}_1, \dim \mathcal{H}_2 < \infty$. For the case that at least one of $\dim \mathcal{H}_1$ and $\dim \mathcal{H}_2$ is infinity, given a frame $\mathbf{f} \otimes \mathbf{g}$ for $\mathcal{H}_1 \otimes \mathcal{H}_2$ (at this time, U and V are determined by Proposition 1.2), Theorem 3.1 tells us that we may obtain all its tensor product dual frames by designing $\widetilde{U}$ and $\widetilde{V}$ satisfying (3.3)–(3.5).

Remark 3.1 shows that Theorem 3.1 cannot be applied to general tensor product frames in a finite-dimensional setting. The following theorem presents an explicit expression of all tensor product dual frames in the finite-dimensional setting.

Theorem 3.2 *Let $\{e_i\}_{i=1}^m$ and $\{u_j\}_{j=1}^n$ be orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume that $\mathbf{f} \otimes \mathbf{g}$ is a frame for $\mathcal{H}_1 \otimes \mathcal{H}_2$ with $\mathbf{f} = \{f_i\}_{i=1}^M$ in $\mathcal{H}_1$ and $\mathbf{g} = \{g_j\}_{j=1}^N$ in $\mathcal{H}_2$. Then the dual frames of $\mathbf{f} \otimes \mathbf{g}$ with the form of tensor product are precisely the families*

$$\Bigl\{\widetilde{\mathbf{f}} \otimes \widetilde{\mathbf{g}} : \widetilde{\mathbf{f}} = \{\widetilde{f}_i\}_{i=1}^M, \widetilde{\mathbf{g}} = \{\widetilde{g}_j\}_{j=1}^N \text{ and } M_{\mathbf{f}} M_{\widetilde{\mathbf{f}}}^* = aI, M_{\mathbf{g}} M_{\widetilde{\mathbf{g}}}^* = bI \\ \text{for some constants } a, b \text{ satisfying } ab = 1\Bigr\}.$$

Proof Since $\dim(\mathcal{H}_1) = m$ and $\dim(\mathcal{H}_2) = n$, we have $T_{\mathbf{f}}, T_{\widetilde{\mathbf{f}}}, T_{\mathbf{g}}$, and $T_{\widetilde{\mathbf{g}}}$ are all linear bounded operators. By Theorem 2.1, $\mathbf{f} \otimes \mathbf{g}$ and $\widetilde{\mathbf{f}} \otimes \widetilde{\mathbf{g}}$ are a pair of dual frames in $\mathcal{H}_1 \otimes \mathcal{H}_2$ if and only if there exist constants a and b with $ab = 1$ such that

$$T_{\mathbf{f}} T_{\widetilde{\mathbf{f}}}^* = aI_{\mathcal{H}_1} \quad \text{and} \quad T_{\mathbf{g}} T_{\widetilde{\mathbf{g}}}^* = bI_{\mathcal{H}_2}. \tag{3.8}$$

By Lemma 3.1, for any $f \in \mathcal{H}_1, g \in \mathcal{H}_2$, (3.8) is equivalent to

$$(e_1, e_2, \ldots, e_m) M_{\mathbf{f}} M_{\widetilde{\mathbf{f}}}^* c(f) = af \quad \text{and} \quad (u_1, u_2, \ldots, u_n) M_{\mathbf{g}} M_{\widetilde{\mathbf{g}}}^* c(g) = bg. \tag{3.9}$$

It follows that (3.9) is equivalent to

$$M_{\mathbf{f}} M_{\widetilde{\mathbf{f}}}^* = aI \quad \text{and} \quad M_{\mathbf{g}} M_{\widetilde{\mathbf{g}}}^* = bI.$$

Then proof is completed. □

4 Conclusions

To construct dual frames with good structure is a fundamental problem in the theory of frames. The tensor product dual frames can provide a rank-one decomposition of bounded antilinear operators between two Hilbert spaces. This paper addresses tensor product dual frames. We characterize tensor product dual frames, give an explicit expression of all dual frames of tensor product frames, and prove the existence of non-tensor product dual frames. We present an explicit expression of all tensor product dual frames of tensor product frames.

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