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General viscosity approximation methods for quasi-nonexpansive mappings with applications

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Abstract

The purpose of this paper is to introduce and study the general viscosity approximation methods for quasi-nonexpansive mappings in the setting of infinite-dimensional Hilbert spaces. Under suitable conditions, we prove that the sequences generated by the proposed new algorithm converge strongly to a fixed point of quasi-nonexpansive mappings in Hilbert spaces, which is also the unique solution of some variational inequality. Then this result is used to study the split equality fixed point problems, the split equality common fixed point problems, the split equality null point problems, etc. Our results improve and generalize many results in the literature and they should have many applications in nonlinear science.

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1 Introduction

Let C be a nonempty subset of a real Hilbert space H. A mapping $T: C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of the fixed points of T is denoted by $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$. $T: C \to C$ is called quasi-nonexpansive if $\mathrm{Fix}(T) \neq \emptyset$, and $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$, $p \in \mathrm{Fix}(T)$.

The viscosity approximation method for nonlinear mappings was first introduced by Moudafi [1]. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by

$$x_{n+1} = \frac{\epsilon_n}{1+\epsilon_n} f(x_n) + \frac{1}{1+\epsilon_n} Tx_n, \quad \forall n \geq 0, \tag{1.1}$$

where f is a contraction with a coefficient $\theta \in [0,1)$ on H, i.e., $\|f(x) - f(y)\| \leq \theta\|x - y\|$ for all $x, y \in H$, and $\{\epsilon_n\}_{n\in\mathbb{N}}$ is a sequence in $(0,1)$ satisfying the following given conditions:

- $\lim_{n\to\infty} \epsilon_n = 0$;
- $\sum_{n=0}^{\infty} \epsilon_n = \infty$;
- $\lim_{n\to\infty}(\frac{1}{\epsilon_n} - \frac{1}{\epsilon_{n+1}}) = 0$.

It is proved that the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by (1.1) converges strongly to the unique solution $p \in \mathrm{Fix}(T)$ of the variational inequality

$$\bigl\langle (I-f)p, x-p \bigr\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{1.2}$$

In [2], Maingé considered the following viscosity approximation method for quasi-nonexpansive mappings. Starting with an arbitrary initial $x_0 \in C \subset H$, $\{x_n\}_{n\in\mathbb{N}}$ generated by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_\omega x_n, \quad \forall n \geq 0, \tag{1.3}$$

where $T_\omega = \omega I + (1-\omega)T$, with T quasi-nonexpansive on $C \subset H$, $\mathrm{Fix}(T) \neq \emptyset$ and $\omega \in (0,1)$, $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence in $(0,1)$ satisfying the following given conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

It is also proved that the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by (1.3) converges strongly to the unique solution of the variational inequality (1.2).

In [3], Tian and Jin considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T_\omega x_n, \quad \forall n \geq 0. \tag{1.4}$$

It is proved that if the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies appropriate conditions, the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by (1.4) converges strongly to the unique solution of the variational inequality

$$\bigl\langle (A-\gamma f)p, x-p \bigr\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T), \tag{1.5}$$

or equivalently $p = P_{\mathrm{Fix}(T)}(I - A + \gamma f)p$, where $\mathrm{Fix}(T)$ is the fixed point set of a quasi-nonexpansive mapping T, and A is a strongly positive linear bounded operator.

In [4], Marino et al. considered the following general viscosity explicit midpoint rule:

$$\begin{aligned} \bar{x}_{n+1} &= \beta_n x_n + (1-\beta_n)Tx_n; \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)T\bigl(s_n x_n + (1-s_n)\bar{x}_{n+1}\bigr), \quad \forall n \geq 0. \end{aligned} \tag{1.6}$$

It is proved that if the sequences $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, and $\{s_n\}_{n\in\mathbb{N}}$ satisfy appropriate conditions, the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by (1.6) converges strongly to the unique solution of the variational inequality (1.2), where $\mathrm{Fix}(T)$ is the fixed point set of a quasi-nonexpansive mapping T.

