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$(h-m)$-convex functions and associated fractional Hadamard and Fejér–Hadamard inequalities via an extended generalized Mittag-Leffler function

Shin Min Kang[1,2], Ghulam Farid[3], Waqas Nazeer[4*] and Sajid Mehmood[5]

*Correspondence:
nazeer.waqas@ue.edu.pk
[4]Division of Science and Technology, University of Education, Lahore, Pakistan
Full list of author information is available at the end of the article

Abstract

The aim of this paper is to present the Hadamard and the Fejér–Hadamard integral inequalities for $(h-m)$-convex functions due to an extended generalized Mittag-Leffler function. These results contain several fractional integral inequalities for the well-known fractional integral operators. Also results for the generalized Mittag-Leffler function are mentioned.

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1 Introduction and preliminaries

Convexity is very important in the field of mathematical inequalities. It is a basic concept in mathematics, its extensions and generalizations have been defined in various ways by using the different techniques. For example one of them is $(h-m)$-convexity, which is a generalization of convexity that contains h-convexity, m-convexity, s-convexity defined on the right half of the real line including zero (see [12, 23] and the references therein).

Definition 1 Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. We say that $f : [0,b] \to \mathbb{R}$ is a $(h-m)$-convex function, if f is non-negative and, for all $x, y \in [0,b]$, $m \in [0,1]$ and $\alpha \in (0,1)$, one has

$$f\bigl(\alpha x + m(1-\alpha)y\bigr) \le h(\alpha)f(x) + mh(1-\alpha)f(y).$$

For suitable choices of h and m, the class of $(h-m)$-convex functions is reduced to different known classes of convex and related functions defined on $[0,b]$ given in the following remark.

Remark 1

(i) If $m = 1$, then we get an h-convex function.

(ii) If $h(\alpha) = \alpha$, then we get an m-convex function.

(iii) If $h(\alpha) = \alpha$ and $m = 1$, then we get a convex function.
(iv) If $h(\alpha) = 1$ and $m = 1$, then we get a p-function.
(v) If $h(\alpha) = \alpha^s$ and $m = 1$, then we get an s-convex function in the second sense.
(vi) If $h(\alpha) = \frac{1}{\alpha}$ and $m = 1$, then we get a Godunova–Levin function.
(vii) If $h(\alpha) = \frac{1}{\alpha^s}$ and $m = 1$, then we get an s-Godunova–Levin function of the second kind.

Convex functions are equivalently defined by the well-known Hadamard inequality stated as follows.

Theorem 1.1 *Let $f : [a,b] \to \mathbb{R}$ be a convex function such that $a < b$, then the following inequality holds*:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

The Fejér–Hadamard inequality is a weighted version of the Hadamard inequality established by Fejér in [8].

Theorem 1.2 *Let $f : [a,b] \to \mathbb{R}$ be a convex function and $g : [a,b] \to \mathbb{R}$ be a non-negative, integrable and symmetric to $\frac{a+b}{2}$. Then the following inequality holds*:

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{2}$$

Many researchers are continuously working on inequalities (1) and (2), and they have produced very interesting results for convex and related functions; for details see [3–7, 12, 16, 18]. In this paper, we wish to prove the Hadamard and the Fejér–Hadamard type integral inequalities for $(h-m)$-convex functions via an extended generalized Mittag-Leffler function.

In [1], M. Andrić et al. defined the extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\cdot;p)$ as follows:

Definition 2 Let $\mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$ is defined by

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n+\alpha)} \frac{t^n}{(l)_{n\delta}}, \tag{3}$$

where β_p is the generalized beta function defined by

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}\,dt$$

and $(c)_{nk}$ is the Pochhammer symbol defined by $(c)_{nk} = \frac{\Gamma(c+nk)}{\Gamma(c)}$.

