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Generalized multivalued Khan-type (ψ,ϕ)-contractions in complete metric spaces

Yun Xiao[1,2*], Zhijian Qiu[1] and Xindong Liu[2]

*Correspondence:
54077957@qq.com
[1]School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu, China
[2]School of Mathematics, Yibin University, Yibin, China

Abstract

In this article, we study a new generalized multivalued Khan-type (ψ,ϕ)-contraction. We obtain some fixed point theorems related to the introduced contraction for multivalued mappings in complete metric spaces. Our theorems extend and improve some previous known results with less conditions. Also, we give some illustrative examples.

MSC: 47H09; 47J25; 65K10

Keywords: Generalized multivalued Khan-type (ψ,ϕ)-contraction; Fixed point; Multivalued mapping; Comparison function

1 Introduction

Throughout this paper, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of non-negative integers, respectively. Similarly, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ represent the set of real numbers, positive real numbers and non-negative real numbers, respectively. Let (X,d) be a metric space and let $T: X \to X$ be a self-mapping. If there is a number $k \in [0,1)$ such that, for all $x,y \in X$, $d(Tx,Ty) \le kd(x,y)$ holds, then T is called a contractive mapping. In 1922, Banach [1] proved a famous result known as the Banach contraction principle, which states that every contractive mapping has a unique fixed point. It is one of the fundamental results in fixed point theory. Due to its importance and simplicity, over the years, many authors made efforts to generalize or extend that result (see [2–12] and the references therein).

Basically, the generalizations go in two directions: The first one is to work out a different expression for the right side of the inequality in the Banach contraction principle. A typical result of this kind is the work of Khan ([13], 1976). Using a symmetric expression, Khan introduced the notion of a Khan-type contraction and proved a corresponding fixed point theorem. In 1978, Fisher [14] modified and improved Khan's work. Four decades later, Piri, Rahrovi and Kumamet [15] extended the work of both Khan [13] and Fisher [14]. They accomplished the work by introducing a new general contractive condition with a symmetric expression and established a corresponding fixed point theorem. Another direction of generalizations of Banach's work is to express both sides of the inequality in the Banach result by forms involving functions. For example, Wardowski introduced the notion of the F-contraction in ([16], 2012) and Jleli and Samet addressed the so-called θ-contraction in ([17, 18], 2014). As a result, they all extended and improved Banach's work.

Combined with the ideas from the F-contraction and the Khan-type contraction, in 2017, Piri, Rahrovi, Marasi and Kumam [19] investigated and developed the so-called F-Khan-contraction and proved the desired fixed point theorem. The results of Piri et al. extended and improved Wardowski's work in [16]. In 2015, by using the Hausdorff–Pompieu metric, Altun, Durmaz and Dag extended the F-contraction to multivalued contractive mappings. They introduced the notion of a multivalued F-contraction and obtained some new fixed point theorems for multivalued mappings in [4]. The purpose of this paper is to further extend the above results. We define and examine some new generalized multivalued Khan-type contraction which extends all of F-contraction and θ-contraction studied previously. The results presented in this paper improve and extend the corresponding results in Piri et al. [19], Jleli et al. [18] and Altun et al. [4].

2 Preliminaries

Let $\mathcal{F}$ be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

($\mathcal{F}_1$) F is non-decreasing;

($\mathcal{F}_2$) for each sequence $\{t_n\} \subset \mathbb{R}^+$, $\lim_{n\to\infty} F(t_n) = -\infty \Leftrightarrow \lim_{n\to\infty} t_n = 0$;

($\mathcal{F}_3$) there exists $r \in (0,1)$ such that $\lim_{t\to 0^+} t^r F(t) = 0$.

Definition 2.1 ([19]) Let (X,d) be a metric space. A mapping $T : X \to X$ is called a F-Khan-contraction, if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x,y \in X$,

(i) if $\max\{d(x,Ty),d(y,Tx)\} \neq 0$, then $Tx \neq Ty$ and

$$\tau + F\bigl(d(Tx,Ty)\bigr) \le F\bigl(M(x,y)\bigr),$$

where

$$M(x,y) = \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{\max\{d(x,Ty),d(y,Tx)\}};$$

(ii) if $\max\{d(x,Ty),d(y,Tx)\} = 0$, then

$$Tx = Ty.$$

In [19], Piri et al. proved the existence and uniqueness theorem of fixed point for F-Khan-contraction.

Inspired by Definition 2.1 and taking the θ-contraction into account, we can define a new Khan-type contraction. Let Θ be the family of all functions $\theta : \mathbb{R}^+ \to (1,\infty)$ satisfying the following conditions:

(Θ_1) θ is non-decreasing;

(Θ_2) for each sequence $\{t_n\} \subset \mathbb{R}^+$, $\lim_{n\to\infty} \theta(t_n) = 1 \Leftrightarrow \lim_{n\to\infty} t_n = 0$;

(Θ_3) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

Definition 2.2 Let (X,d) be a metric spaces. A mapping $T : X \to X$ is called a θ-Khan-contraction, if there exist $\theta \in \Theta$ and $k \in (0,1)$ such that, for all $x,y \in X$,

(i) if $\max\{d(x,Ty),d(y,Tx)\} \neq 0$, then $Tx \neq Ty$ and

$$\theta\bigl(d(Tx,Ty)\bigr) \le \bigl(\theta\bigl(M(x,y)\bigr)\bigr)^k,$$

where

$$M(x,y)=\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{\max\{d(x,Ty),d(y,Tx)\}};$$

(ii) if $\max\{d(x,Ty),d(y,Tx)\}=0$, then

$$Tx=Ty.$$

Note that we get the θ-contraction ([18]) immediately by letting $y=Tx$ in $M(x,y)$.

In 2014, by using comparison functions, Latif, Gordji, Karapinar and Sintunavaratet introduced the notion of generalized (α,ψ)-Meir–Keeler contractive mappings and obtained some new results in [20]. Two years later, Wang and Li extended the results of Latif et al. by introducing (α,ψ)-Meir–Keeler–Khan multivalued mappings in [2]. Inspired by those ideas, we will introduce some new contractive mappings by using comparison functions in this paper.

Definition 2.3 ([20]) Let Ψ be the family of all functions $\psi:\mathbb{R}_0^+\to\mathbb{R}_0^+$ such that
(Ψ_1) ψ is non-decreasing;
(Ψ_2) $\lim_{n\to\infty}\psi^n(t)=0$ for all $t\geq 0$, where ψ^n stands for the nth iterate of ψ.

