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Characterizing the R-duality of g-frames

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Abstract

In this paper, we define the g-Riesz-dual of a given special operator-valued sequence with respect to g-orthonormal bases for a separable Hilbert space. We demonstrate that the g-R-dual keeps some synchronous frame properties with the operator-valued sequence given. We also display some Schauder basis-like properties of the g-R-dual in the light of the properties of the given sequence. In particular, the g-R-dual can be characterized by the use of another sequence, related to the given sequence. Finally, a special sequence is constructed to build the relationship between an operator-valued sequence and a g-Riesz sequence.

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1 Introduction

Duality principles in Gabor theory play a fundamental role in analyzing the Gabor system. In [1], the authors described the concept of the Riesz-dual of a vector-valued sequence and illustrated the common frame properties for the give sequence and its R-dual. The conditions under which a Riesz sequence can be a R-dual of a given frame are investigated in [2]. In this paper, we are interested in the duality principles for g-frames. In [3], the g-R-dual was first defined, and some frame properties of g-R-dual were exhibited by the properties of the given operator-valued sequence. In this paper, our definition of g-R-dual in Sect. 2 is much weaker, and we characterize the g-R-dual with the analysis operator. The properties of the g-completeness, g-w-linearly independent, g-minimality of the g-R-dual is accounted in Sect. 3. In Sect. 4, we construct a sequence with a g-Riesz sequence and a given operator-valued sequence to consider the g-R-dual in a different way.

Throughout this paper, we use $\mathbb{N}$ to denote the set of all natural numbers, and assume that $\{H_i\}_{i\in\mathbb{N}}$ is a sequence of closed subspaces of a separable Hilbert space K, H is a separable Hilbert space. Denote by $\{A_i\}_{i\in\mathbb{N}}$, or for short $\{A_i\}$, a sequence of operators with $A_i \in B(H,H_i)$ for any $i \in \mathbb{N}$. Suppose that $B(H,H_i)$ denotes the collection of all the bounded linear operators from H into H_i, $i \in \mathbb{N}$. Denote by $\bigoplus_{i\in\mathbb{N}} H_i$ the orthogonal direct sum Hilbert space of $\{H_i\}_{i\in\mathbb{N}}$, $\{g_i\} := \{g_i\}_{i\in\mathbb{N}}$ for any $\{g_i\}_{i\in\mathbb{N}} \in \bigoplus_{i\in\mathbb{N}} H_i$.

In [10], Sun raised the concept of a g-frame as follows. Let $A_i \in B(H,H_i)$, $i \in \mathbb{N}$. If there exist two constants a, b such that

$$a\|f\|^2 \le \sum_{i\in\mathbb{N}} \|A_i f\|^2 \le b\|f\|^2, \quad \forall f \in H,$$

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we call $\{A_i\}$ a *g-frame* for H. We call $\{A_i\}$ a *tight g-frame* for H if $a = b$. Specially, if $a = b = 1$, $\{A_i\}$ is called a *Parseval g-frame* for H. If the inequalities above hold only for $f \in \overline{\operatorname{span}}\{A_i^*H_i\}_{i\in\mathbb{N}}$, we call $\{A_i\}$ a *g-frame sequence* for H. If only the right-hand inequality above holds, then we say that $\{A_i\}$ is a *g-Bessel sequence* for H. If $\overline{\operatorname{span}}\{A_i^*H_i\}_{i\in\mathbb{N}} = H$, we say that $\{A_i\}$ is *g-complete* in H. If $\{A_i\}$ is g-complete and such that

$$a\|\{g_i\}\|^2 \le \sum_{i\in\mathbb{N}}\|A_i^*g_i\|^2 \le b\|\{g_i\}\|^2, \quad \forall\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i,$$

we call $\{A_i\}$ a *g-Riesz basis* for H. If the g-completeness is not satisfied, it is called a *g-Riesz sequence* for H. As we know, if $\{A_i\}$ is a g-frame for H, we define $S_Af = \sum_{i\in\mathbb{N}} A_i^*A_if$ for any $f \in H$, then S_A is a well-defined, bounded, positive, invertible operator by [10]. We call S_A a *frame operator* of $\{A_i\}$. Another basic fact is that $\{\widetilde{A}_i\}_{i\in\mathbb{N}} = \{A_iS_A^{-1}\}_{i\in\mathbb{N}}$ is a g-frame for H, we call it a canonical dual g-frame of $\{A_i\}$. Extensively, by [8], if a g-frame $\{B_i\}$ for H such that $f = \sum_{i\in\mathbb{N}} B_i^*A_if$ for every $f \in H$, we say that it is a *dual g-frame* of $\{A_i\}$. Recently, g-frames in Hilbert spaces have been studied intensively; for more details see [4–10] and the references therein.

In the following we introduce some definitions and lemmas connected with the g-bases in Hilbert space which will be needed in the paper.

Definition 1.1 ([10]) If $\{A_i\}$ satisfies

(1) $\{A_i\}$ is a *g-orthonormal sequence* for H, i.e., $\langle A_i^*g_i, A_j^*g_j\rangle = \delta_{ij}\langle g_i, g_j\rangle$ for any $i, j \in \mathbb{N}$, any $g_i \in H_i, g_j \in H_j$.

(2) $\{A_i\}$ is g-complete in H.

We call $\{A_i\}$ a *g-orthonormal basis* for H. Obviously, (2) is equivalent to that $\{A_i\}$ is a Parseval g-frame for H by [5, Corollary 4.4], when (1) holds. Specially, if $\{A_i\}$ only satisfies $A_iA_j^* = 0$ for any $i, j \in \mathbb{N}$, $i \neq j$, $\{A_i\}$ is called a *g-orthogonal sequence* for H.

The g-orthonormal basis is a special case that itself is g-biorthonormal. The following result shows that for the g-Riesz basis there also exists a g-biorthonormal sequence.

Lemma 1.2 ([10], Corollary 3.3) *Let $\{A_i\}$ be a g-Riesz basis for H. Then $\{A_i\}$ and $\{\widetilde{A}_i\}$ are g-biorthonormal, where $\{\widetilde{A}_i\}$ is the canonical dual g-frame of $\{A_i\}$.*

In this paper, we only interested in the case when the g-orthonormal basis for H exists, which is equivalent to the following result.

Lemma 1.3 ([5], Theorem 3.1) *Let H be a separable Hilbert space, $\{H_i\}_{i\in\mathbb{N}}$ be a sequence of separable Hilbert spaces. Then there exists a sequence $\{\Gamma_i\}$, which is a g-orthonormal basis for H if and only if* $\dim H = \sum_{i\in\mathbb{N}} \dim H_i$.

The concept of g-bases in Hilbert space is a generalization of the Schauder basis. Let $\{A_i\}$. If for any $f \in H$, there is a unique sequence $\{g_i\}_{i\in\mathbb{N}}$ with $g_i \in H_i$ for any $i \in \mathbb{N}$ such that $f = \sum_{i\in\mathbb{N}} A_i^*g_i$, we call $\{A_i\}$ a *g-basis* for H. If $\{A_i\}$ is a g-basis for $\overline{\operatorname{span}}\{A_i^*H_i\}_{i\in\mathbb{N}}$, $\{A_i\}$ is called a *g-basic sequence* for H. Moreover, If $\sum_{i\in\mathbb{N}} A_i^*g_i = 0$ for $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$, then $g_i = 0$, we call $\{A_i\}$ *g-w-linearly independent*. If $A_j^*g_j \notin \overline{\operatorname{span}}_{i\neq j}\{A_i^*g_i\}_{i\in\mathbb{N}}$ for any $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$ such that $g_i \in H_i$, $g_i \neq 0$, any $i \in \mathbb{N}$, we call $\{A_i\}$ *g-minimal*. For more details as regards g-bases see [4].

2 Duality for g-frame

Before giving the definition of g-R-dual, we introduce a lemma which is related to the g-Bessel sequence.