Remark 2 Equation (3) is a generalization of the following functions:

(i) setting $p = 0$, it reduces to the function $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t)$ due to Salim et al. defined in [17],
(ii) setting $l = \delta = 1$, it reduces to the function $E^{\gamma,k,c}_{\mu,\alpha}(t;p)$ defined by Rahman et al. in [15],
(iii) setting $p = 0$ and $l = \delta = 1$, it reduces to the function $E^{\gamma,k}_{\mu,\alpha}(t)$ due to Shukla et al. defined in [20]; see also [21],
(iv) setting $p = 0$ and $l = \delta = k = 1$, it reduces to the Prabhakar function $E^{\gamma}_{\mu,\alpha}(t)$ defined in [14].

For more information related to the Mittag-Leffler function we suggest [2, 9]. The corresponding left- and right-sided generalized fractional integral operators $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}$ and $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}$ are defined as follows.

Definition 3 ([1]) Let $\omega,\mu,\alpha,l,\gamma,c \in \mathbb{C}$, $\Re(\mu),\Re(\alpha),\Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the generalized fractional integral operators $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f$ and $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}f$ are defined by

$$\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\big)(x;p) = \int_a^x (x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\big(\omega(x-t)^{\mu};p\big)f(t)\,dt \tag{4}$$

and

$$\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}f\big)(x;p) = \int_x^b (t-x)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\big(\omega(t-x)^{\mu};p\big)f(t)\,dt. \tag{5}$$

Remark 3 Equations (4) and (5) are the generalization of the following fractional integral operators:
(i) setting $p = 0$, it reduces to the fractional integral operators defined by Salim et al. in [17],
(ii) setting $l = \delta = 1$, it reduces to the fractional integral operators defined by Rahman et al. in [15],
(iii) setting $p = 0$ and $l = \delta = 1$, it reduces to the fractional integral operators defined by Srivastava et al. in [21],
(iv) setting $p = 0$ and $l = \delta = k = 1$, it reduces to the fractional integral operators defined by Prabhakar in [14],
(v) setting $p = \omega = 0$, it reduces to the right-sided and left-sided Riemann–Liouville fractional integrals.

Let $f \in L_1[a,b]$. Then the left- and right-sided Riemann–Liouville fractional integrals $I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b^-}f$ of order $\alpha \in \mathbb{R}$ ($\alpha > 0$) are defined by

$$I^{\alpha}_{a^+}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}}, \quad x > a,$$

and

$$I^{\alpha}_{b^-}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha}}, \quad x < b,$$

respectively. Here $\Gamma(\alpha)$ is the Euler Gamma function and $I^{0}_{a^+}f(x) = I^{0}_{b^-}f(x) = f(x)$.

Fractional integral inequalities are useful in establishing the uniqueness of solutions of fractional differential equations. A lot of work dedicated to fractional calculus reflects its importance in almost all fields of mathematics, physics, information technology and other sciences [3, 4, 6, 7, 10, 11, 13, 22]. In the upcoming section, first we prove the Hadamard inequality for $(h-m)$-convex functions via extended generalized fractional integral operators defined in (4) and (5). Then the Fejér–Hadamard inequality for these operators is obtained. Furthermore, the results for fractional integral operators associated with (4), (5) (see Remark 3), and several kinds of convexity (see Remark 1) are highlighted.

2 Main results

First we present the extended generalized fractional integral Hadamard inequality for $(h-m)$-convex functions.

Theorem 2.1 *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a function such that $f\in L_1[a,b]$ with $a<b$ and $m\in(0,1]$. If f is $(h-m)$-convex and $h\in L_1[0,1]$, then the following inequalities for extended generalized fractional integral operators, (4) and (5), hold*:

$$\begin{aligned} &f\left(\frac{bm+a}{2}\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o,a^+}1\right)(mb;p)\\ &\quad\le h\left(\frac{1}{2}\right)\left[m^{\alpha+1}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o m^\mu,b^-}f\right)\left(\frac{a}{m};p\right)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o,a^+}f\right)(mb;p)\right]\\ &\quad\le h\left(\frac{1}{2}\right)(bm-a)^\alpha\left\{m\left[mf\left(\frac{a}{m^2}\right)+f(b)\right]\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,0^+}h\right)(1;p)\right.\\ &\qquad\left.+\left[mf(b)+f(a)\right]\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,1^-}h\right)(0;p)\right\}, \end{aligned}\tag{6}$$

where $\omega^o=\frac{\omega}{(bm-a)^\mu}$.