Remark 2.4 Clearly, if ψ is a comparison function, then $\psi(t)<t$ for each $t>0$ and $\psi(0)=0$.

Example 2.5 Let

$$\psi_1(t)=\alpha t,\quad 0<\alpha<1,\text{for all } t\geq 0;$$

$$\psi_2(t)=\begin{cases}\frac{t}{2}, & \text{if } 0\leq t<1,\\ \frac{t}{3}, & \text{if } 1\leq t,\end{cases}$$

$$\psi_3(t)=\frac{t}{1+t},\quad \text{for all } t\geq 0.$$

It is easy to check that ψ_1, ψ_2 and ψ_3 belong to Ψ.

For more properties and applications of comparison function, we refer the reader to [2, 20–23].

The next definition plays an important role in our work.

Definition 2.6 Let Φ be the family of all functions $\phi:\mathbb{R}_0^+\to\mathbb{R}_0^+$ such that:
(Φ_1) ϕ is non-decreasing and continuous;
(Φ_2) for each sequence $\{t_n\}\subset(0,\infty)$, $\lim_{n\to\infty}\phi(t_n)=0\Leftrightarrow\lim_{n\to\infty}t_n=0$.

Example 2.7 Let

$$\phi_1(t)=t,\quad \text{for all } t>0;$$

$$\phi_2(t)=\ln\theta(t),\quad \theta\in\Theta \text{ for all } t>0;$$

$$\phi_3(t)=e^{F(t)},\quad F\in\mathcal{F} \text{ for all } t>0.$$

It is easy to check that ϕ_1, ϕ_2 and ϕ_3 belong to Φ.

Given a metric space (X,d), by $CB(X)$ and $K(X)$ we denote the family of all nonempty closed and bounded subsets of X, and the family of all nonempty compact subsets of X, respectively. For $A,B \subseteq B(X)$, let

$$H(A,B) = \max\Bigl\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(y,A)\Bigr\},$$

where $D(x,B) = \inf_{y\in B}\{d(x,y)\}$. Then H is a metric on $CB(X)$, which is called the Hausdorff–Pompieu metric. By using the concept of the Hausdorff–Pompieu metric, Nadler introduced the notion of multivalued contraction mappings and he proved a multivalued version of the well known Banach contraction principle ([3], 1969).

By using the Hausdorff–Pompieu metric H, the F-contraction was extended to the multivalued case in [4].

Definition 2.8 ([4]) Let (X,d) be a metric space and $T : X \to CB(X)$ be a multivalued mapping. Then T is called a multivalued F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x,y \in X$,

$$H(Tx,Ty) > 0 \quad \Longrightarrow \quad \tau + F\bigl(H(Tx,Ty)\bigr) \le F\bigl(d(x,y)\bigr).$$

Theorem 2.9 ([4]) *Let (X,d) be a complete metric space and $T : X \to K(X)$ be a multivalued F-contraction. Then T has a fixed point x^* in X.*

Following this direction of research, we introduce a new type of generalized multivalued Khan-type contractive mappings. Via Ψ and Φ, we present the notion of generalized multivalued Khan-type (ψ,ϕ)-contraction mappings.

Definition 2.10 Let (X,d) be a metric space and $T : X \to CB(X)$ be a multivalued mapping. Then T is called a generalized multivalued Khan-type (ψ,ϕ)-contraction, if there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that, for all $x,y \in X$,

(i) if $\max\{D(x,Ty),D(y,Tx)\} \ne 0$, then $Tx \ne Ty$ and

$$\phi\bigl(H(Tx,Ty)\bigr) \le \psi\bigl(\phi\bigl(M(x,y)\bigr)\bigr), \tag{2.1}$$

where

$$M(x,y) = \frac{D(x,Tx)D(x,Ty) + D(y,Ty)D(y,Tx)}{\max\{D(x,Ty),D(y,Tx)\}}; \tag{2.2}$$

(ii) if $\max\{D(x,Ty),D(y,Tx)\} = 0$, then

$$Tx = Ty.$$

Remark 2.11 Let $\psi(t) := e^{-\tau}t$, $\phi(t) := e^{F}(t)$, $F \in \mathcal{F}$, $\mathbb{R}_0^+ \to \mathbb{R}_0^+$ in (2.1). It is easy to check that $\psi \in \Psi$ and $\phi \in \Phi$. Hence we have

$$e^{F(H(Tx,Ty))} \le e^{-\tau}e^{F(M(x,y))},$$

then we get

$$\tau + F\big(H(Tx,Ty)\big) \le F\big(M(x,y)\big). \tag{2.3}$$

Then T is called a generalized multivalued F-Khan-contraction. Let $y \in Tx$ in (2.2), we get

$$M(x,y) = \frac{D(x,Tx)D(x,Ty) + D(y,Ty)D(y,Tx)}{\max\{D(x,Ty),D(y,Tx)\}} = D(x,Tx) \le d(x,y). \tag{2.4}$$

In combination with (2.3) and (2.4), we get a multivalued F-contraction (Definition 2.8, [4]) immediately.

Remark 2.12 Let $\psi(t) := kt$, $k \in (0,1)$, $\phi(t) := \ln\theta(t)$, $\theta \in \Theta$, $\mathbb{R}_0^+ \to \mathbb{R}_0^+$ in (2.1). It is easy to check that $\psi \in \Psi$ and $\phi \in \Phi$. Hence we have

$$\ln\theta\big(H(Tx,Ty)\big) \le k\ln\theta\big(M(x,y)\big),$$

then we have

$$\theta\big(H(Tx,Ty)\big) \le \big(\theta\big(M(x,y)\big)\big)^k. \tag{2.5}$$

Then T is called a generalized multivalued θ-Khan-contraction. Similarly, in combination with (2.4) and (2.5), we obtain

$$\theta\big(H(Tx,Ty)\big) \le \theta\big(d(x,y)\big)^k.$$

We call it a multivalued θ-contraction.

In Sect. 3, we state and prove some new fixed point results for generalized multivalued Khan-type (ψ,ϕ)-contraction. In Sect. 4, we give some applications of the main results of this paper.

3 Main results

Based on the above argument, now we are in a position to give the following results.

Theorem 3.1 *Let (X,d) be a complete metric space and $T : X \to K(X)$ be a generalized multivalued Khan-type (ψ,ϕ)-contraction, then T has a fixed point $x^* \in X$.*

Proof *Case I* Assume that $D(x_{n-1},Tx_n) \neq 0$ for all $n \in \mathbb{N}$.