Lemma 2.1 *The sequence $\{A_i\}$ is a g-Bessel sequence for H if and only if $\sum_{i\in\mathbb{N}} A_i^* g_i$ is convergent for any $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$, and is also equivalent to that $\sum_{i\in\mathbb{N}} \|A_i f\|^2 < \infty$ for every $f \in H$.*

Proof Suppose $\sum_{i\in\mathbb{N}} A_i^* g_i$ is convergent for any $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$. For any $n \in \mathbb{N}$, $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$, we define $T_n : \bigoplus_{i\in\mathbb{N}} H_i \to H, T_n\{g_i\} = \sum_{i=1}^n A_i^* g_i$. Thus T_n is bounded evidently. Since $\{T_n\}_{n\in\mathbb{N}}$ converges to T in the strong operator topology as $n \to \infty$, where $T\{g_i\} = \sum_{i\in\mathbb{N}} A_i^* g_i$ for every $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$. Then T is bounded by the uniform boundedness principle in Banach space. The rest follows directly. □

For a g-Bessel sequence $\{A_i\}$, we can define the analysis operator as $\theta_A : H \to \bigoplus_{i\in\mathbb{N}} H_i, \theta_A f = \{A_i f\}_{i\in\mathbb{N}}$ for any $f \in H$, which is well defined and bounded obviously by Lemma 2.1.

Definition 2.2 Let $\{\Lambda_i\}$, $\{\Gamma_i\}$ be two g-orthonormal bases for H. Suppose a sequence $\{A_i\}$ such that $\sum_{i\in\mathbb{N}} \|A_i \Lambda_j^* g_j\|^2 < \infty$ for any $j \in \mathbb{N}$, any $g_j \in H_j$. We define

$$\mathcal{A}_j^* g_j = \sum_{i\in\mathbb{N}} \Gamma_i^* A_i \Lambda_j^* g_j, \quad \forall j \in \mathbb{N}, g_j \in H_j.$$

We call $\{\mathcal{A}_i\}$ a g-R-dual sequence of $\{A_i\}$.

Remark 2.3 By [4, Theorem 4.4], for any $j \in \mathbb{N}$, $\mathcal{A}_j$ is well defined if and only if $\{A_i \Lambda_j^* g_j\}_{i\in\mathbb{N}} \in \bigoplus_{i\in\mathbb{N}} H_i$ for any $g_j \in H_j$, i.e., $\{A_i Q_j f\}_{i\in\mathbb{N}} \in \bigoplus_{i\in\mathbb{N}} H_i$ for any $f \in H$, i.e., $\{A_i\}$ is a g-Bessel sequence for $\operatorname{ran} Q_j$ by Lemma 2.1, where Q_j is the orthogonal projection from H onto $\overline{\operatorname{ran}}\, \Lambda_j^*$. Obviously, $\{A_i\}$ may not be a g-Bessel sequence for H. The condition of our definition is weaker than that in [3, Definition 1.13]. Thus Definition 2.2 is equivalent to $\mathcal{A}_j = \sum_{i\in\mathbb{N}} \Lambda_j A_i^* \Gamma_i$ for any $j \in \mathbb{N}$. By Definition 1.1, we get $\Gamma_k \mathcal{A}_j^* = A_k \Lambda_j^*$ for every $i, k \in \mathbb{N}$.

The following exhibits that the sequence $\{A_i\}$ satisfying Definition 2.2 shares the common properties with its g-R-dual $\{\mathcal{A}_i\}$. Similar results are referred to in [3, Theorem 2.2].

Theorem 2.4 *Let $\{A_i\}$ satisfy Definition 2.2, $\{\mathcal{A}_i\}$ be its g-R-dual defined in Definition 2.2. Then $\{A_i\}$ is a g-Bessel sequence for H if and only if $\{\mathcal{A}_i\}$ is a g-Bessel sequence for H. Moreover, they have the same upper bound.*

Proof For every $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$, let $f = \sum_{i\in\mathbb{N}} \Lambda_i^* g_i$, $h = \sum_{i\in\mathbb{N}} \Gamma_i^* g_i$. Suppose $\{A_i\}$ is a g-Bessel sequence for H and has an upper bound b. Since $\theta_\Lambda, \theta_\Gamma : H \to \bigoplus_{i\in\mathbb{N}} H_i$ are unitary,

$$\begin{aligned}\Big\|\sum_{j\in\mathbb{N}} \mathcal{A}_j^* g_j\Big\|^2 &= \Big\|\sum_{j\in\mathbb{N}} \theta_\Gamma^* \theta_\Gamma \mathcal{A}_j^* g_j\Big\|^2 = \Big\|\sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} \Gamma_i^* \Gamma_i \mathcal{A}_j^* g_j\Big\|^2 \\ &= \Big\|\sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} \Gamma_i^* A_i \Lambda_j^* g_j\Big\|^2 = \Big\|\sum_{i\in\mathbb{N}} \Gamma_i^* A_i f\Big\|^2\end{aligned}$$

$$= \left\|\theta_\Gamma^* \theta_{\mathcal{A}} f\right\|^2 = \|\theta_{\mathcal{A}} f\|^2 \leq b\|f\|^2$$
$$= b\left\|\theta_\Gamma^* \{g_i\}\right\|^2 = b\left\|\{g_i\}\right\|^2.$$

By Lemma 2.1, $\{\mathcal{A}_i\}$ is a g-Bessel sequence for H and has an upper bound b. The converse is similar. □

When $\{\Lambda_i\}$ is a g-Bessel sequence, there exists a unitary equivalence between $\{\Lambda_i S_\Lambda^{\frac{1}{2}}\}$ and the R-dual $\{\mathcal{A}_i\}$.

Theorem 2.5 *Let $\{\Lambda_i\}$ be a g-Bessel sequence for H, $\{\mathcal{A}_i\}$ be its g-R-dual defined in Definition 2.2. Then*

(1) $\langle \mathcal{A}_i^* g_i, \mathcal{A}_j^* g_k\rangle = \langle S_\Lambda^{\frac{1}{2}} \Lambda_j^* g_j, S_\Lambda^{\frac{1}{2}} \Lambda_i^* g_i\rangle$ *for any $i,j \in \mathbb{N}$, any $g_i \in H_i, g_j \in H_j$.*

(2) $\|\sum_{i\in\mathbb{N}} \mathcal{A}_i^* g_i\| = \|\sum_{i\in\mathbb{N}} S_\Lambda^{\frac{1}{2}} \Lambda_i^* g_i\|$ *for any $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$.*

(3) *there exists an isometric operator T from $\overline{\operatorname{ran}}\, S_\Lambda^{\frac{1}{2}}\theta_\Lambda^*$ onto $\overline{\operatorname{ran}}\,\theta_{\mathcal{A}}^*$ such that $\mathcal{A}_i T = \Lambda_i S_\Lambda^{\frac{1}{2}}$ for any $i \in \mathbb{N}$.*

Proof (1) Since $\{\Lambda_i\}$ is a g-Bessel sequence for H, so is $\{\mathcal{A}_i\}$ by Theorem 2.4. Then, for any $i,j \in \mathbb{N}$, any $g_i \in H_i, g_j \in H_j$, we have

$$\left\langle \mathcal{A}_i^* g_i, \mathcal{A}_j^* g_k\right\rangle = \left\langle \theta_{\mathcal{A}}^* \{\delta_{ik} g_i\}_k, \theta_{\mathcal{A}}^* \{\delta_{jk} g_j\}_k\right\rangle$$
$$= \left\langle \theta_\Gamma^* \theta_\Lambda \theta_\Lambda^* \{\delta_{ik} g_i\}_k, \theta_\Gamma^* \theta_\Lambda \theta_\Lambda^* \{\delta_{jk} g_j\}_k\right\rangle$$
$$= \left\langle S_\Lambda^{\frac{1}{2}} \Lambda_i^* g_i, S_\Lambda^{\frac{1}{2}} \Lambda_j^* g_j\right\rangle.$$

(2) It is direct by (1).

(3) Define $T^* : \operatorname{ran}\theta_{\mathcal{A}}^* \to \operatorname{ran} S_\Lambda^{\frac{1}{2}}$, $T^*(\sum_{i\in\mathbb{N}} \mathcal{A}_i^* g_i) = \sum_{i\in\mathbb{N}} S_\Lambda^{\frac{1}{2}} \Lambda_i^* g_i$ for any $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$. It is easy to verify T^* is well defined by (2). We can extend T to an isometric operator from $\overline{\operatorname{ran}}\, S_\Lambda^{\frac{1}{2}}\theta_\Lambda^*$ onto $\overline{\operatorname{ran}}\,\theta_{\mathcal{A}}^*$. We still denote the operator as T for convenience. □

In the following results we show the properties of g-R-dual in the case that $\{\Lambda_i\}$ is a g-frame sequence by the corresponding analysis operators. The results are similar to the conclusions in [3, Corollary 2.6].