Let H_i, $i = 1,2,3$, be a real Hilbert space, and $T : H_1 \to H_1$ with fixed point set $\mathrm{Fix}(T)$. Let C_1 and C_2 be nonempty closed convex subsets of H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator.

The split feasibility problem (SFP) is the problem of finding

$$x \in H_1 \text{ such that } x \in C_1 \text{ and } Ax \in C_2.$$

In 1994, Censor and Elfving [5] first introduced the (SFP) in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction.

Let $A_1 : H_1 \to H_3$, $A_2 : H_2 \to H_3$ be bounded linear operators. Moudafi [6] introduced the following split equality feasibility problem (SEFP):

$$\text{Find } x_1 \in C_1, x_2 \in C_2 \text{ such that } A_1x_1 = A_2x_2.$$

Obviously, if $A_2 = I$ (identity mapping on H_2) and $H_2 = H_3$, then (SEFP) is reduced to (SFP). Moudafi [6] introduced an iteration process to establish a weak convergence theorem for split equality feasibility problem under suitable assumptions. The (SEFP) has many applications such as decomposition methods for PDEs and applications in game theory and intensity-modulated radiation therapy.

Let $T_1 : H_1 \to H_1$ and $T_2 : H_2 \to H_2$ be firmly quasi-nonexpansive mappings such that $\mathrm{Fix}(T_1) \neq \emptyset$, $\mathrm{Fix}(T_2) \neq \emptyset$, and let $A_1 : H_1 \to H_3$, $A_2 : H_2 \to H_3$ be bounded linear operators. Moudafi [7] introduced an iteration process and established weak convergence theorem for split equality fixed point problem (SEFPP):

$$\text{Find } x_1 \in \mathrm{Fix}(T_1), x_2 \in \mathrm{Fix}(T_2) \text{ such that } A_1x_1 = A_2x_2.$$

When $A_2 = I$ and $H_2 = H_3$, then the (SEFPP) is reduced to the split common fixed point problem (SCFPP):

$$\text{Find } x_1 \in H \text{ such that } x_1 \in \mathrm{Fix}(T_1) \text{ and } A_1x_1 \in \mathrm{Fix}(T_2).$$

When $C_1 \subset H_1$, $C_1 \neq \emptyset$, $C_2 \subset H_2$, $C_2 \neq \emptyset$, let $T_1 = P_{C_1}$ and $T_2 = P_{C_2}$, the (SEFPP) is then reduced to the (SEFP), where P_{C_1} and P_{C_2} denote the metric projection of C_1 and C_2, respectively.

Very recently there have been many works concerning fixed point methods for nonexpansive mappings. For more details, see, e.g., [8–11] and the references therein.

Motivated and inspired by the above works on viscosity approximation method for quasi-nonexpansive mappings and split equality problems, in this paper we first introduce a general viscosity approximation method for quasi-nonexpansive mappings. Under suitable conditions, we prove that the sequences generated by the proposed new algorithm converge strongly to a fixed point of quasi-nonexpansive mappings in Hilbert spaces, which is also the unique solution of some variational inequality. Then this result is used to study the split equality fixed point problems, the split equality common fixed point problems, the split equality null point problems, etc. Our results improve and generalize many results in the literature and they should have many applications in nonlinear science.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $I : H \to H$ be an identity mapping on H. We denote the strong convergence and the weak convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Throughout this paper, we use these notations and assumptions unless specified otherwise.

The following identities are valid in a Hilbert space H: for each $x, y \in H$, $t \in [0, 1]$,

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;
(ii) $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$.