Proof Since f is $(h-m)$-convex, we have

$$f\left(\frac{xm+y}{2}\right)\le h\left(\frac{1}{2}\right)\bigl(mf(x)+f(y)\bigr).\tag{7}$$

Putting in the above $x=(1-t)\frac{a}{m}+tb$ and $y=m(1-t)b+ta$, we get

$$f\left(\frac{bm+a}{2}\right)\le h\left(\frac{1}{2}\right)\left(mf\left((1-t)\frac{a}{m}+tb\right)+f\bigl(m(1-t)b+ta\bigr)\right).\tag{8}$$

Multiplying (8) by $t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)$ on both sides, then integrating over $[0,1]$, we have

$$\begin{aligned} &f\left(\frac{bm+a}{2}\right)\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)\,dt\\ &\quad\le h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)mf\left((1-t)\frac{a}{m}+tb\right)dt\right.\\ &\qquad\left.+\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)f\bigl(m(1-t)b+ta\bigr)\,dt\right]. \end{aligned}$$

Putting in the above $x = (1-t)\frac{a}{m} + tb$ and $y = m(1-t)b + ta$, then, by using (4) and (5), we get

$$\begin{aligned} &f\left(\frac{bm+a}{2}\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o,a^+}1\right)(mb;p) \\ &\quad\le h\left(\frac{1}{2}\right)\left[m^{\alpha+1}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o m^\mu,b^-}f\right)\left(\frac{a}{m};p\right) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o,a^+}f\right)(mb;p)\right]. \end{aligned} \tag{9}$$

Again by using $(h-m)$-convexity of f, we have

$$\begin{aligned} &mf\left((1-t)\frac{a}{m} + tb\right) + f\big(m(1-t)b + ta\big) \\ &\quad\le m^2 h(1-t)f\left(\frac{a}{m^2}\right) + mh(t)f(b) + mh(1-t)f(b) + h(t)f(a) \\ &\quad= m\left[mf\left(\frac{a}{m^2}\right) + f(b)\right]h(1-t) + \big[mf(b) + f(a)\big]h(t). \end{aligned} \tag{10}$$

Multiplying (10) by $h(\frac{1}{2})t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)$ on both sides, then integrating over $[0,1]$, we have

$$\begin{aligned} &h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)mf\left((1-t)\frac{a}{m} + tb\right)dt \right.\\ &\qquad\left. + \int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)f\big(m(1-t)b + ta\big)\,dt\right] \\ &\quad\le h\left(\frac{1}{2}\right)\left\{m\left[mf\left(\frac{a}{m^2}\right) + f(b)\right]\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)h(1-t)\,dt \right.\\ &\qquad\left. + \big[mf(b) + f(a)\big]\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)h(t)\,dt\right\}. \end{aligned}$$

By using (4) and (5), we get

$$\begin{aligned} &h\left(\frac{1}{2}\right)\left[m^{\alpha+1}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o m^\mu,b^-}f\right)\left(\frac{a}{m};p\right) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o,a^+}f\right)(mb;p)\right] \\ &\quad\le h\left(\frac{1}{2}\right)(bm-a)^\alpha\left\{m\left[mf\left(\frac{a}{m^2}\right) + f(b)\right]\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,0^+}h\right)(1;p) \right.\\ &\qquad\left. + \big[mf(b) + f(a)\big]\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,1^-}h\right)(0;p)\right\}. \end{aligned}$$

From the above inequality and (9), we get the required inequality (6). □

Several known results are special cases of the above generalized fractional Hadamard inequality comprised in the following remark.

Remark 4

(i) If we put $p = 0$ in (6), then [16, Theorem 2.1] is obtained.
(ii) If we put $h(t) = t$, $m = 1$ and $p = 0$ in (6), then [5, Theorem 2.1] is obtained.
(iii) If we put $h(t) = t$ and $p = 0$ in (6), then [6, Theorem 3] is obtained.