Theorem 2.6 *Let $\{\Lambda_i\}$ satisfy Definition 2.2, $\{\mathcal{A}_i\}$ be its g-R-dual defined in Definition 2.2. Then $\{\Lambda_i\}$ is a g-frame sequence for H if and only if $\{\mathcal{A}_i\}$ is a g-frame sequence for H with the same frame bounds. Specially, in this case the following are equivalent:*

(1) *$\{\Lambda_i\}$ is a g-frame for H with the frame bounds a, b.*

(2) *$\{\mathcal{A}_i\}$ is a g-Riesz sequence for H with the frame bounds a, b.*

(3) *There exists $0 < b_1 < \infty$ such that $\sum_{i\in\mathbb{N}} \|\Lambda_i P f\|^2 \leq b_1 \sum_{i\in\mathbb{N}} \|\Lambda_i f\|^2$ for any $f \in H$, where P is an arbitrary orthogonal projection on H.*

(4) *There exists $0 < b_1 < \infty$ such that $\sum_{i\in\mathbb{N}} \|\Lambda_i P_n f\|^2 \leq b_1 \sum_{i\in\mathbb{N}} \|\Lambda_i f\|^2$ for any $f \in H$, where P_n is the orthogonal projection from H onto $\overline{\operatorname{span}}\,\{\Lambda_i^* H_i\}_{i=1}^n$ for any $n \in \mathbb{N}$.*

Proof The case of the g-Bessel upper bound we get easily by Theorem 2.4. We now show the case of the lower bound in a similar way as the proof of Theorem 2.4.

Because $\{A_i\}$, $\{\mathcal{A}_i\}$ are g-Bessel sequences, we easily have $\theta_A = \theta_\Gamma\theta_{\mathcal{A}}^*\theta_\Lambda$. Then $g \in \ker\theta_A$ if and only if $g \in \ker\theta_{\mathcal{A}}^*\theta_\Lambda$, i.e., $\theta_\Lambda g \in \ker\theta_{\mathcal{A}}^*$. Hence, $g \in (\ker\theta_A)^\perp$ if and only if $\theta_\Lambda g \in (\ker\theta_{\mathcal{A}}^*)^\perp$ since θ_Λ is unitary.

Evidently, $\{A_i\}$ is a g-frame sequence for H if and only if for any $f \in \operatorname{ran}\theta_A^*$, one has $a\|f\|^2 \le \sum_{i\in\mathbb{N}}\|A_if\|^2 = \|\theta_A f\|^2 \le b\|f\|^2$, i.e.,

$$a\|\theta_\Lambda f\|^2 = \left\|\theta_{\mathcal{A}}^*\theta_\Lambda f\right\|^2 \le b\|f\|^2 = b\|\theta_\Lambda f\|^2,$$

which is equivalent to $\{\mathcal{A}_i\}$ is a g-frame sequence for H.

The equivalence of (1) and (2) is obvious since $(\ker\theta_A)^\perp = \{0\}$ if and only if $(\ker\theta_{\mathcal{A}}^*)^\perp = \{0\}$ by the proof above.

(1) ⇒ (3). Let $\{A_i\}$ be a g-frame for H with the frame bounds a, b. Take P as an arbitrary orthogonal projection on H. For any $f = f_1 + f_2 \in H$, where $f_1 \in \operatorname{ran}P, f_2 \in \ker P$, we have

$$\sum_{i\in\mathbb{N}}\|A_iPf\|^2 = \sum_{i\in\mathbb{N}}\|A_if_1\|^2 \le b\|f\|^2 \le a^{-1}b\sum_{i\in\mathbb{N}}\|A_if\|^2.$$

(3) ⇒ (4) is direct.

(4) ⇒ (2). It is obvious by Theorem 3.3. □

The following result was given in [3, Theorem 4.1], we here give a simple illustration by the use of the analysis operators.

Lemma 2.7 *Let $\{A_i\}, \{B_i\}$ be two g-frames for H, $\{\mathcal{A}_i\}$, $\{\mathcal{B}_i\}$ be their g-R-dual sequences defined in Definition 2.2, respectively. Then $\{A_i\}$ is a dual g-frame of $\{B_i\}$ if and only if $\langle\mathcal{A}_i^*g_i, \mathcal{B}_j^*g_j\rangle = \delta_{ij}\langle g_i, g_j\rangle$ for any $i, j \in \mathbb{N}$, any $g_i \in H_i$, $g_j \in H_j$.*

Proof By Definition 2.2, we get $\theta_{\mathcal{A}} = \theta_\Lambda\theta_A^*\theta_\Gamma$, $\theta_{\mathcal{B}} = \theta_\Lambda\theta_B^*\theta_\Gamma$. Then $\theta_{\mathcal{A}}\theta_{\mathcal{B}}^* = \theta_\Lambda\theta_A^*\theta_B\theta_\Lambda^*$. Obviously, $\theta_A^*\theta_B = I$ if and only if $\theta_{\mathcal{A}}\theta_{\mathcal{B}}^* = I_{\bigoplus_{i\in\mathbb{N}}H_i}$, i.e., $\langle\mathcal{A}_i^*g_i, \mathcal{B}_j^*g_j\rangle = \delta_{ij}\langle g_i, g_j\rangle$ for any $i, j \in \mathbb{N}$, any $g_i \in H_i$, $g_j \in H_j$. □

The following shows that the g-R-dual of the canonical dual g-frame is the "minimal" and has the "smallest distance" with $\{A_i\}$ among the g-R-duals of all the alternate dual g-frames, which is a generalization of the result in [3, Theorem 4.5].

Theorem 2.8 *Let $\{A_i\}$ be a g-frame for H, $\{\widetilde{A}_i\}$ be the canonical dual g-frame of $\{A_i\}$, $\{B_i\}$ be a dual g-frame of $\{A_i\}$. $\{\mathcal{A}_i\}$ and $\{\mathcal{B}_i\}$ are the corresponding g-R-duals defined in Definition 2.2, respectively. Then the following are equivalent:*

(1) *$B_i = \widetilde{A}_i$ for every $i \in \mathbb{N}$.*

(2) *$\|\mathcal{B}^*g_i\| \le \|\mathcal{C}_i^*g_i\|$ for every $i \in \mathbb{N}$, $g_i \in H_i$, where $\{C_i\}$ is an arbitrary dual g-frame of $\{A_i\}$, $\{\mathcal{C}_i\}$ is the g-R-dual of $\{C_i\}$.*

(3) *$\|\mathcal{B}_i^*g_i - \mathcal{A}_i^*g_i\| \le \|\mathcal{C}_i^*g_i - \mathcal{A}_i^*g_i\|$ for every $i \in \mathbb{N}$, $g_i \in H_i$, where $\{C_i\}$ is an arbitrary dual g-frame of $\{A_i\}$, $\{\mathcal{C}_i\}$ is the g-R-dual of $\{C_i\}$.*

Proof (1) ⇔ (2). By [3, Theorem 4.4], we obtain $\mathcal{B}_i = \widetilde{\mathcal{A}}_i + \Delta_i$ for any $i \in \mathbb{N}$, where $\{\Delta_i\}$ is a g-Bessel sequence for H such that $\operatorname{ran}\theta_\Delta^* \subset (\operatorname{ran}\theta_{\mathcal{A}}^*)^\perp$. Then, for every $\{g_i\} \in \bigoplus_{i\in\mathbb{N}}H_i$, we get

$$\left\|\theta_{\mathcal{B}}^*\{g_i\}\right\|^2 = \left\|\theta_{\widetilde{\mathcal{A}}}^*\{g_i\} + \theta_\Delta^*\{g_i\}\right\|^2 \ge \left\|\theta_{\widetilde{\mathcal{A}}}^*\{g_i\}\right\|^2.$$

Specially, if we take $\{\delta_{ij}g_i\}_{j\in\mathbb{N}}$, then $\|\mathcal{B}_i^*g_i\| \geq \|\widetilde{\mathcal{A}}_i^*g_i\|$. Hence, $B_i = \widetilde{A}_i$ if and only if $\Delta_i = 0$ for any $i \in \mathbb{N}$.

