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Improvements of bounds for the Sándor–Yang means

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Abstract

In the article, we provide new bounds for two Sándor–Yang means in terms of the arithmetic and contraharmonic means. Our results are the improvements of the previously known results.

MSC: 26E60

Keywords: Sándor–Yang mean; Arithmetic mean; Quadratic mean; Contra-harmonic mean; Neuman–Sándor mean; Seiffert mean; Power mean; Schwab–Borchardt mean

1 Introduction

Let $p \in \mathbb{R}$ and $x, y > 0$ with $x \neq y$. Then the arithmetic mean $A(x,y)$, quadratic mean $Q(x,y)$, contraharmonic mean $C(x,y)$, Neuman–Sándor mean $NS(x,y)$ [1], Seiffert mean $T(x,y)$ [2–5], pth power mean $M_p(x,y)$ [6–13], and Schwab–Borchardt mean $SB(x,y)$ [14, 15] are defined by

$$A(x,y) = \frac{x+y}{2}, \qquad Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}, \qquad C(x,y) = \frac{x^2+y^2}{x+y}, \tag{1.1}$$

$$NS(x,y) = \frac{x-y}{2\sinh^{-1}(\frac{x-y}{x+y})}, \qquad T(x,y) = \frac{x-y}{2\arctan(\frac{x-y}{x+y})}, \tag{1.2}$$

$$M_p(x,y) = \begin{cases} (\frac{x^p+y^p}{2})^{1/p}, & p \neq 0, \\ \sqrt{xy}, & p = 0, \end{cases}$$

and

$$SB(x,y) = \begin{cases} \frac{\sqrt{y^2-x^2}}{\arccos(x/y)}, & x < y, \\ \frac{\sqrt{x^2-y^2}}{\cosh^{-1}(x/y)}, & x > y, \end{cases}$$

respectively, where $\sinh^{-1}(t) = \log(t + \sqrt{t^2+1})$ and $\cosh^{-1}(t) = \log(t + \sqrt{t^2-1})$ are the inverse hyperbolic sine and cosine functions.

Let $U(x,y)$ and $V(x,y)$ be the symmetric bivariate means. Then Yang [16] introduced the Sándor–Yang mean

$$R_{UV}(x,y) =: V(x,y)e^{\frac{U(x,y)}{SB[U(x,y),V(x,y)]}-1},$$

and provided the explicit formulas for $R_{AQ}(x,y)$ and $R_{QA}(x,y)$ as follows:

$$R_{AQ}(x,y) = Q(x,y)e^{A(x,y)/T(x,y)-1}, \tag{1.3}$$

$$R_{QA}(x,y) = A(x,y)e^{Q(x,y)/NS(x,y)-1}. \tag{1.4}$$

Recently, the bounds and properties for certain bivariate means and related special functions have attracted the attention of many researchers [17–28].

Zhao, Qian, and Song [29] proved that the double inequalities

$$M_\alpha(a,b) < R_{QA}(a,b) < M_\beta(a,b),$$
$$M_\lambda(a,b) < R_{AQ}(a,b) < M_\mu(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha \leq \log 2/[1+\log 2-\log(1+\sqrt{2})] = 1.5517\ldots$, $\beta \geq 5/3$, $\lambda \leq 4\log 2/(4+2\log 2-\pi) = 1.2351\ldots$, and $\mu \geq 4/3$.

Xu [30], and Xu, Chu, and Qian [31] proved that the two-sided inequalities

$$C^{1/6}(x,y)A^{5/6}(x,y) < R_{AQ}(x,y) < \frac{1}{6}C(x,y) + \frac{5}{6}A(x,y), \tag{1.5}$$

$$C^{1/3}(x,y)A^{2/3}(x,y) < R_{QA}(x,y) < \frac{1}{3}C(x,y) + \frac{2}{3}A(x,y) \tag{1.6}$$

are valid for all $x,y>0$ with $x\neq y$.

The main purpose of this paper is to improve the bounds for $R_{AQ}(x,y)$ and $R_{QA}(x,y)$ given by (1.5) and (1.6).

2 Lemmas

In order to prove our main results, we need four lemmas which we present in this section.