(iv) If we put $h(t) = t$ and $p = \omega = 0$ in (6), then [7, Theorem 2.1] is obtained.
(v) If we put $h(t) = t$, $m = 1$ and $p = \omega = 0$ in (6), then [18, Theorem 2] is obtained.
(vi) If we put $h(t) = t$, $m = 1$, $\alpha = 1$ and $p = \omega = 0$ in (6), then the Hadamard inequality is obtained.

In the following we prove an analog version of the Hadamard inequality for generalized fractional integrals.

Theorem 2.2 *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a function such that $f \in L_1[a,b]$ with $a < b$ and $m \in (0,1]$. If f is $(h-m)$-convex, then the following inequalities for extended generalized fractional integral operators, (4) and (5), hold*:

$$
\begin{aligned}
& f\left(\frac{a+bm}{2}\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o 2^\mu,(\frac{a+bm}{2})^+}1\right)(mb;p) \\
&\quad \leq h\left(\frac{1}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o 2^\mu,(\frac{a+bm}{2})^+}f\right)(mb;p) + m^{\alpha+1}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o (2m)^\mu,(\frac{a+bm}{2m})^-}f\right)\left(\frac{a}{m};p\right)\right] \\
&\quad \leq h\left(\frac{1}{2}\right)\frac{(bm-a)^\alpha}{2^\alpha}\left\{m\left[mf\left(\frac{a}{m^2}\right)+f(b)\right]\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)h\left(\frac{2-t}{2}\right)dt\right. \\
&\qquad \left. + \left[mf(b)+f(a)\right]\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)h\left(\frac{t}{2}\right)dt\right\},
\end{aligned}
\tag{11}
$$

where $\omega^o = \frac{\omega}{(bm-a)^\mu}$.

Proof Putting $x = \frac{t}{2}b + \frac{(2-t)}{2}\frac{a}{m}$ and $y = \frac{t}{2}a + m\frac{(2-t)}{2}b$ in (7), we get

$$
f\left(\frac{a+bm}{2}\right) \leq h\left(\frac{1}{2}\right)\left[mf\left(\frac{t}{2}b + \frac{(2-t)}{2}\frac{a}{m}\right) + f\left(\frac{t}{2}a + m\frac{(2-t)}{2}b\right)\right]. \tag{12}
$$

Multiplying (12) by $t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)$ on both sides, then integrating over $[0,1]$, we have

$$
\begin{aligned}
& f\left(\frac{a+bm}{2}\right)\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)\,dt \\
&\quad \leq h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)mf\left(\frac{t}{2}a + m\frac{(2-t)}{2}b\right)dt\right. \\
&\qquad \left. + \int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)f\left(\frac{t}{2}b + \frac{(2-t)}{2}\frac{a}{m}\right)dt\right].
\end{aligned}
$$

Putting $x = \frac{t}{2}b + \frac{(2-t)}{2}\frac{a}{m}$ and $y = \frac{t}{2}a + m\frac{(2-t)}{2}b$, then, by using (4) and (5), we get

$$
\begin{aligned}
& f\left(\frac{a+bm}{2}\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o 2^\mu,(\frac{a+bm}{2})^+}1\right)(mb;p) \\
&\quad \leq h\left(\frac{1}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o 2^\mu,(\frac{a+bm}{2})^+}f\right)(mb;p) + m^{\alpha+1}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o (2m)^\mu,(\frac{a+bm}{2m})^-}f\right)\left(\frac{a}{m};p\right)\right].
\end{aligned}
\tag{13}
$$

Again by using $(h-m)$-convexity of f, we have

$$\begin{aligned} & f\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)+mf\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right) \\ & \quad \le h\left(\frac{t}{2}\right)f(a)+mh\left(\frac{2-t}{2}\right)f(b)+mh\left(\frac{t}{2}\right)f(b)+m^2h\left(\frac{2-t}{2}\right)f\left(\frac{a}{m^2}\right) \\ & \quad = m\left[mf\left(\frac{a}{m^2}\right)+f(b)\right]h\left(\frac{2-t}{2}\right)+\left[mf(b)+f(a)\right]h\left(\frac{t}{2}\right). \end{aligned} \tag{14}$$