(2) ⇔ (3). By Lemma 2.7, for any $i \in \mathbb{N}$, we obtain

$$\left\|\mathcal{B}_i^*g_i - \mathcal{A}_i^*g_i\right\|^2 = \left\|\mathcal{B}_i^*g_i\right\|^2 + \left\|\mathcal{A}_i^*g_i\right\|^2 - 2.$$

Similarly, $\|\widetilde{\mathcal{A}}_i^*g_i - \mathcal{A}_i^*g_i\| = \|\widetilde{\mathcal{A}}_i^*g_i\|^2 + \|\mathcal{A}_i^*g_i\|^2 - 2$. Thus the equivalence is direct. □

3 Characterization of the Schauder basis-like properties of g-R-dual

Suppose $\{A_i\}$ is a g-Bessel sequence for H, $\{\mathcal{A}_i\}$ is its g-R-dual defined in Definition 2.2. We will characterize the Schauder basis-like properties (g-completeness, g-w-linearly independence, g-minimality) of $\{\mathcal{A}_i\}$ in terms of $\{A_i\}$.

Theorem 3.1 *Let $\{A_i\}$ be a g-Bessel sequence for H, $\{\mathcal{A}_i\}$ be its g-R-dual defined in Definition 2.2. Then the following are equivalent*:

(1) *$\{A_i\}$ is g-complete.*

(2) *$\{\mathcal{A}_i\}$ is g-w-linearly independent.*

(3) *If $\lim_{n\to\infty}\|\theta_A x_n\|^2 = 0$, then $\{g_i\} = 0$, where $x_n = \sum_{i=1}^n \Lambda_i^*g_i \in H$ for any $n \in \mathbb{N}$ and any $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$.*

Proof (1) ⇔ (2). By Definition 2.2, $\theta_{\mathcal{A}}^* = \theta_\Gamma^*\theta_A\theta_\Lambda^*$. For arbitrary $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$, we have $\{g_i\} \in \ker\theta_{\mathcal{A}}^*$ if and only if $\theta_\Lambda^*\{g_i\} \in \ker\theta_A$. Then $\{A_i\}$ is g-complete if and only if $\ker\theta_{\mathcal{A}}^* = \{0\}$, i.e., $\{\mathcal{A}_i\}$ is g-w-linearly independent.

(2) ⇔ (3). It is evident as $\|\theta_A x_n\|^2 = \|\theta_{\mathcal{A}}^*\theta_\Lambda x_n\|^2$. □

Now we have the next special result. By [4, Theorem 5.2], if $\{A_i\}$ is a g-frame sequence for H, the existing condition of the g-biorthonormal sequence means the minimality of $\{A_i\}$.

Theorem 3.2 *Let $\{A_i\}$ be a g-Bessel sequence for H, $\{\mathcal{A}_i\}$ defined in Definition 2.2 be its g-R-dual. If there exists a sequence $\{\Delta_i\}$ which is g-biorthonormal with $\{\mathcal{A}_i\}$ such that Δ_i^* is injective for any $i \in \mathbb{N}$, then*

(1) *there are constants $0 < c_i \leq 1$ for arbitrary $i \in \mathbb{N}$ such that $\|c_ig_i\| \leq \|\sum_{j\in\mathbb{N}} \mathcal{A}_j^*g_j\|$ for any $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$;*

(2) *there are constants $0 < a_i$ for arbitrary $i \in \mathbb{N}$ such that*

$$\left\|\{a_ig_i\}_{i\in\mathbb{N}}\right\|^2 \leq \sum_{j\in\mathbb{N}}\left\|A_j\theta_\Lambda^*\{g_i\}\right\|^2, \quad \forall\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i.$$

Moreover, (1) *and* (2) *are equivalent.*

Proof Take arbitrary $h_i \in H_i$ and $\|h_i\| = 1$ and let $c_i = \min\{1, \frac{1}{\|\Delta_i\|}\}$ for every $i \in \mathbb{N}$. Since $\langle\mathcal{A}_i^*g_i, \Delta_j^*g_j\rangle = \delta_{ij}\langle g_i, g_j\rangle$ for any $i, j \in \mathbb{N}$, $g_i \in H_i$ $g_j \in H_j$, we have

$$\left\|\sum_{j\in\mathbb{N}} \mathcal{A}_j^*g_j\right\| = \sup_{\|f\|=1, f\in H}\left|\left\langle\sum_{j\in\mathbb{N}} \mathcal{A}_j^*g_j, f\right\rangle\right|$$

$$\geq \left|\left\langle \sum_{j\in\mathbb{N}} \mathcal{A}_j^* g_j, \frac{1}{\|\Delta_i^* h_i\|}\Delta_i^* h_i\right\rangle\right|$$
$$\geq \left|\left\langle \sum_{j\in\mathbb{N}} \mathcal{A}_j^* g_j, \frac{1}{\|\Delta_i\|}\Delta_i^* h_i\right\rangle\right|$$
$$\geq |c_i|\left|\left\langle \sum_{j\in\mathbb{N}} \mathcal{A}_j^* g_j, \Delta_i^* h_i\right\rangle\right| = |c_i|\left|\langle g_i, h_i\rangle\right|.$$

By the arbitrariness of h_i, we have $|c_i|\|g_i\| \leq \|\sum_{j\in\mathbb{N}} \mathcal{A}_j^* g_j\|$.

Take $a_i = \frac{c_i}{2^i}$ for every $i \in \mathbb{N}$. For any $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$, we obtain

$$\begin{aligned}\|\{a_i g_i\}\|^2 &= \sum_{i\in\mathbb{N}} \left\|\frac{c_i}{2^i} g_i\right\|^2 = \sum_{i\in\mathbb{N}} \frac{1}{2^{2i}}\|c_i g_i\|^2 \\ &\leq \sum_{i\in\mathbb{N}} \frac{1}{2^{2i}} \sup_{i\in\mathbb{N}} \|c_i g_i\|^2 \\ &\leq \left\|\sum_{j\in\mathbb{N}} \mathcal{A}_j^* g_j\right\| = \sum_{j\in\mathbb{N}} \left\|A_j \theta_\Lambda^* \{g_i\}\right\|^2.\end{aligned}$$

The converse is evident since $\|a_i g_i\|^2 \leq \|\{a_i g_i\}\|^2$. □

In the following we illustrate that the g-R-dual $\{\mathcal{A}_i\}$ is a g-basic sequence by the properties of $\{A_i\}$, which also shows the conclusion of Theorem 2.6 from another perspective. It can be realized as a kind of g-completeness of $\{\mathcal{A}_i\}$.

Theorem 3.3 *Let $\{A_i\}$ be a g-frame sequence for H, $\{\mathcal{A}_i\}$ defined in Definition 2.2 be its g-R-dual. Let P_n be the orthogonal projection from H onto $N_n := \overline{\text{span}}\{\Lambda_i^* H_i\}_{i=1}^n$ for any $n \in \mathbb{N}$. Then the following are equivalent*:

(1) *$\{\mathcal{A}_i\}$ a g-basic sequence for H.*

(2) *There exists a constant $0 < b < \infty$ such that $\sum_{i\in\mathbb{N}} \|A_i P_n f\|^2 \leq b \sum_{i\in\mathbb{N}} \|A_i f\|^2$ for any $n \in \mathbb{N}$, any $f \in H$.*

(3) *There exists a constant $0 < b < \infty$ such that $S_{AP_n} \leq b S_A$ for any $n \in \mathbb{N}$, where S_{AP_n} is the frame operator of the g-Bessel sequence $\{A_i P_n\}_{i\in\mathbb{N}}$.*

In this case, we have

$$\operatorname{ran}\theta_A^* = \overline{\text{span}}\left\{\Lambda_i^* g_i : \sum_{i\in\mathbb{N}} \left\|A_i \Lambda_i^* g_i\right\|^2 \neq 0, \forall i \in \mathbb{N}, g_i \in H_i\right\}.$$

Proof Let $\mathbb{I} = \{j \in \mathbb{N} : \mathcal{A}_j^* = \theta_\Gamma^* \theta_A \Lambda_j^* \neq 0\}$. Without loss of generality, we can suppose $\mathcal{A}_i \neq 0$ for any $i \in \mathbb{N}$.

(1) ⇔ (2). By [4, Theorem 3.3], $\{\mathcal{A}_i\}$ is a g-basic sequence for H if and only if there exists a constant $0 < b < \infty$ such that, for arbitrary $n \leq m$, any $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$, one has

$$\left\|\sum_{i=1}^n \mathcal{A}_i^* g_i\right\|^2 \leq b\left\|\sum_{i=1}^m \mathcal{A}_i^* g_i\right\|^2 = b\sum_{i\in\mathbb{N}} \|A_i x\|^2,$$

where $x = \sum_{i=1}^m \Lambda_i^* g_i$. Since $P_n \Lambda_i^* = 0$ for every $i \in \mathbb{N}$ such that $n < i \leq m$, $\sum_{i=1}^n \Lambda_i^* g_i = P_n x$. Similarly, we have $\|\sum_{i=1}^n \mathcal{A}_i^* g_i\|^2 = \sum_{i\in\mathbb{N}} \|A_i P_n x\|^2$.