Let C be a nonempty subset of the real Hilbert space H, and let $T: C \to H$ be a single-valued mapping. Then T is called

(1) nonexpansive if $\|Tx - Ty\| \leq \|x-y\|$ for all $x, y \in C$;
(2) L-Lipschitz continuous if there exists $L > 0$ such that $\|Tx - Ty\| \leq L\|x-y\|$ for all $x, y \in C$;
(3) quasi-nonexpansive if $\mathrm{Fix}(T) \neq \emptyset$, and

$$\|Tx - p\| \leq \|x-p\| \quad \text{for all } x \in C \text{ and for all } p \in \mathrm{Fix}(T);$$

(4) ρ-strongly quasi-nonexpansive, where $\rho \geq 0$, if $\mathrm{Fix}(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \leq \|x-p\|^2 - \rho\|Tx - x\|$$

for all $x \in C$ and for all $p \in \mathrm{Fix}(T)$;
(5) strictly quasi-nonexpansive if $\mathrm{Fix}(T) \neq \emptyset$ and $\|Tx - p\| < \|x-p\|$ for all $p \in \mathrm{Fix}(T)$ and for all $x \in C \setminus \mathrm{Fix}(T)$;
(6) monotone if $\langle x-y, Tx-Ty\rangle \geq 0$ for all $x, y \in C$;
(7) η-strongly monotone if there exists $\eta > 0$ such that $\langle x-y, Tx-Ty\rangle \geq \eta\|x-y\|^2$ for all $x, y \in C$;
(8) α-inverse-strongly monotone (in short α-ism) if there exists $\alpha > 0$ such that $\langle x-y, Tx-Ty\rangle \geq \eta\|Tx-Ty\|^2$ for all $x, y \in C$;
(9) demiclosed if for each sequence $\{x_n\}_{n\in\mathbb{N}}$ and $x \in C$ with $x_n \rightharpoonup x$ and $(I-T)x_n \to 0$ implies $(I-T)x = 0$;
(10) firmly nonexpansive if $\|Tx-Ty\|^2 \leq \langle Tx-Ty, x-y\rangle$ for all $x, y \in C$;
(11) α-averaged if there exist $\alpha \in (0,1)$ and nonexpansive $S: C \to H$ such that $T = (1-\alpha)I + \alpha S$.

It is easy to see that every strongly quasi-nonexpansive mapping is a strictly quasi-nonexpansive mapping and every strictly quasi-nonexpansive mapping is a quasi-nonexpansive mapping.

Lemma 2.1 *Let H be a Hilbert space and let $T_\rho: H \to H$ with $\mathrm{Fix}(T_\rho) \neq \emptyset$. Then T_ρ is a ρ-strongly quasi-nonexpansive mapping with $\rho \geq 0$ if and only if there exists a quasi-nonexpansive mapping T such that $T_\rho = (1-\frac{1}{1+\rho})I + \frac{1}{1+\rho}T$ and $\mathrm{Fix}(T_\rho) = \mathrm{Fix}(T)$.*

Proof Since the statement $\mathrm{Fix}(T_\rho) = \mathrm{Fix}(T)$ is evident, we only prove that T_ρ is a ρ-strongly quasi-nonexpansive mapping if and only if T is a quasi-nonexpansive mapping.

(Necessity) For all $x \in H$ and for all $p \in \mathrm{Fix}(T)$,

$$\begin{aligned}\|T_\rho x - p\|^2 &= \left\|\left(1-\frac{1}{1+\rho}\right)(x-p) + \frac{1}{1+\rho}(Tx-p)\right\|^2 \\ &= \left(1-\frac{1}{1+\rho}\right)\|x-p\|^2 + \frac{1}{1+\rho}\|Tx-p\|^2 \\ &\quad - \left(1-\frac{1}{1+\rho}\right)\left(\frac{1}{1+\rho}\right)\|Tx-x\|^2\end{aligned}$$

$$\begin{aligned} &= \left(1 - \frac{1}{1+\rho}\right)\|x-p\|^2 + \frac{1}{1+\rho}\|Tx-p\|^2 \\ &\quad - \rho\left\|\left(\left(1-\frac{1}{1+\rho}\right)x + \frac{1}{1+\rho}Tx\right) - x\right\|^2 \\ &= \left(1 - \frac{1}{1+\rho}\right)\|x-p\|^2 + \frac{1}{1+\rho}\|Tx-p\|^2 - \rho\|T_\rho x - x\|^2. \end{aligned} \tag{2.1}$$

Since T_ρ is ρ-strongly quasi-nonexpansive, we have

$$\|T_\rho x - p\|^2 \le \|x-p\|^2 - \rho\|T_\rho x - x\|^2. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\|Tx-p\|^2 \le \|x-p\|^2,$$

i.e., T is a quasi-nonexpansive mapping.