Lemma 2.1 (see [32, Theorem 1.25]) *Let $a,b\in\mathbb{R}$ with $a<b$, $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2 (see [33, Lemma 1.1]) *Suppose that the power series $f(x)=\sum_{n=0}^{\infty}a_nx^n$ and $g(x)=\sum_{n=0}^{\infty}b_nx^n$ have the radius of convergence $r>0$, and $b_n>0$ for all $n=0,1,2,\ldots$. If there exists $n_0\geq 1$ such that the non-constant sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing) for $0\leq n\leq n_0$ and decreasing (increasing) for $n\geq n_0$, then there exists $x_0\in(0,r)$ such that the function $f(x)/g(x)$ is strictly increasing (decreasing) on $(0,x_0)$ and decreasing (increasing) on (x_0,r).*

Lemma 2.3 *The function*

$$f(t) = \frac{\frac{2}{3}\log[\sec(t)] + \frac{t}{\tan(t)} - 1}{\log[\frac{\sec^2(t)+5}{6}] - \frac{\log[\sec(t)]}{3}} \tag{2.1}$$

is strictly decreasing from $(0, \pi/4)$ *onto* $(3\pi + 4\log 2 - 12)/[2(6\log 7 - 7\log 2 - 6\log 3)], 12/25)$.

Proof Let

$$f_1(t) = \frac{2}{3}\log\bigl[\sec(t)\bigr] + \frac{t}{\tan(t)} - 1,$$
$$f_2(t) = \log\left[\frac{\sec^2(t)+5}{6}\right] - \frac{\log[\sec(t)]}{3},$$
$$f_3(t) = 2\tan(t) + \sin(t)\cos(t) - 3t, \qquad f_4(t) = \frac{5\sin^5(t)}{\cos(t)[1+5\cos^2(t)]}.$$

Then it is not difficult to verify that

$$f_1(0^+) = f_2(0^+) = f_3(0^+) = f_4(0^+) = 0, \tag{2.2}$$
$$f(t) = \frac{f_1(t)}{f_2(t)}, \qquad \frac{f_1'(t)}{f_2'(t)} = \frac{f_3(t)}{f_4(t)}, \tag{2.3}$$
$$\frac{f_3'(t)}{f_4'(t)} = \frac{2[1+5\cos^2(t)]^2}{5[10\cos^4(t)+19\cos^2(t)+1]}, \tag{2.4}$$
$$\left[\frac{f_3'(t)}{f_4'(t)}\right]' = -\frac{2\sin(2t)[1+5\cos^2(t)][75\cos^2(t)-9]}{5[10\cos^4(t)+19\cos^2(t)+1]^2} < 0 \tag{2.5}$$

for $t \in (0, \pi/4)$.

Therefore, the function $f(t)$ is strictly decreasing on $(0, \pi/4)$ follows easily from Lemma 2.1, (2.2), (2.3), and (2.5).

It follows from (2.1)–(2.4) that

$$f(0^+) = \lim_{t\to 0^+}\frac{f_3'(t)}{f_4'(t)} = \frac{12}{25}$$

and

$$f\left(\frac{\pi}{4}\right) = \lim_{t\to\pi/4}\frac{\frac{2}{3}\log[\sec(t)] + \frac{t}{\tan(t)} - 1}{\log[\frac{\sec^2(t)+5}{6}] - \frac{\log[\sec(t)]}{3}} = \frac{3\pi + 4\log 2 - 12}{2(6\log 7 - 7\log 2 - 6\log 3)} = 0.4258\ldots. \qquad \square$$

Lemma 2.4 *The function*

$$g(t) = \frac{t\coth(t) - \frac{2\log[\cosh(t)]}{3} - 1}{\log[\frac{\cosh^2(t)+2}{3}] - \frac{2\log[\cosh(t)]}{3}} \tag{2.6}$$

is strictly decreasing from $(0, \log(1+\sqrt{2}))$ *onto* $([3\sqrt{2}\log(1+\sqrt{2}) - \log 2 - 3]/(5\log 2 - 3\log 3), 3/10)$.