(2) ⇔ (3). (2) is equivalent to $\langle S_{AP_n}f,f\rangle = \langle\theta_A P_n f,\theta_A P_n f\rangle \le b\langle Sf,f\rangle$ for any $f\in H$, which is obvious.

By [4, Lemma 2.16], $\{\mathcal{A}_i\}$ is a g-Riesz sequence for H. Then $\mathcal{A}_i\neq 0$ for any $i\in\mathbb{N}$. By Definition 2.2, we have $\mathcal{A}_i^* = \theta_\Gamma^*\theta_A\Lambda_i^*$. Then $\theta_A\Lambda_i^*\neq 0$, i.e., $\sum_{i\in\mathbb{N}}\|A_i\Lambda_i^*g_i\|^2\neq 0$ for any $i\in\mathbb{N}$, $g_i\in H_i$. Hence,

$$\overline{\operatorname{span}}\left\{\Lambda_i^*g_i:\sum_{i\in\mathbb{N}}\left\|A_i\Lambda_i^*g_i\right\|^2\neq 0,\forall i\in\mathbb{N},g_i\in H_i\right\} = H.$$

Therefore, we only need to show the g-completeness of $\{A_i\}$ in H.

Suppose there exists $f\in H, f\neq 0$ such that $\langle A_i^*g_i,f\rangle = 0$ for arbitrary $i\in\mathbb{N}$, $g_i\in H_i$. Obviously, there is a sequence $\{f_i\}\in\bigoplus_{i\in\mathbb{N}}H_i$ such that $f=\sum_{i\in\mathbb{N}}\Lambda_i^*f_i$. Assume $k\in\mathbb{N}$ is the smallest positive integer such that $f_i\neq 0$. Then $P_kf = \Lambda_k^*f_k$. We get

$$0\neq\sum_{i\in\mathbb{N}}\left\|A_i\Lambda_k^*f_k\right\|^2 = \sum_{i\in\mathbb{N}}\|A_iP_kf\|^2\le b\sum_{i\in\mathbb{N}}\|A_if\|^2 = 0,$$

which is a contradiction. □

Now we give some equivalent characterizations for a g-frame to be a g-Riesz basis.

Theorem 3.4 *Let $\{A_i\}$ be a g-frame for H. Then the following are equivalent*:

(1) *$\{A_i\}$ is a g-basis for H.*
(2) *$\{A_i\}$ is g-w-linearly independent.*
(3) *$\{A_i\}$ is a g-Riesz basis for H.*
(4) *The g-R-dual $\{\mathcal{A}_i\}$ defined in Definition 2.2 is a g-Riesz basis for H.*
(5) *If $\lim_{n\to\infty}\sum_{i\in\mathbb{N}}\|\mathcal{A}_ix_n\|^2 = 0$, then $\{g_i\} = 0$, where $x_n = \sum_{i=1}^n\Gamma_i^*g_i$ for any $n\in\mathbb{N}$, $\{g_i\}\in\bigoplus_{i\in\mathbb{N}}H_i$.*
(6) *$\{A_i\}$ is exact (i.e., if it ceases to be a g-frame whenever any one of its elements is removed), and the canonical dual g-frame is biorthonormal with $\{A_i\}$.*

Proof The equivalence of (1), (2), (3) can be obtained by [4, Lemma 2.16]. By [9, Corollary 2.6], we get the equivalence of (3) and (6). Since $\{A_i\}$ is a g-frame, we get $\sum_{i\in\mathbb{N}}\|\mathcal{A}_ix_n\|^2 = \|\theta_A^*\theta_\Gamma x_n\|^2$. Then (5) holds if and only if θ_A^* is injective, i.e., (3) holds.

Similarly, by Definition 2.2, we have $\theta_{\mathcal{A}} = \theta_\Lambda\theta_A^*\theta_\Gamma$. For any $f\in H$, we obtain $f\in\ker\theta_{\mathcal{A}}$ if and only if $\theta_\Gamma f\in\ker\theta_A^*$. Thus we get the equivalence of (3), (4) by Theorem 2.6. □

4 G-R-dual and the g-orthogonal sequence

4.1 The characterization of g-R-dual

Let $\{\Lambda_i\}$ be a g-orthonormal basis for H. In this section we mainly investigate the conditions under which a g-Riesz sequence $\{\mathcal{A}_i\}$ is the g-R-dual of a g-frame $\{A_i\}$. We denote $\{\widetilde{\mathcal{A}}_i\}$ as the canonical dual g-frame of $\{\mathcal{A}_i\}$, which is also a g-Riesz sequence. Define $C_i = A_i\theta_\Lambda^*\theta_{\widetilde{\mathcal{A}}}$ for any $i\in\mathbb{N}$. Then

$$C_i^*g_i = \sum_{j\in\mathbb{N}}\widetilde{\mathcal{A}}_j^*\Lambda_jA_i^*g_i,\quad\forall g_i\in H_i.$$

Evidently, $\{C_i\}$ is a g-Bessel sequence for H. Let $M = \operatorname{ran}\theta_{\mathcal{A}}^*$. Thus $\operatorname{ran}\theta_C^*\subset M$. By Lemma 1.2, we also get $\mathcal{A}_jC_i^* = \Lambda_jA_i^*$ for any $i\in\mathbb{N}$.

Proposition 4.1 *Let $\{\Lambda_i\}$ be a g-orthonormal basis for H, $\{\mathcal{A}_i\}$ be a g-Riesz basis for M, $\{\widetilde{\mathcal{A}}_i\}$ be the canonical dual g-frame of $\{\mathcal{A}_i\}$ in M, where M is a closed subspace of H. For any sequence $\{A_i\}$, we have the following*:

(1) *There exists a sequence $\{\Gamma'_i\}$ such that $A_i = \Gamma'_i\theta^*_{\mathcal{A}}\theta_\Lambda$ for any $i \in \mathbb{N}$, i.e., $A^*_i g_i = \sum_{j\in\mathbb{N}} \Lambda^*_j \mathcal{A}_j \Gamma'^*_i g_i$ for any $g_i \in H_i$.*

(2) *The sequence $\{\Gamma'_i\}$ satisfying $A_i = \Gamma'_i\theta^*_{\mathcal{A}}\theta_\Lambda$ can be written as $\Gamma'_i = C_i + D_i$ for every $i \in \mathbb{N}$, where $C_i = A_i\theta^*_\Lambda\theta_{\widetilde{\mathcal{A}}}$, $D_i \in B(H, H_i)$ and $\operatorname{ran} D^*_i \subset M^\perp$.*

(3) *If $H = M$, the sequence $\{\Gamma'_i\}$ satisfying $A_i = \Gamma'_i\theta^*_{\mathcal{A}}\theta_\Lambda$ has the unique solution $\Gamma'_i = C_i$ for any $i \in \mathbb{N}$, where $C_i = A_i\theta^*_\Lambda\theta_{\widetilde{\mathcal{A}}}$.*

Proof (1) Since $A^*_i g_i = \sum_{j\in\mathbb{N}} \Lambda^*_j \Lambda_j A^*_i g_i$ for any $i \in \mathbb{N}$, $g_i \in H_i$ and $\mathcal{A}_j C^*_i = \Lambda_j A^*_i$, we have $A^*_i g_i = \sum_{j\in\mathbb{N}} \Lambda^*_j \mathcal{A}_j C^*_i g_i$. We take $\Gamma'_i = C_i$.

(2) For any $i \in \mathbb{N}$, take arbitrary operator $D_i \in B(M^\perp, H_i)$. Obviously, $\operatorname{ran} D^*_i \subset M^\perp$ is satisfied. Let $\Gamma'_i = C_i + D_i$. Since $M = \operatorname{ran}\theta^*_{\mathcal{A}}$, by (1), we have

$$\Gamma'_i\theta^*_{\mathcal{A}}\theta_\Lambda = (C_i + D_i)\theta^*_{\mathcal{A}}\theta_\Lambda = C_i\theta^*_{\mathcal{A}}\theta_\Lambda = A_i.$$

For the converse, suppose $A_i = \Gamma'_i\theta^*_{\mathcal{A}}\theta_\Lambda$ for any $i \in \mathbb{N}$. By (1), $C_i\theta^*_{\mathcal{A}}\theta_\Lambda = A_i$. Let $D_i = \Gamma'_i - C_i$. Hence, $D_i\theta^*_{\mathcal{A}}\theta_\Lambda = 0$. Since $M = \operatorname{ran}\theta^*_{\mathcal{A}}$, $M \subset \ker D_i$. Thus $\operatorname{ran} D^*_i \subset M^\perp$.