We construct a sequence starting from $x_0 \in X$. If $x_0 \in Tx_0$, then x_0 is a fixed point of T and the proof is completed. Suppose that $x_0 \notin Tx_0$. Because Tx_0 is a compact subset of X, then $D(x_0,Tx_0) > 0$ and we can choose $x_1 \in Tx_0$ such that $d(x_0,x_1) = D(x_0,Tx_0)$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T, and subsequently, the proof is completed. Assume that $x_1 \notin Tx_1$, then it is clear that $D(x_1,Tx_1) > 0$ because Tx_1 is a compact subset of X. On the other hand, from $D(x_1,Tx_1) \le H(Tx_0,Tx_1)$ and (Φ_1), we obtain

$$\phi\big(D(x_1,Tx_1)\big) \le \phi\big(H(Tx_0,Tx_1)\big).$$

It follows from (2.1) and Remark 2.4 that

$$\begin{aligned}
\phi\bigl(D(x_1,Tx_1)\bigr) &\le \phi\bigl(H(Tx_0,Tx_1)\bigr) \le \psi\bigl(\phi\bigl(M(x_0,x_1)\bigr)\bigr) \\
&= \psi\left(\phi\left(\frac{D(x_0,Tx_0)D(x_0,Tx_1)+D(x_1,Tx_1)D(x_1,Tx_0)}{\max\{D(x_0,Tx_1),D(x_1,Tx_0)\}}\right)\right) \\
&= \psi\bigl(\phi\bigl(D(x_0,Tx_0)\bigr)\bigr) \\
&< \phi\bigl(D(x_0,Tx_0)\bigr).
\end{aligned} \tag{3.1}$$

Since Tx_1 is a compact subset of X, we can choose $x_2 \in Tx_1$ such that $d(x_1,x_2) = D(x_1,Tx_1)$. Then from (3.1) we get

$$\phi\bigl(d(x_1,x_2)\bigr) = \phi\bigl(D(x_1,Tx_1)\bigr) < \phi\bigl(D(x_0,Tx_0)\bigr) = \phi\bigl(d(x_0,x_1)\bigr). \tag{3.2}$$

It follows from (3.2) and (Φ_1) that

$$d(x_1,x_2) \le d(x_0,x_1).$$

We continue constructing the sequence similarly. If $x_2 \in Tx_2$, then this proof is done. Thus, we assume that $x_2 \notin Tx_2$. Then $D(x_2,Tx_2) > 0$ since Tx_2 is a compact subset of X, and from $D(x_2,Tx_2) \le H(Tx_1,Tx_2)$, we have

$$\begin{aligned}
\phi\bigl(D(x_2,Tx_2)\bigr) &\le \phi\bigl(H(Tx_1,Tx_2)\bigr) \le \psi\bigl(\phi\bigl(M(x_1,x_2)\bigr)\bigr) \\
&= \psi\left(\phi\left(\frac{D(x_1,Tx_1)D(x_1,Tx_2)+D(x_2,Tx_2)D(x_2,Tx_1)}{\max\{D(x_1,Tx_2),D(x_2,Tx_1)\}}\right)\right) \\
&= \psi\bigl(\phi\bigl(D(x_0,Tx_0)\bigr)\bigr) \\
&< \phi\bigl(D(x_1,Tx_1)\bigr).
\end{aligned} \tag{3.3}$$

In addition, the compactness of Tx_2 implies that there exists $x_3 \in Tx_2$ such that $d(x_2,x_3) = D(x_2,Tx_2)$. Then from (3.3) we get

$$\phi\bigl(d(x_2,x_3)\bigr) = \phi\bigl(D(x_2,Tx_2)\bigr) < \phi\bigl(D(x_1,Tx_1)\bigr) = \phi\bigl(d(x_1,x_2)\bigr). \tag{3.4}$$

It follows from (2.4) and (Φ_1) that

$$d(x_2,x_3) \le d(x_1,x_2).$$

By induction, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}_0}$ satisfying

$$x_{n+1} \in Tx_n, x_{n+1} \notin Tx_{n+1}, \quad d(x_n,x_{n+1}) = D(x_n,Tx_n) > 0, \tag{3.5}$$

and

$$d(x_n,x_{n+1}) \le d(x_{n-1},x_n),$$

for all $n \in \mathbb{N}$. Therefore the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}_0}$ is a positive and non-increasing sequence, and hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) \geq 0.$$

Now, we claim that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

In fact, from (3.5) and (Φ_1), by using (2.1), we get

$$\begin{aligned}
0 &\leq \phi\bigl(d(x_n, x_{n+1})\bigr) = \phi\bigl(D(x_n, Tx_n)\bigr) \\
&\leq \phi\bigl(H(Tx_{n-1}, Tx_n)\bigr) \leq \psi\bigl(\phi\bigl(M(x_{n-1}, x_n)\bigr)\bigr) \\
&= \psi\left(\phi\left(\frac{D(x_{n-1}, Tx_{n-1})D(x_{n-1}, Tx_n) + D(x_n, Tx_n)D(x_n, Tx_{n-1})}{\max\{D(x_{n-1}, Tx_n), D(x_n, Tx_{n-1})\}}\right)\right) \\
&= \psi\bigl(\phi\bigl(D(x_{n-1}, Tx_{n-1})\bigr)\bigr) \leq \psi^2\bigl(\phi\bigl(D(x_{n-2}, Tx_{n-2})\bigr)\bigr) \\
&\leq \psi^3\bigl(\phi\bigl(D(x_{n-3}, Tx_{n-3})\bigr)\bigr) \leq \cdots \\
&\leq \psi^n\bigl(\phi\bigl(D(x_0, Tx_0)\bigr)\bigr).
\end{aligned}$$

From (Ψ_2) we have

$$\lim_{n\to\infty} \psi^n\bigl(\phi\bigl(D(x_0, Tx_0)\bigr)\bigr) = 0.$$

By using the sandwich theorem, we get

$$\lim_{n\to\infty} \phi\bigl(d(x_n, x_{n+1})\bigr) = 0.$$

Therefore, from (Φ_2) we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

and hence

$$\lim_{n\to\infty} D(x_n, Tx_n) = 0. \tag{3.6}$$

Now, we claim that

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Arguing by contradiction, we assume that there exists a $\varepsilon > 0$ for which we can seek two sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that, for all $n \in \mathbb{N}$, $p(n)$ is the smallest index for which

$$p(n) > q(n) > n, \qquad d(x_{p(n)}, x_{q(n)}) \geq \varepsilon, \qquad d(x_{p(n)-1}, x_{q(n)}) < \varepsilon. \tag{3.7}$$

Thus, for all $n \in \mathbb{N}$, by using the triangle inequality, we have

$$\varepsilon \leq d(x_{p(n)}, x_{q(n)}) \leq D(x_{p(n)}, Tx_{q(n)}) + D(Tx_{q(n)}, x_{q(n)}). \tag{3.8}$$

It follows from (3.6) and (3.8) and by using the sandwich theorem again, we have

$$\liminf_{n\to\infty} D(x_{p(n)}, Tx_{q(n)}) \geq \varepsilon.$$

Thus, there exists $n_1 \in \mathbb{N}$, such that

$$D(x_{p(n)}, Tx_{q(n)}) > \frac{\varepsilon}{2},$$

for all $n > n_1$.