Multiplying (14) by $h(\frac{1}{2})t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)$ on both sides, then integrating over $[0,1]$, we have

$$\begin{aligned} & h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)f\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)dt \right. \\ & \qquad \left. +\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)mf\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right)dt\right] \\ & \quad \le h\left(\frac{1}{2}\right)\left\{m\left[mf\left(\frac{a}{m^2}\right)+f(b)\right]\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)h\left(\frac{2-t}{2}\right)dt \right. \\ & \qquad \left. +\left[mf(b)+f(a)\right]\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)h\left(\frac{t}{2}\right)dt\right\}. \end{aligned}$$

By using (4) and (5), we get

$$\begin{aligned} & h\left(\frac{1}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^{o}2^{\mu},(\frac{a+bm}{2})^+}f\right)(mb;p)+m^{\alpha+1}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^{o}(2m)^{\mu},(\frac{a+bm}{2m})^-}f\right)\left(\frac{a}{m};p\right)\right] \\ & \quad \le h\left(\frac{1}{2}\right)\frac{(bm-a)^{\alpha}}{2^{\alpha}}\left\{m\left[mf\left(\frac{a}{m^2}\right)+f(b)\right]\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)h\left(\frac{2-t}{2}\right)dt \right. \\ & \qquad \left. +\left[mf(b)+f(a)\right]\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)h\left(\frac{t}{2}\right)dt\right\}. \end{aligned}$$

From the above inequality and (13), we get the required inequality (11). □

Remark 5 If we put $p=0$ in (11), then [16, Theorem 2.2] is obtained.

Corollary 2.3 *If we put* $h(t)=t$, $m=1$ *and* $p=0$ *in* (11), *then the following inequality analog to the Hadamard inequality* [5, Theorem 2.1] *for convex functions via generalized fractional integrals is obtained*:

$$\begin{aligned} f\left(\frac{b+a}{2}\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^{o},a^+}1\right)(b) & \le \frac{1}{2}\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^{o},(\frac{a+b}{2})^-}f\right)(a)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^{o},(\frac{a+b}{2})^+}f\right)(b)\right] \\ & \le \frac{1}{2}\left[f(a)+f(b)\right]\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^{o},b^-}1\right)(a). \end{aligned}$$

If we put $h(t)=t$, $m=1$ and $p=\omega=0$ in (11), then we get the following result for Riemann–Liouville fractional integral.

Corollary 2.4 ([19]) *Let $f:[a,b]\to\mathbb{R}$ be a function with $0\le a<b$ and $f\in L_1[a,b]$. If f is convex on $[a,b]$, then the following inequalities for fractional integrals hold*:

$$f\left(\frac{b+a}{2}\right)\le\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I^\alpha_{(\frac{a+b}{2})^+}f(b)+I^\alpha_{(\frac{a+b}{2})^-}f(a)\right]\le\frac{1}{2}\left[f(a)+f(b)\right].$$

In the following we present a Fejér–Hadamard inequality for $(h-m)$-convex functions via generalized fractional integral operators.

Theorem 2.5 *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a function such that $f\in L_1[a,b]$ with $a<b$ and $m\in(0,1]$. Also, let $g:[a,b]\to\mathbb{R}$ be a function which is non-negative and integrable. If f is $(h-m)$-convex, $f(x)=f(a+mb-mx)$ and $h\in L_1[0,1]$, then the following inequalities for extended generalized fractional integral operators, (4) and (5), hold*:

$$\begin{aligned}
&f\left(\frac{bm+a}{2}\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o m^\mu,b^-}g\right)\left(\frac{a}{m};p\right)\\
&\quad\le h\left(\frac{1}{2}\right)(m+1)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o m^\mu,b^-}fg\right)\left(\frac{a}{m};p\right)\\
&\quad\le h\left(\frac{1}{2}\right)\frac{(bm-a)^\alpha}{m^\alpha}\left\{m\left[mf\left(\frac{a}{m^2}\right)+f(b)\right]\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,0^+}h\right)(1;p)\right.\\
&\qquad\left.+\left[mf(b)+f(a)\right]\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,1^-}h\right)(0;p)\right\},
\end{aligned}\tag{15}$$

where $\omega^o=\frac{\omega}{(bm-a)^\mu}$.