(3) If $H = M$, we have $D_i = 0$ for any $i \in \mathbb{N}$ from (2). □

Proposition 4.1 did not have any assumption on $\{A_i\}$ or use any relationship between $\{A_i\}$ and $\{\mathcal{A}_i\}$.

The next result exhibits that $\{C_i\}$ and $\{A_i\}$ have the common properties.

Proposition 4.2 *Let $\{\Lambda_i\}$ be a g-orthonormal basis for H, $\{\mathcal{A}_i\}$ be a g-Riesz basis for M with the frame bounds c and d, $\{\widetilde{\mathcal{A}}_i\}$ be the canonical dual g-frame of $\{\mathcal{A}_i\}$ in M, where M is a closed subspace of H. For a sequence $\{A_i\}$, define $C_i = A_i\theta^*_\Lambda\theta_{\widetilde{\mathcal{A}}}$, for any $i \in \mathbb{N}$, we have*

(1) *If $\{A_i\}$ is a g-Bessel sequence for H with the upper bound b, then $\{C_i\}$ is a g-Bessel sequence for H with the upper bound bc^{-1}. Moreover, for any $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$, we have*

$$c\left\|\sum_{i\in\mathbb{N}} C^*_i g_i\right\|^2 \le \left\|\sum_{i\in\mathbb{N}} A^*_i g_i\right\|^2 \le d\left\|\sum_{i\in\mathbb{N}} C^*_i g_i\right\|^2.$$

Specially, $\{A_i\}$ is g-w-linearly independent if and only if $\{C_i\}$ is g-w-linearly independent.

(2) *If $\{A_i\}$ is a g-frame for H with the frame bounds a, b, then $\{C_i\}$ is a g-frame for M with the frame bounds ad^{-1}, bc^{-1}.*

(3) *If $\{A_i\}$ is a g-Riesz basis for H with the frame bounds a, b, then $\{C_i\}$ is a g-Riesz basis for M with the frame bounds ad^{-1}, bc^{-1}.*

(4) *If $\{C_i\}$ is a g-Bessel sequence for H with the upper bound b_1, then $\{A_i\}$ is a g-Bessel sequence for H with the upper bound $b_1 d$.*

(5) *If $\{C_i\}$ is a g-frame for M with the frame bounds a_1, b_1, then $\{A_i\}$ is a g-frame for H with the frame bounds $a_1 c, b_1 d$.*

(6) *If $\{C_i\}$ is a g-Riesz basis for M with the frame bounds a_1, b_1, then $\{A_i\}$ is a g-Riesz basis for H with the frame bounds $a_1 c, a_1 d$.*

Proof (1) Since $C_i = A_i\theta_\Lambda^*\theta_{\widetilde{\mathcal{A}}}$ for any $i \in \mathbb{N}$, for every $f \in H$, we have

$$\sum_{i\in\mathbb{N}} \|C_i f\|^2 = \sum_{i\in\mathbb{N}} \left\|A_i\theta_\Lambda^*\theta_{\widetilde{\mathcal{A}}} f\right\|^2 \le bc^{-1}\|f\|^2.$$

Moreover, because $\theta_C^* = \theta_{\widetilde{\mathcal{A}}}^*\theta_\Lambda\theta_A^*$, for any $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$, we have

$$\left\|\sum_{i\in\mathbb{N}} C_i^* g_i\right\|^2 = \left\|\sum_{i\in\mathbb{N}} \widetilde{\mathcal{A}}_i^*\theta_\Lambda\theta_A^* g_i\right\|^2 \le c^{-1}\left\|\sum_{i\in\mathbb{N}} A_i^* g_i\right\|^2.$$

As $\theta_A^* = \theta_\Lambda^*\theta_{\mathcal{A}}\theta_C^*$, for every $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$, we get

$$\left\|\sum_{i\in\mathbb{N}} A_i^* g_i\right\|^2 = \sum_{i\in\mathbb{N}} \left\|\mathcal{A}_i\theta_C^* g_i\right\|^2 \le d\left\|\sum_{i\in\mathbb{N}} C_i^* g_i\right\|^2.$$

Obviously, $\{A_i\}$ is g-w-linearly independent if and only if $\{C_i\}$ is g-w-linearly independent from the above.

(2) The case of upper bound was obtained by (1). Similarly as (1), for every $f \in M$, we get

$$ad^{-1}\|f\|^2 \le a\left\|\theta_\Lambda^*\theta_{\widetilde{\mathcal{A}}} f\right\|^2 \le \sum_{i\in\mathbb{N}} \left\|A_i\theta_\Lambda^*\theta_{\widetilde{\mathcal{A}}} f\right\|^2 = \sum_{i\in\mathbb{N}} \|C_i f\|^2.$$

(3) Suppose $\{A_i\}$ is a g-Riesz basis for H. Since $\{C_i\}$ is a g-frame for M by (2) and is g-w-linearly independent by (1), $\{C_i\}$ is a g-Riesz basis for M by [4, Lemma 2.16]. The frame bounds can be obtained by (2).

The rest is similar to the above. □

From the above, $\{C_i\}$, $\{A_i\}$ have the same properties, but the bounds may not be common.

Corollary 4.3 *Let $\{\Lambda_i\}$ be a g-orthonormal basis for H, $\{\mathcal{A}_i\}$ be a g-orthonormal basis for M, where M is a closed subspace of H. For a sequence $\{A_i\}$, define $C_i = A_i\theta_\Lambda^*\theta_{\widetilde{\mathcal{A}}}$ for any $i \in \mathbb{N}$, we have*:

1. *$\{C_i\}$ is a g-Bessel sequence for H if and only if $\{A_i\}$ is a g-Bessel sequence for H with the same bound.*
2. *$\{C_i\}$ is a g-frame for M if and only if $\{A_i\}$ is a g-frame for H with the same bounds.*
3. *$\{C_i\}$ is a g-Riesz basis for M if and only if $\{A_i\}$ is a g-Riesz basis for H with the same bounds.*

Proof Take $c = d = 1$ by the proof of Proposition 4.2, which can be obtained directly. □

Let $\{\mathcal{A}_i\}$ be a g-Riesz basis for M, where M is a closed subspace of H. Let $\mathscr{A}_i = \mathcal{A}_i S_{\mathcal{A}}^{-\frac{1}{2}}$ for any $i \in \mathbb{N}$, where $S_{\mathcal{A}}$ is the frame operator of $\{\mathcal{A}_i\}$. Then $\{\mathscr{A}_i\}$ is a g-orthonormal basis for M. Let $\{\Lambda_i\}$ be a g-orthonormal basis for H and $\Theta = \theta_\Lambda^*\theta_{\mathscr{A}}$. Obviously, $\Theta : M \to H$ is unitary and $\mathscr{A}_i = \Lambda_i\Theta$. Then we have the following result.

Proposition 4.4 *Let $\{\Lambda_i\}$ be a g-orthonormal basis for H, $\{\mathcal{A}_i\}$ be a g-Riesz basis for M with the frame bounds c, d, where M is a closed subspace of H, $\{A_i\}$ be a g-frame for H with the frame bounds a, b. Define $C_i = A_i\theta_\Lambda^*\theta_{\widetilde{\mathcal{A}}}$ for every $i \in \mathbb{N}$. Then the following are equivalent:*

(1) *$\{C_i\}$ is a Parseval g-frame for M.*

(2) *$S_{\mathcal{A}} = \Theta^* S_A \Theta$, where $\Theta = \theta_\Lambda^*\theta_{\widetilde{\mathcal{A}}} S_{\mathcal{A}}^{\frac{1}{2}}$.*

Proof By Proposition 4.2, $\{C_i\}$ is a g-frame for M. Since $\theta_C = \theta_A\theta_\Lambda^*\theta_{\widetilde{\mathcal{A}}}$ and $\theta_{\widetilde{\mathcal{A}}} = \theta_\Lambda\Theta S_{\mathcal{A}}^{-\frac{1}{2}}$, we have $S_C = S_{\mathcal{A}}^{-\frac{1}{2}}\Theta^* S_A \Theta S_{\mathcal{A}}^{-\frac{1}{2}}$. Obviously, $S_C = P$ if and only if $S_{\mathcal{A}} = \Theta^* S_A\Theta$, where P is the orthogonal projection from H onto M. □

If $\{A_i\}$ is a tight g-frame for H with the bound a. Let $\{\mathcal{A}_i\}$ be a tight g-Riesz basis for M with frame bound a. Then $S_A = aI$, $S_{\mathcal{A}} = aP$. Thus Proposition 4.4(2) holds obviously. Then we get Corollary 4.6 directly.