This implies that

$$\max\bigl\{D(x_{p(n)}, Tx_{q(n)}), D(Tx_{p(n)}, x_{q(n)})\bigr\} > \frac{\varepsilon}{2}, \tag{3.9}$$

for all $n > n_1$.

From (3.7) and by using the triangle inequality again, we have

$$\begin{aligned} \varepsilon &\leq d(x_{p(n)}, x_{q(n)}) \\ &\leq D(x_{p(n)}, Tx_{p(n)}) + H(Tx_{p(n)}, Tx_{q(n)}) + D(Tx_{q(n)}, x_{q(n)}). \end{aligned} \tag{3.10}$$

In combination with (3.6) and (3.10), we get

$$\liminf_{n\to\infty} H(Tx_{p(n)}, Tx_{q(n)}) \geq \varepsilon.$$

Thus, there exists $n_2 \in \mathbb{N}$, such that

$$H(Tx_{p(n)}, Tx_{q(n)}) > \frac{\varepsilon}{2}, \tag{3.11}$$

for all $n > n_2$.

It follows from (3.11), (Φ_1) and (2.1) that

$$\begin{aligned} \phi\left(\frac{\varepsilon}{2}\right) &\leq \phi\bigl(H(Tx_{p(n)}, Tx_{q(n)})\bigr) \leq \psi\bigl(\phi\bigl(M(x_{p(n)}, x_{q(n)})\bigr)\bigr) \\ &\leq \psi\left(\phi\left(\frac{D(x_{p(n)}, Tx_{p(n)})D(x_{p(n)}, Tx_{q(n)}) + D(x_{q(n)}, Tx_{q(n)})D(x_{q(n)}, Tx_{p(n)})}{\max\{D(x_{p(n)}, Tx_{q(n)}), D(Tx_{p(n)}, x_{q(n)})\}}\right)\right) \\ &< \phi\left(\frac{D(x_{p(n)}, Tx_{p(n)})D(x_{p(n)}, Tx_{q(n)}) + D(x_{q(n)}, Tx_{q(n)})D(x_{q(n)}, Tx_{p(n)})}{\max\{D(x_{p(n)}, Tx_{q(n)}), D(Tx_{p(n)}, x_{q(n)})\}}\right), \end{aligned} \tag{3.12}$$

for all $n > \max\{n_1, n_2\}$.

On the other hand, from (3.9) we know that

$$\begin{aligned}0 &\le \frac{D(x_{p(n)}, Tx_{p(n)})D(x_{p(n)}, Tx_{q(n)}) + D(x_{q(n)}, Tx_{q(n)})D(x_{q(n)}, Tx_{p(n)})}{\max\{D(x_{p(n)}, Tx_{q(n)}), D(Tx_{p(n)}, x_{q(n)})\}} \\ &= \frac{D(x_{p(n)}, Tx_{p(n)})D(x_{p(n)}, Tx_{q(n)})}{\max\{D(x_{p(n)}, Tx_{q(n)}), D(Tx_{p(n)}, x_{q(n)})} \\ &\quad + \frac{D(x_{q(n)}, Tx_{q(n)})D(x_{q(n)}, Tx_{p(n)})}{\max\{D(x_{p(n)}, Tx_{q(n)}), D(Tx_{p(n)}, x_{q(n)})} \\ &\le D(x_{p(n)}, Tx_{p(n)}) + D(x_{q(n)}, Tx_{q(n)}).\end{aligned}$$

Let $n \to \infty$ in the above inequality and by taking (3.6) into account, we obtain

$$\lim_{n\to\infty} \frac{D(x_{p(n)}, Tx_{p(n)})D(x_{p(n)}, Tx_{q(n)}) + D(x_{q(n)}, Tx_{q(n)})D(x_{q(n)}, Tx_{p(n)})}{\max\{D(x_{p(n)}, Tx_{q(n)}), D(Tx_{p(n)}, x_{q(n)})\}} = 0.$$

So, there exists $n_3 \in \mathbb{N}$ such that

$$\frac{D(x_{p(n)}, Tx_{p(n)})D(x_{p(n)}, Tx_{q(n)}) + D(x_{q(n)}, Tx_{q(n)})D(x_{q(n)}, Tx_{p(n)})}{\max\{D(x_{p(n)}, Tx_{q(n)}), D(Tx_{p(n)}, x_{q(n)})\}} < \frac{\varepsilon}{2},$$

for all $n > n_3$

This implies that

$$\phi\left(\frac{D(x_{p(n)}, Tx_{p(n)})D(x_{p(n)}, Tx_{q(n)}) + D(x_{q(n)}, Tx_{q(n)})D(x_{q(n)}, Tx_{p(n)})}{\max\{D(x_{p(n)}, Tx_{q(n)}), D(Tx_{p(n)}, x_{q(n)})\}}\right) \le \phi\left(\frac{\varepsilon}{2}\right), \tag{3.13}$$

for all $n > n_3$.

In combination with (3.12) and (3.13) we get

$$\phi\left(\frac{\varepsilon}{2}\right) < \phi\left(\frac{D(x_{p(n)}, Tx_{p(n)})D(x_{p(n)}, Tx_{q(n)}) + D(x_{q(n)}, Tx_{q(n)})D(x_{q(n)}, Tx_{p(n)})}{\max\{D(x_{p(n)}, Tx_{q(n)}), D(Tx_{p(n)}, x_{q(n)})\}}\right) \le \phi\left(\frac{\varepsilon}{2}\right),$$

for all $n > \max\{n_1, n_2, n_3\}$, which is a contradiction. Hence

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Therefore, we conclude that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, so there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d\bigl(x_n, x^*\bigr) = 0$$

and

$$\lim_{n\to\infty} D\bigl(x_{n+1}, Tx^*\bigr) = d\bigl(x^*, Tx^*\bigr). \tag{3.14}$$

Now, we claim that $x^* \in Tx^*$.