Proof Let

$$g_1(t) = t\coth(t) - \frac{2\log[\cosh(t)]}{3} - 1,$$

$$
\begin{aligned}
g_2(t) &= \log\left[\frac{\cosh^2(t)+2}{3}\right] - \frac{2\log[\cosh(t)]}{3},\\
g_3(t) &= \left[3\sinh(t)+\sinh^3(t)-3t\cosh(t)\right]\left[\cosh^2(t)+2\right],\\
g_4(t) &= 4\sinh^5(t).
\end{aligned}
$$

Then we clearly see that

$$g_1(0^+) = g_2(0^+) = g_3(0^+) = g_4(0^+) = 0, \tag{2.7}$$

$$g(t) = \frac{g_1(t)}{g_2(t)}, \qquad \frac{g_1'(t)}{g_2'(t)} = \frac{g_3(t)}{g_4(t)}. \tag{2.8}$$

Elaborate computations lead to

$$
\begin{aligned}
\frac{g_3'(t)}{g_4'(t)} &= \frac{5\sinh(4t)+50\sinh(2t)-84t-36t\cosh(2t)}{20[\sinh(4t)-2\sinh(2t)]}\\
&= \frac{5\sum_{n=0}^{\infty}\frac{(4t)^{2n+1}}{(2n+1)!}+50\sum_{n=0}^{\infty}\frac{(2t)^{2n+1}}{(2n+1)!}-84t-36t\sum_{n=0}^{\infty}\frac{(2t)^{2n}}{(2n)!}}{20\sum_{n=0}^{\infty}\frac{(4t)^{2n+1}}{(2n+1)!}-40\sum_{n=0}^{\infty}\frac{(2t)^{2n+1}}{(2n+1)!}} =: \frac{\sum_{n=0}^{\infty}a_n}{\sum_{n=0}^{\infty}b_n},
\end{aligned}
\tag{2.9}
$$

where

$$a_n = \frac{(5\times 2^{2n+1}-9n-1)2^{2n+5}}{(2n+3)!}, \qquad b_n = \frac{10(2^{2n+2}-1)2^{2n+5}}{(2n+3)!}. \tag{2.10}$$

From (2.10) we clearly see that

$$\frac{a_1}{b_1} - \frac{a_0}{b_0} = -\frac{1}{10} < 0, \tag{2.11}$$

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = \frac{9[3(2n-1)2^{2n+1}+1]}{10(2^{2n+2}-1)(2^{2n+4}-1)} > 0 \tag{2.12}$$

for all $n \geq 1$, and

$$b_n > 0 \tag{2.13}$$

for all $n \geq 0$.

It follows from Lemma 2.2 and (2.9)–(2.13) that there exists $t_0 \in (0,\infty)$ such that the function $g_3'(t)/g_4'(t)$ is strictly decreasing on $(0,t_0)$ and strictly increasing on (t_0,∞).

Note that

$$
\begin{aligned}
\left[\frac{g_3'(t)}{g_4'(t)}\right]' &= \frac{4\cosh(4t)+16\cosh(2t)-18t\sinh(2t)-21}{5[\sinh(4t)-2\sinh(2t)]}\\
&\quad - \frac{[5\sinh(4t)+50\sinh(2t)-84t-36t\cosh(2t)][\cosh(4t)-\cosh(2t)]}{5[\sinh(4t)-2\sinh(2t)]^2},\\
\left[\frac{g_3'(t)}{g_4'(t)}\right]'_{t=\log(1+\sqrt{2})} &= \frac{(13{,}464\sqrt{2}+19{,}041)\log(1+\sqrt{2})-12{,}117\sqrt{2}-17{,}136}{5770+4080\sqrt{2}}\\
&= -0.0613\ldots < 0.
\end{aligned}
\tag{2.14}
$$

From (2.14) and piecewise monotonicity of the function $g_3'(t)/g_4'(t)$, we clearly see that $t_0 > \log(1+\sqrt{2})$ and the function $g_3'(t)/g_4'(t)$ is strictly decreasing on $(0, \log(1+\sqrt{2}))$. Then Lemma 2.1 together with (2.7) and (2.8) leads to the conclusion that $g(t)$ is strictly decreasing on $(0, \log(1+\sqrt{2}))$.