Proof Multiplying (8) by $t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)g(tb+(1-t)\frac{a}{m})$ on both sides, then integrating over $[0,1]$, we have

$$\begin{aligned}
&f\left(\frac{bm+a}{2}\right)\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)g\left(tb+(1-t)\frac{a}{m}\right)dt\\
&\quad\le h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)g\left(tb+(1-t)\frac{a}{m}\right)mf\left((1-t)\frac{a}{m}+tb\right)dt\right.\\
&\qquad\left.+\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)g\left(tb+(1-t)\frac{a}{m}\right)f\left(m(1-t)b+ta\right)dt\right].
\end{aligned}$$

Putting $x=(1-t)\frac{a}{m}+tb$ in the above and then, by using the condition $f(x)=f(a+mb-mx)$, we get

$$f\left(\frac{bm+a}{2}\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o m^\mu,b^-}g\right)\left(\frac{a}{m};p\right)\le h\left(\frac{1}{2}\right)(m+1)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega^o m^\mu,b^-}fg\right)\left(\frac{a}{m};p\right).\tag{16}$$

Now multiplying (10) by $h(\frac{1}{2})t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)g(tb+(1-t)\frac{a}{m})$ on both sides, then integrating over $[0,1]$, we have

$$\begin{aligned}
&h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)g\left(tb+(1-t)\frac{a}{m}\right)mf\left((1-t)\frac{a}{m}+tb\right)dt\right.\\
&\qquad\left.+\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)g\left(tb+(1-t)\frac{a}{m}\right)f\left(m(1-t)b+ta\right)dt\right]
\end{aligned}$$

$$\leq h\left(\frac{1}{2}\right)\left\{\left[m^2 f\left(\frac{a}{m^2}\right) + mf(b)\right]\int_0^1 t^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(\omega t^{\mu};p\right)h(1-t)\,dt\right.$$
$$\left.+\left[mf(b)+f(a)\right]\int_0^1 t^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(\omega t^{\mu};p\right)h(t)\,dt\right\}.$$

By using (4) and (5), we get

$$h\left(\frac{1}{2}\right)(m+1)\left(\epsilon_{\mu,\alpha,l,\omega^{o}m^{\mu},b^{-}}^{\gamma,\delta,k,c}fg\right)\left(\frac{a}{m};p\right)$$
$$\leq h\left(\frac{1}{2}\right)\frac{(bm-a)^{\alpha}}{m^{\alpha}}\left\{m\left[mf\left(\frac{a}{m^2}\right)+f(b)\right]\left(\epsilon_{\mu,\alpha,l,\omega,0^{+}}^{\gamma,\delta,k,c}h\right)(1;p)\right.$$
$$\left.+\left[mf(b)+f(a)\right]\left(\epsilon_{\mu,\alpha,l,\omega,1^{-}}^{\gamma,\delta,k,c}h\right)(0;p)\right\}.$$

From the above inequality and (16), we get the required inequality (15). □

Remark 6
(i) If we put $p = 0$ in (15), then [16, Theorem 2.5] is obtained.
(ii) If we put $h(t) = t$, $m = 1$ and $p = 0$ in (15), then [5, Theorem 2.2] is obtained.

3 Concluding remarks

The aim of this paper is to extend the generalized fractional integral inequalities via $(h-m)$-convexity. It is worthy of note that the presented results in particular contain a number of fractional integral inequalities for h-convex, m-convex, s-convex, convex and related functions (see Remark 1 and Remark 3). The Fejér–Hadamard inequality summarizes all the discussed results in a very nice compact form. We hope this work will attract researchers working in mathematical analysis, fractional calculus and other related fields.

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Author details
[1]Department of Mathematics and Research Institute of Natural Science, Gyeongsang National University, Jinju, Korea. [2]Center for General Education, China Medical University, Taichung, Taiwan. [3]Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan. [4]Division of Science and Technology, University of Education, Lahore, Pakistan. [5]Govt Boys Primary School Sherani, Attock, Pakistan.

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