Arguing by contradiction, we assume that $D(x^*, Tx^*) > 0$, then there are two cases:

(a) for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$, $n_0 = 1$, $n_k > n_{k-1}$ and $x_{n_k} \in Tx^*$;

(b) there exists $m \in \mathbb{N}$, such that $D(x_n, Tx^*) > 0$ for each $n \ge m$.

From (a), we get

$$x^* = \lim_{k\to\infty} x_{n_k} \in Tx^*,$$

which is a contraction.

From (b), we get

$$\max\bigl\{D\bigl(x_n, Tx^*\bigr), D\bigl(x^*, Tx_n\bigr)\bigr\} > 0. \tag{3.15}$$

Since T is a generalized multivalued Khan-type (ψ,ϕ)-contraction, from (3.15) we obtain

$$\begin{aligned}\phi\bigl(D\bigl(x_{n+1}, Tx^*\bigr)\bigr) &\le \phi\bigl(H\bigl(Tx_n, Tx^*\bigr)\bigr) \le \psi\bigl(\phi\bigl(M\bigl(x_n,x^*\bigr)\bigr)\bigr)\\ &= \psi\left(\phi\left(\frac{D(x_n,Tx_n)D(x_n,Tx^*)+D(x^*,Tx^*)D(x^*,Tx_n)}{\max\{D(x_n,Tx^*),D(Tx_n,x^*)\}}\right)\right)\\ &< \phi\left(\frac{D(x_n,Tx_n)D(x_n,Tx^*)+D(x^*,Tx^*)D(x^*,Tx_n)}{\max\{D(x_n,Tx^*),D(Tx_n,x^*)\}}\right).\end{aligned} \tag{3.16}$$

On the other hand, in combination with (3.6), (3.14) and (3.15), we get

$$\lim_{n\to\infty}\frac{D(x_n,Tx_n)D(x_n,Tx^*)+D(x^*,Tx^*)D(x^*,Tx_n)}{\max\{D(x_n,Tx^*),D(Tx_n,x^*)\}} = 0.$$

Thus, taking $D(x^*,Tx^*) > 0$ into account, there exists $n_4 \in \mathbb{N}$, such that

$$\frac{D(x_n,Tx_n)D(x_n,Tx^*)+D(x^*,Tx^*)D(x^*,Tx_n)}{\max\{D(x_n,Tx^*),D(Tx_n,x^*)\}} < \frac{1}{2}D\bigl(x^*,Tx^*\bigr),$$

for all $n > n_4$. And this implies that

$$\phi\left(\frac{D(x_n,Tx_n)D(x_n,Tx^*)+D(x^*,Tx^*)D(x^*,Tx_n)}{\max\{D(x_n,Tx^*),D(Tx_n,x^*)\}}\right) \le \phi\left(\frac{1}{2}D\bigl(x^*,Tx^*\bigr)\right).$$

It follows from (3.16) that

$$\phi\bigl(D\bigl(x_{n+1},Tx^*\bigr)\bigr) < \phi\left(\frac{1}{2}D\bigl(x^*,Tx^*\bigr)\right),$$

for all $n > n_4$.

Since $\phi \in \Phi$, we obtain

$$D\bigl(x_{n+1},Tx^*\bigr) \le \frac{1}{2}D\bigl(x^*,Tx^*\bigr),$$

for all $n > n_4$.

Letting $n \to \infty$ in the above inequality and taking (3.14) into account, we get

$$D\bigl(x^*,Tx^*\bigr) \le \frac{1}{2}D\bigl(x^*,Tx^*\bigr),$$

which is a contraction. So we have

$$D\bigl(x^*,Tx^*\bigr) = 0,$$

this implies that

$$x^* \in Tx^*.$$

Hence x^* is a fixed point of T.

Case II Assume that there exist $i \in \mathbb{N}$ such that $D(x_{i-1}, Tx_i) = 0$.

Since $D(x_{i-1}, Tx_i) = 0$, we get

$$\max\{D(x_{i-1}, Tx_i), D(Tx_{i-1}, x_i)\} = 0.$$

By condition (ii) of Definition 2.10, it follows that

$$Tx_{i-1} = Tx_i,$$

and hence

$$x_i \in Tx_i,$$

and this implies that x_i is a fixed point of T. This complete the proof. □

Remark 3.2 Note that the continuity of T is not supposed in Theorem 3.1.

From Definition 2.10 and Theorem 3.1, we get the result of single-valued mappings as follows.

Definition 3.3 Let (X, d) be a metric space. A mapping $T : X \to X$ is called a generalized Khan-type (ψ, ϕ)-contraction if there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that, for all $x, y \in X$,

(i) if $\max\{d(x, Ty), d(y, Tx)\} \neq 0$, then $Tx \neq Ty$ and

$$\phi(d(Tx, Ty)) \leq \psi(\phi(M(x, y)),$$

where

$$M(x, y) = \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{\max\{d(x, Ty), d(y, Tx)\}};$$

(ii) if $\max\{d(x, Ty), d(y, Tx)\} = 0$ then

$$Tx = Ty.$$

Remark 3.4 Using the same methods of Remark 2.11 and Remark 2.12, we get the F-Khan-contraction (Definition 2.1, [19]) and θ-Khan-contraction (Definition 2.2), respectively.

Theorem 3.5 *Let (X, d) be a complete metric space and $T : X \to X$ be a generalized Khan-type (ψ, ϕ)-contraction, then T has a unique fixed point $x^* \in X$, and for every $x \in X$ the sequence $\{T^n x\}_{n \in \infty}$ converges to x^*.*

Proof We only need to prove the uniqueness. For this purpose, we assume that y^* is another fixed point of T in X such that $d(x^*, y^*) > 0$. Therefore

$$\max\{d(x^*, Ty^*), d(Tx^*, y^*)\} > 0.$$

So, from condition (i) of Definition 3.3, we get

$$\begin{aligned} 0 &< \phi(Tx^*, Ty^*) \\ &\leq \psi\left(\phi\left(\frac{D(x^*, Tx^*)D(x^*, Ty^*) + D(y^*, Ty^*)D(y^*, Tx^*)}{\max\{D(x^*, Ty^*), D(Tx^*, y^*)\}}\right)\right) \\ &< \phi\left(\frac{D(x^*, Tx^*)D(x^*, Ty^*) + D(y^*, Ty^*)D(y^*, Tx^*)}{\max\{D(x^*, Ty^*), D(Tx^*, y^*)\}}\right). \end{aligned}$$

But

$$\frac{D(x^*, Tx^*)D(x^*, Ty^*) + D(y^*, Ty^*)D(y^*, Tx^*)}{\max\{D(x^*, Ty^*), D(Tx^*, y^*)\}} = 0,$$

this leads to a contraction and hence $x^* = y^*$. This completes the proof. □

4 Applications

Corollary 4.1 *Let (X, d) be a complete metric space and $T : X \to K(X)$ be a mapping. If there exists $\lambda \in (0, 1)$ such that, for all $x, y \in X$,*

$$H(Tx, Ty) \leq \lambda M(x, y),$$

where

$$M(x, y) = \frac{D(x, Tx)D(x, Ty) + D(y, Ty)D(y, Tx)}{\max\{D(x, Ty), D(y, Tx)\}},$$

then there exists a fixed point x^* of T in X.