It follows from (2.6)–(2.10) that

$$g(0^+) = \frac{a_0}{b_0} = \frac{3}{10}, \qquad g(\log(1+\sqrt{2})) = \frac{3\sqrt{2}\log(1+\sqrt{2}) - \log 2 - 3}{5\log 2 - 3\log 3} = 0.2719\ldots. \qquad \square$$

3 Main results

Theorem 3.1 *The double inequality*

$$\begin{aligned}
&\left[\frac{1}{6}C(x,y) + \frac{5}{6}A(x,y)\right]^{\alpha_1}\left[C^{1/6}(x,y)A^{5/6}(x,y)\right]^{1-\alpha_1} \\
&\quad < R_{AQ}(x,y) \\
&\quad < \left[\frac{1}{6}C(x,y) + \frac{5}{6}A(x,y)\right]^{\beta_1}\left[C^{1/6}(x,y)A^{5/6}(x,y)\right]^{1-\beta_1}
\end{aligned}$$

holds for all $x, y > 0$ with $x \neq y$ if and only if $\alpha_1 \leq (3\pi + 4\log 2 - 12)/[2(6\log 7 - 7\log 2 - 6\log 3)] = 0.4258\ldots$ and $\beta_1 \geq 12/25$.

Proof Since $A(x,y)$, $R_{AQ}(x,y)$, and $C(x,y)$ are symmetric and homogenous of degree one, without loss of generality, we assume that $x > y > 0$. Let $v = (x-y)/(x+y) \in (0,1)$ and $t = \arctan(v) \in (0, \pi/4)$. Then (1.1)–(1.3) lead to

$$\begin{aligned}
&\frac{\log[R_{AQ}(x,y)] - \log[C^{1/6}(x,y)A^{5/6}(x,y)]}{\log[C(x,y)/6 + 5A(x,y)/6] - \log[C^{1/6}(x,y)A^{5/6}(x,y)]} \\
&\quad = \frac{\log(\sqrt{1+v^2}) + \arctan(v)/v - 1 - \log(\sqrt[6]{1+v^2})}{\log[(1+v^2)/6 + 5/6] - \log(\sqrt[6]{1+v^2})} \\
&\quad = \frac{\frac{2}{3}\log[\sec(t)] + \frac{t}{\tan(t)} - 1}{\log[\frac{\sec^2(t)+5}{6}] - \frac{\log[\sec(t)]}{3}}.
\end{aligned} \tag{3.1}$$

Therefore, Theorem 3.1 follows easily from Lemma 2.3 and (3.1). $\square$

Theorem 3.2 *The two-sided inequalities*

$$\begin{aligned}
&\left[\frac{1}{3}C(x,y) + \frac{2}{3}A(x,y)\right]^{\alpha_2}\left[C^{1/3}(x,y)A^{2/3}(x,y)\right]^{1-\alpha_2} \\
&\quad < R_{QA}(x,y) \\
&\quad < \left[\frac{1}{3}C(x,y) + \frac{2}{3}A(x,y)\right]^{\beta_2}\left[C^{1/3}(x,y)A^{2/3}(x,y)\right]^{1-\beta_2}
\end{aligned}$$

are valid for all $x, y > 0$ with $x \neq y$ if and only if $\alpha_2 \leq [3\sqrt{2}\log(1+\sqrt{2}) - \log 2 - 3]/(5\log 2 - 3\log 3) = 0.2719\ldots$ and $\beta_2 \geq 3/10$.