(Sufficiency) Since

$$\|Tx-p\| \le \|x-p\|,$$

then

$$\begin{aligned} \|T_\rho x - p\|^2 &= \left\|\left(1-\frac{1}{1+\rho}\right)(x-p) + \frac{1}{1+\rho}(Tx-p)\right\|^2 \\ &= \left(1 - \frac{1}{1+\rho}\right)\|x-p\|^2 + \frac{1}{1+\rho}\|Tx-p\|^2 \\ &\quad - \left(1-\frac{1}{1+\rho}\right)\left(\frac{1}{1+\rho}\right)\|Tx-x\|^2 \\ &\le \left(1 - \frac{1}{1+\rho}\right)\|x-p\|^2 + \frac{1}{1+\rho}\|x-p\|^2 \\ &\quad - \rho\left\|\left(\frac{1}{1+\rho}Tx + \left(1-\frac{1}{1+\rho}\right)x\right) - x\right\|^2 \\ &= \|x-p\|^2 - \rho\|T_\rho x - x\|^2, \end{aligned}$$

i.e., T_ρ is a ρ-strongly quasi-nonexpansive mapping. □

Lemma 2.2 *Let $F : H \to H$ be L-Lipschitz continuous and η-strongly monotone with $L > 0$ and $\eta > 0$. Then, for all $\mu \in (0, \frac{2\eta}{L^2})$ and $\beta \in (0,1)$, $I - \beta\mu F$ is $(1-\beta\tau)$-contraction, where $\tau = \mu(\eta - \frac{1}{2}\mu L^2)$.*

Proof By assumption, it is easy to see that $\eta \le L$. Since $\mu \in (0, \frac{2\eta}{L^2})$, $\tau = \mu(\eta - \frac{1}{2}\mu L^2) > 0$, we have $1 - \beta\tau < 1$. On the other hand, $\tau = \mu(\eta - \frac{1}{2}\mu L^2) = -\frac{1}{2}L^2(\mu - \frac{\eta}{L^2})^2 + \frac{\eta^2}{2L^2} \le \frac{\eta^2}{2L^2} \le \frac{1}{2}$. Hence $1 - \beta\tau > 0$.

For all $x, y \in H$,

$$\begin{aligned} \bigl\|(I-\beta\mu F)x - (I-\beta\mu F)y\bigr\|^2 &= \bigl\|(x-y) - \beta\mu(Fx-Fy)\bigr\|^2 \\ &= \|x-y\|^2 + \beta^2\mu^2\|Fx-Fy\|^2 - 2\beta\mu\langle x-y, Fx-Fy\rangle \end{aligned}$$

$$\begin{aligned}
&\leq \|x-y\|^2 + \beta^2\mu^2L^2\|x-y\|^2 - 2\beta\mu\eta\|x-y\|^2 \\
&= \bigl[\bigl(1-2\beta\tau+\beta^2\tau^2\bigr) - \beta^2\tau^2 - \beta\mu^2L^2 + \beta^2\mu^2L^2\bigr]\|x-y\|^2 \\
&\leq (1-\beta\tau)^2\|x-y\|^2,
\end{aligned}$$

i.e.,

$$\bigl\|(I-\beta\mu F)x-(I-\beta\mu F)y\bigr\| \leq (1-\beta\tau)\|x-y\|. \qquad \square$$

Lemma 2.3 ([12]) *Let C be a nonempty subset of H, and let $T_1, T_2 : C \to C$ be quasi-nonexpansive operators. Suppose that either T_1 or T_2 is strictly quasi-nonexpansive, and* $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \neq \emptyset$. *Then the following hold*:

(i) $\mathrm{Fix}(T_1T_2) = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$;

(ii) *T_1T_2 is quasi-nonexpansive*;

(iii) *When both T_1 and T_2 are strictly quasi-nonexpansive, T_1T_2 is strictly quasi-nonexpansive.*

Lemma 2.4 ([12]) *Let C be a nonempty subset of a Hilbert space H, and let $T : C \to H$ be nonexpansive operators, and $\alpha \in (0,1)$. Then the following are equivalent*:

(i) *T is averaged*;

(ii) $\|Tx-Ty\|^2 \leq \|x-y\|^2 - \frac{1-\alpha}{\alpha}\|(I-T)x-(I-T)y\|^2$, $\forall x, y \in C$.

Lemma 2.5 *Let C be a nonempty subset of H, and let $T_1 : C \to C$ be a quasi-nonexpansive mapping and $T_2 : C \to C$ be a firmly nonexpansive mapping such that* $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \neq \emptyset$. *Then T_1T_2 is quasi-nonexpansive and* $\mathrm{Fix}(T_1T_2) = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$.

Proof Since each firmly nonexpansive mapping is nonexpansive and $\frac{1}{2}$-averaged, by Lemma 2.4, for all $p \in \mathrm{Fix}(T_2)$ and for all $x \in C \setminus \mathrm{Fix}(T_2)$, we have

$$\|T_2x-p\|^2 \leq \|x-p\|^2 - \|x-T_2x\|^2 < \|x-p\|^2,$$

i.e., T_2 is strictly quasi-nonexpansive. Then Lemma 2.5 follows from Lemma 2.3. $\square$

Recall that the metric projection P_K from a Hilbert space H to a closed convex subset K of H is defined as follows: for each $x \in H$, there exists a unique element $P_Kx \in K$ such that

$$\|x-P_Kx\| = \inf\bigl\{\|x-y\| : y \in K\bigr\}.$$

Lemma 2.6 ([13]) *Let K be a closed convex subset of H. Given $x \in H$ and $z \in K$. Then $z = P_Kx$ if and only if there holds the inequality*

$$\langle z-x, y-z\rangle \geq 0, \quad \forall y \in K.$$

Lemma 2.7 ([14]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive self-mapping on C. Then $I-T$ is demiclosed, i.e., for each sequence $\{x_n\}_{n\in\mathbb{N}}$ and $x \in C$ with $x_n \rightharpoonup x$ and $(I-T)x_n \to 0$ implies $(I-T)x = 0$.*

Lemma 2.8 ([15]) *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0,$$

with
- $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$, $\sum_{n=0}^{\infty}\alpha_n = \infty$;
- $\limsup_{n\to\infty}\sigma_n \leq 0$;
- $\gamma_n \geq 0$, $\sum_{n=0}^{\infty}\gamma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9 ([16]) *Let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_k}\}_{k\in\mathbb{N}}$ of $\{\Gamma_n\}_{n\in\mathbb{N}}$ which satisfies $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also consider the sequence of integers $\{\delta(n)\}_{n\in\mathbb{N}}$ defined by*

$$\delta(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\delta(n)\}_{n\in\mathbb{N}}$ is a nondecreasing sequence verifying $\lim_{n\to\infty}\delta(n) = \infty$, *and for all $n \geq n_0$, it holds that $\Gamma_{\delta(n)} < \Gamma_{\delta(n)+1}$ and we have*

$$\Gamma_n < \Gamma_{\delta(n)+1}.$$

3 General viscosity approximation methods for quasi-nonexpansive mappings

Let $f : H \to H$ be θ-contractive with $\theta \in (0,1)$, $F : H \to H$ be L-Lipschitz continuous and η-strongly monotone with $L > 0$, and $\eta > 0$. Choose $\mu \in (0, \frac{2\eta}{L^2})$ and $\theta \in (0,\tau)$, where $\tau = \mu(\eta - \frac{1}{2}\mu L^2)$. Throughout this section, we use these notations and assumptions unless specified otherwise.