Proof Let $\psi(t) := \lambda t$ and $\phi(t) = t$, $\mathbb{R}_0^+ \to \mathbb{R}_0^+$ in Theorem 3.1. It is easy to check that $\psi \in \Psi$ and $\phi \in \Phi$. The conclusion can be obtained immediately. □

Corollary 4.2 *Let (X, d) be a complete metric space and $T : X \to K(X)$ be a generalized multivalued θ-Khan-contraction, that is, if there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that, for all $x, y \in X$,*

(i) *if* $\max\{D(x, Ty), D(y, Tx)\} \neq 0$, *then* $Tx \neq Ty$ *and*

$$\theta(H(Tx, Ty)) \leq (\theta(M(x, y)))^k,$$

where

$$M(x, y) = \frac{D(x, Tx)D(x, Ty) + D(y, Ty)D(y, Tx)}{\max\{D(x, Ty), D(y, Tx)\}};$$

(ii) *if* $\max\{D(x,Ty),D(y,Tx)\}=0$, *then*

$$Tx = Ty;$$

then there exists a fixed point x^* *of* T *in* X.

Proof By using Remark 2.12 the conclusion can be obtained immediately. □

Corollary 4.3 *Let* (X,d) *be a complete metric space and* $T:X\to K(X)$ *be a generalized multivalued* F*-Khan-contraction, that is, if there exist* $F\in\mathcal{F}$ *and* $\tau>0$ *such that, for all* $x,y\in X$,

(i) *if* $\max\{D(x,Ty),D(y,Tx)\}\neq 0$, *then* $Tx\neq Ty$ *and*

$$\tau + F\big(H(Tx,Ty)\big)\le F\big(M(x,y)\big),$$

where

$$M(x,y)=\frac{D(x,Tx)D(x,Ty)+D(y,Ty)D(y,Tx)}{\max\{D(x,Ty),D(y,Tx)\}};$$

(ii) *if* $\max\{D(x,Ty),D(y,Tx)\}=0$, *then*

$$Tx = Ty;$$

then there exists a fixed point x^* *of* T *in* X.

Proof By using Remark 2.11 the conclusion can be obtained immediately. □

Remark 4.4 Let $y\in Tx$ in condition (i) of Corollary 4.3, we have

$$M(x,y)=\frac{D(x,Tx)D(x,Ty)+D(y,Ty)D(y,Tx)}{\max\{D(x,Ty),D(y,Tx)\}}=D(x,Tx)\le d(x,y).$$

Then we get Theorem 2.9 ([4]) immediately.

Corollary 4.5 *Let* (X,d) *be a complete metric space and* $T:X\to K(X)$ *be a multivalued mapping. Suppose that, for all* $x,y\in X$,

$$H(Tx,Ty)\le\frac{M(x,y)}{1+M(x,y)}.$$

Then there exists a fixed point x^* *of* T *in* X.

Proof Let $\psi(t):=\frac{t}{1+t}$, $\phi(t):=t$, $\mathbb{R}_0^+\to\mathbb{R}_0^+$ in Theorem 3.1. It is easy to check that $\psi\in\Psi$ and $\phi\in\Phi$, and the conclusion can be obtained immediately. □

Example 4.6 Now, we present an application where Theorem 3.1 can be applied.

Let $X = [0,1]$ with the metric $d(x,y) = |x-y|$, $x,y \in X$. Obviously (X,d) is a complete metric space. Define a mapping

$$T : X \to K(X)$$

by

$$Tx = \left[0, \frac{x}{3}\right]$$

for all $x \in X$. Then

$$H(Tx, Ty) = \frac{|x-y|}{3}, \quad x,y \in X.$$

Let $\psi(t) := \frac{3}{4}t$ and $\phi(t) := t$, $t \geq 0$, then

$$\phi\bigl(H(Tx,Ty)\bigr) = H(Tx,Ty) = \frac{|x-y|}{3},$$

and

$$\begin{aligned}
&\psi\left(\phi\left(\frac{D(x,Tx)D(x,Ty)+D(y,Ty)D(y,Tx)}{\max\{D(x,Ty),D(y,Tx)\}}\right)\right) \\
&\quad = \psi\left(\frac{D(x,Tx)D(x,Ty)+D(y,Ty)D(y,Tx)}{\max\{D(x,Ty),D(y,Tx)\}}\right) \\
&\quad = \frac{3}{4}\frac{\frac{2}{3}xD(x,Ty)+\frac{2}{3}yD(y,Tx)}{\max\{D(x,Ty),D(y,Tx)\}}.
\end{aligned}$$

Since $x \neq y$, the inequalities $x \leq \frac{y}{3}$ and $y \leq \frac{x}{3}$ cannot be simultaneously true.

If $x > \frac{y}{3}$ and $y \leq \frac{x}{3}$, then $D(y,Tx) = 0$, and hence

$$\begin{aligned}
&\psi\left(\phi\left(\frac{D(x,Tx)D(x,Ty)+D(y,Ty)D(y,Tx)}{\max\{D(x,Ty),D(y,Tx)\}}\right)\right) \\
&\quad = \frac{3}{4}\frac{\frac{2}{3}xD(x,Ty)+\frac{2}{3}yD(y,Tx)}{\max\{D(x,Ty),D(y,Tx)\}} = \frac{x}{2}.
\end{aligned}$$

Note that if $x > \frac{y}{3}$ and $y \leq \frac{x}{3}$, then $\frac{|x-y|}{3} \leq \frac{x}{2}$. Hence

$$\phi\bigl(H(Tx,Ty)\bigr) \leq \psi\left(\phi\left(\frac{D(x,Tx)D(x,Ty)+D(y,Ty)D(y,Tx)}{\max\{D(x,Ty),D(y,Tx)\}}\right)\right).$$

Similarly, we see that if $y > \frac{x}{3}$ and $x \leq \frac{y}{3}$, the above inequality holds.