Proof Since $A(x,y)$, $R_{QA}(x,y)$, and $C(x,y)$ are symmetric and homogenous of degree one, without loss generality, we assume that $x > y > 0$. Let $v = (x-y)/(x+y) \in (0,1)$ and $t = \sinh^{-1}(v) \in (0, \log(1+\sqrt{2})$. Then from (1.1), (1.3), and (1.4) we clearly see that

$$\begin{aligned}
&\frac{\log[R_{QA}(x,y)] - \log[C^{1/3}(x,y)A^{2/3}(x,y)]}{\log[C(x,y)/3 + 2A(x,y)/3] - \log[C^{1/3}(x,y)A^{2/3}(x,y)]} \\
&\quad= \frac{[\sqrt{1+v^2}\sinh^{-1}(v)]/v - 1 - \log(\sqrt[3]{1+v^2})}{\log[(1+v^2)/3 + 2/3] - \log(\sqrt[3]{1+v^2})} \\
&\quad= \frac{t\coth(t) - \frac{2\log[\cosh(t)]}{3} - 1}{\log[\frac{\cosh^2(t)+2}{3}] - \frac{2\log[\cosh(t)]}{3}}.
\end{aligned} \tag{3.2}$$

Therefore, Theorem 3.2 follows easily from Lemma 2.4 and (3.2). □

From (1.3), (1.4), and Theorems 3.1 and 3.2 we get Corollary 3.3 immediately.

Corollary 3.3 *Let*

$$\begin{aligned}
\lambda(\alpha;a,b) &= 6\alpha\log\bigl[C(a,b) + 5A(a,b)\bigr] \\
&\quad + (1-\alpha)\bigl[\log C(a,b) + 5\log A(a,b)\bigr] - 6\log Q(a,b) + 6(1-\alpha\log 6), \\
\mu(\alpha;a,b) &= 3\alpha\log\bigl[C(a,b) + 2A(a,b)\bigr] \\
&\quad + (1-\alpha)\log C(a,b) - (1+2\alpha)\log A(a,b) + 3(1-\alpha\log 3).
\end{aligned}$$

Then the double inequalities

$$\begin{aligned}
&\frac{6A(a,b)}{\lambda(\beta_1;a,b)} < T(a,b) < \frac{6A(a,b)}{\lambda(\alpha_1;a,b)}, \\
&\frac{3Q(a,b)}{\mu(\beta_2;a,b)} < NS(a,b) < \frac{3Q(a,b)}{\mu(\alpha_2;a,b)}
\end{aligned}$$

hold for all $a,b > 0$ *with* $a \neq b$ *if and only if* $\alpha_1 \leq (3\pi + 4\log 2 - 12)/[2(6\log 7 - 7\log 2 - 6\log 3)] = 0.4258\ldots$, $\beta_1 \geq 12/25$, $\alpha_2 \leq [3\sqrt{2}\log(1+\sqrt{2}) - \log 2 - 3]/(5\log 2 - 3\log 3) = 0.2719\ldots$, *and* $\beta_2 \geq 3/10$.

4 Results and discussion

In the article, we present the best possible parameters α_1, β_1, α_2, and β_2 such that the double inequalities

$$\begin{aligned}
&\left[\frac{1}{6}C(x,y) + \frac{5}{6}A(x,y)\right]^{\alpha_1}\bigl[C^{1/6}(x,y)A^{5/6}(x,y)\bigr]^{1-\alpha_1} \\
&\quad < R_{AQ}(x,y) \\
&\quad < \left[\frac{1}{6}C(x,y) + \frac{5}{6}A(x,y)\right]^{\beta_1}\bigl[C^{1/6}(x,y)A^{5/6}(x,y)\bigr]^{1-\beta_1}, \\
&\left[\frac{1}{3}C(x,y) + \frac{2}{3}A(x,y)\right]^{\alpha_2}\bigl[C^{1/3}(x,y)A^{2/3}(x,y)\bigr]^{1-\alpha_2}
\end{aligned}$$

$$< R_{QA}(x,y)$$
$$< \left[\frac{1}{3}C(x,y) + \frac{2}{3}A(x,y)\right]^{\beta_2}\left[C^{1/3}(x,y)A^{2/3}(x,y)\right]^{1-\beta_2}$$

hold for all $x, y > 0$ with $x \neq y$. Our results are the improvements of the inequalities given by (1.5) and (1.6).

5 Conclusion

We present sharp upper and lower bounds for the Sándor–Yang means R_{AQ} and R_{QA} in terms of the arithmetic and contraharmonic means and provide new bounds for the Seiffert mean T and Neuman–Sándor mean NS. Our approach may have further applications in the theory of bivariate means and special functions.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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