Theorem 3.1 *Let $T : H \to H$ be a quasi-nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$, *with $I - T$ demiclosed at* 0. *For any given $x_0 \in H$, the iterative sequence $\{x_n\}_{n\in\mathbb{N}} \subset H$ is generated by*

$$\begin{aligned} v_n &= \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} &= \alpha_n f(x_n) + (I - \alpha_n\mu F)v_n, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ are two sequences in $(0,1)$ *satisfying the following conditions*:

(i) $\lim_{n\to\infty}\alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(iii) $\limsup_{n\to\infty}\beta_n(1-\beta_n) > 0$.

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by (3.1) *converges strongly to $p \in$* $\mathrm{Fix}(T)$, *which is the unique solution in* $\mathrm{Fix}(T)$ *of the variational inequality* (*VI*)

$$\langle \mu F(p) - f(p), x - p\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{3.2}$$

Proof Clearly, $\mathrm{Fix}(T)$ is closed and convex.

Step 1. There exists unique $p \in \mathrm{Fix}(T)$ such that

$$\langle \mu F(p) - f(p), x - p\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

Take $\lambda = \frac{\mu\eta-\theta}{(\mu L+\theta)^2}$, for $\forall x, y \in H$, we have

$$\begin{aligned}
&\big\| \big(I-\lambda(\mu F-f)\big)x - \big(I-\lambda(\mu F-f)\big)y\big\|^2 \\
&\quad = \big\|(x-y) - \lambda\big((\mu F-f)x-(\mu F-f)y\big)\big\|^2 \\
&\quad = \|x-y\|^2 + \lambda^2\big\|(\mu F-f)x-(\mu F-f)y\big\|^2 \\
&\qquad - 2\lambda\big\langle x-y, (\mu F-f)x-(\mu F-f)y\big\rangle \\
&\quad = \|x-y\|^2 + \lambda^2\big\|\mu\big(F(x)-F(y)\big) - \big(f(x)-f(y)\big)\big\|^2 \\
&\qquad - 2\lambda\mu\big\langle x-y, F(x)-F(y)\big\rangle + 2\lambda\big\langle x-y, f(x)-f(y)\big\rangle \\
&\quad \le \|x-y\|^2 + \lambda^2\big(\mu\big\|F(x)-F(y)\big\| + \big\|f(x)-f(y)\big\|\big)^2 \\
&\qquad - 2\lambda\mu\eta\|x-y\|^2 + 2\lambda\|x-y\|\cdot\big\|f(x)-f(y)\big\| \\
&\quad \le \big[1+\lambda^2(\mu L+\theta)^2 - 2\lambda(\mu\eta-\theta)\big]\|x-y\|^2 \\
&\quad = \big[1-\lambda(\mu\eta-\theta)\big]\|x-y\|^2,
\end{aligned}$$

i.e., $I-\lambda(\mu F-f)$ is a contractive mapping. So is $P_{\mathrm{Fix}(T)}(I-\lambda(\mu F-f))$. Hence, there exists unique $p \in \mathrm{Fix}(T)$ such that $p = P_{\mathrm{Fix}(T)}(I-\lambda(\mu F-f))p$, then

$$\big\langle \mu F(p) - f(p), x-p\big\rangle \ge 0, \quad \forall x \in \mathrm{Fix}(T).$$

Step 2. $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence.