If $x > \frac{y}{3}$ and $y > \frac{x}{3}$, then

$$\begin{aligned}
&\psi\left(\phi\left(\frac{D(x,Tx)D(x,Ty)+D(y,Ty)D(y,Tx)}{\max\{D(x,Ty),D(y,Tx)\}}\right)\right) \\
&\quad = \frac{3}{4}\frac{\frac{2}{3}x(x-\frac{y}{3})+\frac{2}{3}y(y-\frac{x}{3})}{\max\{x-\frac{y}{3},y-\frac{x}{3}\}}
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{3}{4}\frac{\frac{2}{3}x(x-\frac{y}{3})+\frac{2}{3}y(y-\frac{x}{3})}{(x-\frac{y}{3})+(y-\frac{x}{3})} \\
&= \frac{3(x^2+y^2)-2xy}{4(x+y)}.
\end{aligned}$$

Notice that

$$\frac{|x-y|}{3} \leq \frac{3(x^2+y^2)-2xy}{4(x+y)},$$

whenever $x > y$ or $x < y$. So we get

$$\phi\big(H(Tx,Ty)\big) \leq \psi\left(\phi\left(\frac{D(x,Tx)D(x,Ty)+D(y,Ty)D(y,Tx)}{\max\{D(x,Ty),D(y,Tx)\}}\right)\right).$$

Thus, conditions in Theorem 3.1 hold. Therefore, by Theorem 3.1, it follows that there exists a fixed point of T in X. In fact, 0 is a fixed point of T.

Example 4.7 In this example, we present an application where Theorem 3.5 can be applied. This application is inspired by [7, 19].

Let $X = C[0,1]$ be the set of all real continuous functions on $[0,1]$, and d is defined by

$$d(f,g) = \|f-g\| = \max_{t\in[0,1]}\big|f(t)-g(t)\big|, \quad f,g \in X.$$

Let

$$Y = \left\{f \in X : 0 \leq f(t) \leq \frac{1}{8}, t \in [0,1] \text{ or } f(t) = 1, t \in [0,1]\right\},$$

and $G : [0,1] \times [0,1] \times Y \to X$ be defined by

$$G\big(t,s,f(r)\big) = \begin{cases} \frac{1}{2}, & 0 \leq f(r) \leq \frac{1}{8}, \\ \frac{1}{4}, & f(r) = 1, \end{cases}$$

for all $r,s,t \in [0,1]$ and $f \in Y$. Obviously (Y,d) is complete metric space, and $G(t,s,f(r))$ is integrable with respect to r on $[0,1]$.

Let T be defined on Y by

$$Tf(s) = \int_0^1 G\big(s,r,f(r)\big)\,dr,$$

for all $s \in [0,1]$. We have

$$Tf(s) = \begin{cases} \int_0^1 G(s,r,f(r)) = \int_0^1 \frac{1}{2}\,dr = \frac{1}{2}, & 0 \leq f(r) \leq \frac{1}{8}, \\ \int_0^1 G(s,r,f(r)) = \int_0^1 \frac{1}{4}\,dr = \frac{1}{4}, & f(r) = 1, \end{cases}$$

this proves that $Tf \in Y$ for all $f \in Y$.

So for all $r, s \in [0,1]$ and $f, g \in Y$, we have

$$\bigl|G\bigl(s,r,f(r)\bigr) - G\bigl(s,r,g(r)\bigr)\bigr| = \begin{cases} 0, & f(r) = g(r) = 1 \text{ or } 0 \leq f(r), g(r) \leq \frac{1}{8}, \\ \frac{1}{4}, & \text{otherwise}, \end{cases}$$

and

$$\bigl|f(r) - Tf(r)\bigr| = \begin{cases} |f(r) - \frac{1}{2}|, & 0 \leq f(r) \leq \frac{1}{8}, \\ |1 - \frac{1}{4}| = \frac{3}{4}, & f(r) = 1, \end{cases}$$

and

$$\bigl|f(r) - Tg(r)\bigr| = \begin{cases} |f(r) - \frac{1}{2}|, & 0 \leq f(r), g(r) \leq \frac{1}{8}, \\ |f(r) - \frac{1}{4}|, & 0 \leq f(r) \leq \frac{1}{8}, g(r) = 1, \\ |1 - \frac{1}{2}| = \frac{1}{2}, & f(r) = 1, 0 \leq g(r) \leq \frac{1}{8}, \\ |1 - \frac{1}{4}| = \frac{3}{4}, & f(r) = g(r) = 1. \end{cases}$$

According to symmetry the above relations are established for $|g(r) - Tg(r)|$ and $|g(r) - Tf(r)|$. Obviously $0 \leq f(r) \leq \frac{1}{8}$ implies that $\frac{3}{8} \leq |f(r) - \frac{1}{2}| \leq \frac{1}{2}$ and $\frac{3}{8} \leq |f(r) - \frac{1}{4}| \leq \frac{1}{4}$.

Therefore,

$$\frac{1}{8} \leq \max\bigl\{\|f - Tg\|, \|g - Tf\|\bigr\} \leq \frac{1}{4}.$$

Noticing that

$$\begin{aligned} &\frac{|f(r) - Tf(r)||f(r) - Tg(r)| + |g(r) - Tg(r)||g(r) - Tf(r)|}{\max\{\|f - Tg\|, \|g - Tf\|\}} \\ &\quad \geq 4\bigl[\bigl|f(r) - Tf(r)\bigr|\bigl|f(r) - Tg(r)\bigr| + \bigl|g(r) - Tg(r)\bigr|\bigl|g(r) - Tf(r)\bigr|\bigr] \\ &\quad \geq 4\left[\frac{3}{8} \cdot \frac{1}{8} + \frac{3}{8} \cdot \frac{1}{8}\right] = \frac{3}{8}, \end{aligned}$$

we get

$$\begin{aligned} &\bigl|G\bigl(s,r,f(r)\bigr) - G\bigl(s,r,g(r)\bigr)\bigr| \\ &\quad \leq \frac{2}{3} \cdot \frac{|f(r) - Tf(r)||f(r) - Tg(r)| + |g(r) - Tg(r)||g(r) - Tf(r)|}{\max\{\|f - Tg\|, \|g - Tf\|\}}. \end{aligned}$$

Now, we prove that the integral equation

$$f(s) = \int_0^1 G\bigl(s,r,f(r)\bigr)\,dr$$

has a unique solution $f^* \in Y$.