Proposition 4.5 *Let $\{\Lambda_i\}$ be a g-orthonormal basis for H, $\{\mathcal{A}_i\}$ be a g-Riesz basis for M, where M is a closed subspace of H. If $\{A_i\}$ is a g-frame for H, define $C_i = A_i\theta_\Lambda^*\theta_{\widetilde{\mathcal{A}}}$ for any $i \in \mathbb{N}$. Then the following are equivalent:*

(1) *If $\{\mathcal{A}_i\}$ is the g-R-dual sequence of $\{A_i\}$ with respect to two g-orthonormal bases $\{\Lambda_i\}$, $\{\Gamma_i\}$.*

(2) *There exists a g-orthonormal basis $\{\Gamma_i\}$ for H such that $A_i = \Gamma_i\theta_{\mathcal{A}}^*\theta_\Lambda$ for every $i \in \mathbb{N}$.*

(3) *There exists a g-orthonormal basis $\{\Gamma_i\}$ for H such that $C_i = \Gamma_i P$ for every $i \in \mathbb{N}$, where P is the orthogonal projection from H onto M.*

(4) *$\{C_i\}$ is a Parseval g-frame for M and $\dim\ker\theta_C^* = \dim M^\perp$.*

(5) *$S_{\mathcal{A}} = \Theta^* S_A\Theta$ and $\dim\ker\theta_C^* = \dim M^\perp$, where $\Theta = \theta_\Lambda^*\theta_{\widetilde{\mathcal{A}}} S_{\mathcal{A}}^{\frac{1}{2}}$.*

Proof (1) ⇒ (2) By Definition 2.2, we have $\mathcal{A}_i^* = \theta_\Gamma^*\theta_A\Lambda_i^*$ for every $i \in \mathbb{N}$. Hence, $A_i = \Gamma_i\theta_{\mathcal{A}}^*\theta_\Lambda$.

(2) ⇒ (1) It is obvious by Definition 2.2. The equivalence of (2) and (3) can be obtained by Proposition 4.1.

(3) ⇒ (4) For any $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$, we have

$$\theta_C^*\{g_i\} = \sum_{i\in\mathbb{N}} C_i^* g_i = \sum_{i\in\mathbb{N}} P\Gamma_i^* g_i = P\theta_\Gamma^*\{g_i\}.$$

Obviously, $\{g_i\} \in \ker\theta_C^*$ if and only if $\theta_\Gamma^*\{g_i\} \in M^\perp$. Then $\dim\ker\theta_C^* = \dim M^\perp$ as θ_Γ is unitary. Evidently, $\{C_i\}$ is a Parseval g-frame for M.

(4) ⇒ (3) Suppose $\{C_i\}$ is a Parseval g-frame for M. Let $K = M \oplus (\operatorname{ran}\theta_C)^\perp$, $T_i = C_i \oplus P_i Q^\perp$ for any $i \in \mathbb{N}$, where Q, P_i are the orthogonal projection from $\bigoplus_{i\in\mathbb{N}} H_i$ onto $\operatorname{ran}\theta_C$, H_i, respectively, for every $i \in \mathbb{N}$. It is easy to get $\{T_i\}$ is a g-orthonormal basis for K by [7, Theorem 4.1].

Since $\dim\ker\theta_C^* = \dim M^\perp$, there exists a unitary operator $V : M^\perp \to \ker\theta_C^*$. Let $\Gamma_i = T_i(P \oplus V) = C_i \oplus P_i Q^\perp V$ for every $i \in \mathbb{N}$. As $P \oplus V : M \oplus M^\perp \to M \oplus (\operatorname{ran}\theta_C)^\perp$ is unitary, where P is the orthogonal projection from H onto M, we see that $\{\Gamma_i\}$ is a g-orthonormal basis for H by [6, Theorem 3.5]. Obviously, we have $C_i = \Gamma_i P$. The equivalence of (4), (5) is direct by Proposition 4.4. □

By Proposition 4.5, we can also get the following corollary, which was showed in [3, Theorem 2.7].

Corollary 4.6 *Let $\{\Lambda_i\}$ be a g-orthonormal basis for H, $\{\mathcal{A}_i\}$ be a tight g-Riesz basis for M with the frame bound a, where M is a closed subspace of H. If $\{A_i\}$ is a tight g-frame with the frame bound a. Then there exists a g-orthonormal basis $\{\Gamma_i\}$ for H such that $\{\mathcal{A}_i\}$ is the g-R-dual of $\{A_i\}$ with respect to two g-orthonormal bases $\{\Lambda_i\}$, $\{\Gamma_i\}$ if and only if* $\dim\ker\theta_C^* = \dim M^\perp$, *where $C_i = A_i\theta_\Lambda^*\theta_{\widetilde{\mathcal{A}}}$ for any $i \in \mathbb{N}$.*

Proof By Proposition 4.2(3), $\{C_i\}$ is a Parseval g-frame for M. It is obvious by Proposition 4.5. □

Corollary 4.7 *Let $\{\Lambda_i\}$ be a g-orthonormal basis for H, $\{\mathcal{A}_i\}$ be a g-Riesz basis for M, $\{\widetilde{\mathcal{A}}_i\}$ be the canonical dual g-frame of $\{\mathcal{A}_i\}$ in M, where M is a closed subspace of H. If $\{A_i\}$ is a g-frame for H. Define $C_i = A_i\theta_\Lambda^*\theta_{\widetilde{\mathcal{A}}}$ for any $i \in \mathbb{N}$. For any $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$, let $g = \theta_\Lambda^*\{g_i\} \in H$, $h = \theta_{\mathcal{A}}^*\{g_i\} \in M$. Then there exists a g-orthonormal basis $\{\Gamma_i\}$ for H such that $\{\mathcal{A}_i\}$ is the g-R-dual of $\{A_i\}$ with respect to two g-orthonormal bases $\{\Lambda_i\}$, $\{\Gamma_i\}$ if and only if $\sum_{i\in\mathbb{N}} \|A_i g\|^2 = \|h\|^2$ and* $\dim\ker\theta_C^* = \dim M^\perp$.

Proof Obviously, we have

$$\sum_{i\in\mathbb{N}} \|A_i g\|^2 = \left\|\theta_A\theta_\Lambda^*\{g_i\}\right\|^2 = \left\|\theta_{\mathcal{A}}^*\{g_i\}\right\|^2 = \|h\|^2.$$

The result now follows from Proposition 4.5 directly. □

4.2 The construction of orthogonal sequence

Now we will construct a sequence $\{\Gamma_i'\}$ such $A_i = \sum_{j\in\mathbb{N}} \Gamma_i'\widetilde{\mathcal{A}}_j^*\Lambda_j$, which is characterized in Proposition 4.1.