For all $f, g \in Y$ and $s \in [0,1]$, we have

$$\bigl|Tf(s) - Tg(s)\bigr| = \left|\int_0^1 G\bigl(s,r,f(r)\bigr)\,dr - \int_0^1 G\bigl(s,r,g(r)\bigr)\,dr\right|$$

$$
\begin{aligned}
&\leq \int_0^1 \left|G\big(s,r,f(r)\big) - G\big(s,r,g(r)\big)\right| dr \\
&\leq \frac{2}{3} \cdot \int_0^1 \frac{|f(r) - Tf(r)||f(r) - Tg(r)| + |g(r) - Tg(r)||g(r) - Tf(r)|}{\max\{\|f - Tg\|, \|g - Tf\|\}} dr \\
&\leq \frac{2}{3} \cdot \int_0^1 \frac{\|f - Tf\|\|f - Tg\| + \|g - Tg\|\|g - Tf\|}{\max\{\|f - Tg\|, \|g - Tf\|\}} dr \\
&= \frac{2}{3} \cdot \frac{\|f - Tf\|\|f - Tg\| + \|g - Tg\|\|g - Tf\|}{\max\{\|f - Tg\|, \|g - Tf\|\}}.
\end{aligned}
$$

So, for all $f,g \in Y$, we have

$$
\|Tf - Tg\| \leq \frac{2}{3} \cdot \frac{\|f - Tf\|\|f - Tg\| + \|g - Tg\|\|g - Tf\|}{\max\{\|f - Tg\|, \|g - Tf\|\}}.
$$

Let $\psi(t) := \frac{2}{3}t$, $\phi(t) := t$, $\mathbb{R}^+ \to \mathbb{R}^+$.

Then we get

$$
\phi\big(\|Tf - Tg\|\big) \leq \psi\left(\phi\left(\frac{\|f - Tf\|\|f - Tg\| + \|g - Tg\|\|g - Tf\|}{\max\{\|f - Tg\|, \|g - Tf\|\}}\right)\right).
$$

Consequently, all the conditions of Theorem 3.5 are satisfied. Therefore T has a fixed point which is the solution of the integral equation $f(s) = \int_0^1 G(s,r,f(r))\,dr$.

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References
1. Banach, B.: Sur les opérations dons les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. **3**, 133–181 (1992)
2. Wang, Z., Li, H.: Fixed point theorems and endpoint theorems for (α, ψ)-Meir–Keeler–Khan multivalued mapping. Fixed Point Theory Appl. **2016**, 12 (2016). https://doi.org/10.1186/s13663-016-0500-0
3. Nadler, S.B.: Multivalued contraction mappings. Pac. J. Math. **30**, 475–488 (1969)
4. Altun, I., Durmaz, G., Dag, H.: Multivalued *F*-contractions on complete metric spaces. J. Nonlinear Convex Anal. **16**(4), 659–666 (2015)
5. Ciric, L.B.: A generalization of Banach contraction principle. Proc. Am. Math. Soc. **45**, 267–273 (1974)
6. Piri, H., Kumam, P.: Some fixed point theorems concerning *F*-contraction in complete metric spaces. Fixed Point Theory Appl. **2014**, 210 (2014). https://doi.org/10.1186/1687-1812-2014-210
7. Dung, N.V., Hang, V.T.L.: A fixed point theorem for generalized *F*-contraction on complete metric space. Vietnam J. Math. **43**, 743–753 (2015)
8. Chang, S.S., Tang, Y., Wang, L., Xu, Y., Zhao, Y., Wang, G.: Convergence theorems for some multi-valued generalized nonexpansive mappings. Fixed Point Theory Appl. **2014**, 33 (2014). https://doi.org/10.1186/1687-1812-2014-33
9. Haghi, R.H., Rezapour, S.H., Shahzad, N.: Be careful on partial metric fixed point results. Topol. Appl. **160**, 450–454 (2013)
10. Samet, B., Karapinar, E., Aydi, H., Rajic, V.C.: Discussion on some coupled fixed point theorems. Fixed Point Theory Appl. **2013**, Article ID 50 (2013)
11. Aydi, H., Karapinar, E., Samet, B.: Remarks on some recent fixed point theorems. Fixed Point Theory Appl. **2012**, Article ID 76 (2012)
12. Haghi, R.H., Rezapour, S.H., Shahzad, N.: Some fixed point generalizations are not real generalizations. Nonlinear Anal. **74**, 1799–1803 (2011)

13. Khan, M.S.: A fixed point theorem for metric spaces. Riv. Mat. Univ. Parma **3**(1), 69–72 (1976)
14. Fisher, B.: On a theorem of Khan. Riv. Mat. Univ. Parma **4**, 135–137 (1978)
15. Piri, H., Rahrovi, S., Kumam, P.: Generalization of Khan fixed point theorem. J. Math. Comput. Sci. **17**, 76–83 (2017)
16. Wardowski, D.: Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl. **2012**, 94 (2012). https://doi.org/10.1186/1687-1812-2012-94
17. Jleli, M., Karapinar, E., Samet, B.: Further generalizations of the Banach contraction principle. J. Inequal. Appl. **2014**, 439 (2014). https://doi.org/10.1186/1029-242X-2014-439
18. Jleli, M., Samet, B.: A new generalization of the Banach contraction principle. J. Inequal. Appl. **2014**, 38 (2014). https://doi.org/10.1186/1029-242X-2014-38
19. Piri, H., Rahrovi, S., Marasi, H., Kumam, P.: A fixed point theorem for F-Khan-contractions on complete metric spaces and application to integral equations. J. Nonlinear Sci. Appl. **10**, 4564–4573 (2017)
20. Latif, A., Gordji, M.E., Karapinar, E., Sintunavarat, W.: Fixed point results for generalized (α,ψ)-Meir–Keeler contractive mappings and applications. J. Inequal. Appl. **2014**, Article ID 68 (2014)
21. Hussain, H., Kadelbrug, Z., Radenovic, S., Al-Solamy, F.: Comparison functions and fixed point results in partial metric spaces. Abstr. Appl. Anal. **2012**, 605781 (2012). https://doi.org/10.1155/2012/605781
22. Pansuwan, A., Sintunavarat, W., Parvaneh, V., Cho, Y.J.: Some fixed point theorems for (α,θ,k)-contractive multi-valued mappings with some applications. Fixed Point Theory Appl. **2015**, 132 (2015). https://doi.org/10.1186/s13663-015-0385-3
23. Karapinar, E., Samet, B.: Generalized α–ψ contractive type mappings and related fixed point theorems with applications. Abstr. Appl. Anal. **2012**, 793486 (2012). https://doi.org/10.1155/2012/793486