Proposition 4.8 *Let $\{\Lambda_i\}$ be a g-orthonormal basis for H, $\{\mathcal{A}_i\}$ be a g-Riesz basis for M, $\{\widetilde{\mathcal{A}}_i\}$ be the canonical dual g-frame of $\{\mathcal{A}_i\}$ in M, where M is a closed subspace of H. If* $\dim M^\perp = \sum_i \dim H_i = \infty$, *we have:*

(1) *For any sequence $\{A_i\}$, there exists a g-w-linearly independent sequence $\{\Gamma_i'\}$ such that $A_i = \sum_{j\in\mathbb{N}} \Gamma_i'\widetilde{\mathcal{A}}_j^*\Lambda_j$ for every $i \in \mathbb{N}$.*

(2) *For any g-Bessel sequence $\{A_i\}$, there exists a norm-bounded and g-w-linearly independent sequence $\{\Gamma_i'\}$ such that $A_i = \sum_{j\in\mathbb{N}} \Gamma_i'\widetilde{\mathcal{A}}_j^*\Lambda_j$ for every $i \in \mathbb{N}$.*

(3) *For any operator sequence $\{A_i\}$, there exists a g-orthogonal sequence $\{\Gamma_i'\}$ such that $A_i = \sum_{j\in\mathbb{N}} \Gamma_i'\widetilde{\mathcal{A}}_j^*\Lambda_j$ for every $i \in \mathbb{N}$.*

Proof (1) Since $\dim M^\perp = \sum_{i\in\mathbb{N}} \dim H_i$, there exists a g-orthonormal basis $\{E_i\}$ for $M^\perp$ by [5, Theorem 3.1] with $E_i \in B(M^\perp, H_i)$ for any $i \in \mathbb{N}$. Let $W_i = \overline{\operatorname{ran}} E_i^*$ for any $i \in \mathbb{N}$. Then $M^\perp = \bigoplus_{i\in\mathbb{N}} W_i$ and $E_i : W_i \to H_i$ is unitary. Let $C_i = A_i\theta_\Lambda^*\theta_{\widetilde{\mathcal{A}}}$ for any $i \in \mathbb{N}$. Then $\mathcal{A}_i E_j^* = 0$ and $C_i E_j^* = \sum_{k\in\mathbb{N}} A_i\Lambda_k^*\widetilde{\mathcal{A}}_k E_j^* = 0$.

Since there exists an invertible operator $D_i : W_i \to H_i$ for any $i \in \mathbb{N}$, we see that $D_i E_i^* + C_i E_i^* = D_i E_i^* \in B(H, H_i)$ is invertible. Let $\Gamma_i' = D_i + C_i \in B(H, H_i)$. Obviously, $\Gamma_i' \neq 0$.

For any $\{g_i\} \in \bigoplus_{i\in\mathbb{N}} H_i$, if $\sum_{i\in\mathbb{N}} \Gamma_i'^{*} g_i = 0$, then, for any $j \in \mathbb{N}$, we have

$$E_j \sum_{i\in\mathbb{N}} \Gamma'^{*}_i g_i = \sum_{i\in\mathbb{N}} \left(E_j C_i^* + E_j D_i^*\right) g_i = E_j D_j^* g_j = 0.$$

Then $g_j = 0$.

(2) By the proof of (1), we can choose D_i such that $\|D_i\| = 1$ (if not, we choose $D_i' = \frac{D_i}{\|D_i\|}$) for any $i \in \mathbb{N}$. By Proposition 4.2, $\{C_i\}$ is a g-Bessel sequence for M. Suppose the upper bound of $\{C_i\}$ is b. Then $\|C_i\| \le b$. Hence, for every $i \in \mathbb{N}$, $g_i \in H_i$, we have

$$\left\| \Gamma'^{*}_i g_i \right\|^2 = \left\| C_i^* g_i \right\|^2 + \left\| D_i^* g_i \right\|^2 \le \left(b^2 + 1\right) \|g_i\|^2.$$

(3) By Proposition 4.1, the sequence $\{\Gamma_i'\}$ such that $A_i = \sum_{j\in\mathbb{N}} \Gamma_i' \widetilde{\mathcal{A}}_j^* \Lambda_j = \Gamma_i' \theta_{\widetilde{\mathcal{A}}}^* \theta_\Lambda$ can be written as $\Gamma_i' = C_i + D_i$, where $C_i = A_i \theta_\Lambda^* \theta_{\widetilde{\mathcal{A}}}$, $\overline{\operatorname{ran}}\, D_i^* \subset M^\perp$ for any $i \in \mathbb{N}$. For every $i,j \in \mathbb{N}$, $i \ne j$, $g_i \in H_i$, $g_j \in H_j$, we have

$$\left\langle \Gamma'^{*}_i g_i, \Gamma_j'^{*} g_j \right\rangle = 0 \quad \text{if and only if} \quad \left\langle C_i^* g_i, C_j^* g_j \right\rangle + \left\langle D_i^* g_i, D_j^* g_j \right\rangle = 0.$$

We will use the following inductive procedure to construct $\{D_i\}$ such that $\overline{\operatorname{ran}}\, D_i^* \subset M^\perp$ and $D_j D_i^* = -C_j C_i^*$ for every $i,j \in \mathbb{N}$, $i \ne j$. Let $T_{ij} = -C_i C_j^* \in B(H_j, H_i)$. Then $T_{ij}^* = T_{ji}$. Let I_i be the identity on H_i.

(1) Let $D_1^* = E_1^*$.

(2) Let $D_2^* = E_1^* X_{1,2}^* + E_2^*$, where $X_{1,2}^* = T_{12}$.

Obviously, $D_1 D_2^* = E_1 E_1^* X_{1,2}^* + E_1 E_2^* = T_{12}$. Then $\Gamma_1' \Gamma_2'^{*} = 0$.

3) For any $k \in \mathbb{N}$, assuming that we have gotten operators $D_1, D_2, \dots, D_k$ in terms of $X_{i,k} \in B(H_i, H_k)$ $(i = 1, \dots, k-1)$ such that $D_k^* = \sum_{i=1}^{k-1} E_i^* X_{i,k}^* + E_k^*$. Then, for $k+1$, we define D_{k+1} by $D_{k+1}^* = \sum_{i=1}^{k} E_i^* X_{i,k+1}^* + E_{k+1}^*$, where operators $X_{i,k+1}$ $(i = 1, 2, \dots, k)$ are given by the following equation:

$$\begin{pmatrix} I_1 & & & \\ X_{12} & I_2 & & \\ \vdots & & \ddots & \\ X_{1k} & X_{2k} & \cdots & I_k \end{pmatrix} \begin{pmatrix} X_{1,k+1}^* \\ X_{2,k+1}^* \\ \vdots \\ X_{k,k+1}^* \end{pmatrix} = \begin{pmatrix} T_{1,k+1} \\ T_{2,k+1} \\ \vdots \\ T_{k,k+1} \end{pmatrix}.$$

Obviously, we can obtain $X_{i,k+1} \in B(H_i, H_{k+1})$ $(i = 1, \dots, k)$. Thus we have constructed the sequence $\{D_i\}$ and obtained $\{\Gamma_i'\}$ by $\Gamma_i' = C_i + D_i$ for any $i \in \mathbb{N}$. Then $\{\Gamma_i'\}$ such that $\Gamma_i' \Gamma_j'^{*} = 0$ for every $i,j \in \mathbb{N}$ with $i \ne j$.

Lastly, we show the sequence $\{\Gamma_i'\}$ satisfies the desired condition: $A_i = \sum_{j\in\mathbb{N}} \Gamma_i' \mathcal{A}_j^* \Lambda_j$ for all $i \in \mathbb{N}$.

Since $(\ker D_i)^\perp = \overline{\operatorname{ran}}\, D_i^* \subset M^\perp$ and $\overline{\operatorname{ran}}\, \widetilde{\mathcal{A}}_j^* \subset M$ for any $i,j \in \mathbb{N}$, we have

$$\overline{\operatorname{ran}}\, \widetilde{\mathcal{A}}_j^* \subset M \subset \ker D_i.$$

Hence, $D_i \widetilde{\mathcal{A}}_j^* = 0$ for any $i,j \in \mathbb{N}$. On the other hand, since $C_i = A_i \theta_\Lambda^* \theta_{\widetilde{\mathcal{A}}}$ for any $i \in \mathbb{J}$, we get $\mathcal{A}_j C_i^* = \Lambda_j A_i^*$. By $A_i^* g_i = \sum_{j\in\mathbb{N}} \Lambda_j^* \Lambda_j A_i^* g_i$ for any $g_i \in H_i$, any $i \in \mathbb{N}$, we have $A_i^* g_i =$

$\sum_{j\in\mathbb{N}} \Lambda_j^* \mathcal{A}_j C_i^* g_i$. So $\sum_{j\in\mathbb{N}} C_i \widetilde{\mathcal{A}}_j^* \Lambda_j = A_i$ for any $i \in \mathbb{N}$. Then

$$\sum_{j\in\mathbb{N}} \Gamma_i' \widetilde{\mathcal{A}}_j^* \Lambda_j = \sum_{j\in\mathbb{N}} (C_i + D_i) \widetilde{\mathcal{A}}_j^* \Lambda_j = \sum_{j\in\mathbb{N}} C_i \widetilde{\mathcal{A}}_j^* \Lambda_j = A_i, \quad \forall i \in \mathbb{N}. \qquad \square$$

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