Let $p \in \mathrm{Fix}(T)$ be the unique solution of the variational inequality (3.2). Then $Tp = p$ and

$$\|Tp-p\| \le \|x-p\|. \tag{3.3}$$

From Lemma 2.2 and (3.3), it follows that

$$\begin{aligned}
\|x_{n+1}-p\| &= \big\|\alpha_n f(x_n) + (I-\alpha_n\mu F)v_n - p\big\| \\
&= \big\|\alpha_n\big(f(x_n)-f(p)\big) + (I-\alpha_n\mu F)v_n - (I-\alpha_n\mu F)p + \alpha_n\big(f(p)-\mu F(p)\big)\big\| \\
&\le \alpha_n\theta\|x_n-p\| + (1-\alpha_n\tau)\|v_n-p\| + \alpha_n\big\|f(p)-\mu F(p)\big\| \\
&= \alpha_n\theta\|x_n-p\| + (1-\alpha_n\tau)\big\|\beta_n x_n + (1-\beta_n)Tx_n - p\big\| + \alpha_n\big\|f(p)-\mu F(p)\big\| \\
&\le \alpha_n\theta\|x_n-p\| + (1-\alpha_n\tau)\big(\beta_n\|x_n-p\| + (1-\beta_n)\|Tx_n-p\|\big) \\
&\quad + \alpha_n\big\|f(p)-\mu F(p)\big\| \\
&\le \alpha_n\theta\|x_n-p\| + (1-\alpha_n\tau)\big(\beta_n\|x_n-p\| + (1-\beta_n)\|x_n-p\|\big) \\
&\quad + \alpha_n\big\|f(p)-\mu F(p)\big\| \\
&= \big(1-\alpha_n(\tau-\theta)\big)\|x_n-p\| + \alpha_n(\tau-\theta)\frac{\|f(p)-\mu F(p)\|}{\tau-\theta} \\
&\le \max\left\{\|x_n-p\|, \frac{\|f(p)-\mu F(p)\|}{\tau-\theta}\right\}.
\end{aligned}$$

Then, by induction on n, $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence. So are $\{v_n\}_{n\in\mathbb{N}}$, $\{f(x_n)\}_{n\in\mathbb{N}}$, and $\{F(x_n)\}_{n\in\mathbb{N}}$.

Step 3. $\lim_{n\to\infty}\|x_n - p\| = 0$.

From the well-known inequality

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle,$$

which holds for all $x, y \in H$, it follows that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \bigl\|\alpha_n f(x_n) + (I-\alpha_n\mu F)v_n - p\bigr\|^2 \\
&= \bigl\|\alpha_n\bigl(f(x_n) - \mu F(p)\bigr) + (I-\alpha_n\mu F)v_n - (I-\alpha_n\mu F)p\bigr\|^2 \\
&\le \bigl\|(I-\alpha_n\mu F)v_n - (I-\alpha_n\mu F)p\bigr\|^2 + 2\alpha_n\bigl\langle f(x_n) - \mu F(p), x_{n+1}-p\bigr\rangle \\
&\le (1-\alpha_n\tau)^2\|v_n - p\|^2 + 2\alpha_n\bigl\langle f(x_n) - f(p), x_{n+1}-p\bigr\rangle \\
&\quad + 2\alpha_n\bigl\langle f(p) - \mu F(p), x_{n+1}-p\bigr\rangle \\
&\le (1-\alpha_n\tau)^2\bigl\|\beta_n(x_n-p) + (1-\beta_n)(Tx_n - p)\bigr\|^2 \\
&\quad + 2\alpha_n\bigl\|f(x_n) - f(p)\bigr\| \cdot \|x_{n+1}-p\| + 2\alpha_n\bigl\langle f(p) - \mu F(p), x_{n+1}-p\bigr\rangle \\
&= (1-\alpha_n\tau)^2\bigl(\beta_n\|x_n-p\|^2 + (1-\beta_n)\|Tx_n-p\|^2 - \beta_n(1-\beta_n)\|Tx_n - x_n\|^2\bigr) \\
&\quad + 2\alpha_n\bigl\|f(x_n) - f(p)\bigr\| \cdot \|x_{n+1}-p\| + 2\alpha_n\bigl\langle f(p) - \mu F(p), x_{n+1}-p\bigr\rangle \\
&\le (1-\alpha_n\tau)^2\bigl(\|x_n-p\|^2 - \beta_n(1-\beta_n)\|Tx_n - x_n\|^2\bigr) \\
&\quad + \alpha_n\theta\bigl(\|x_n-p\|^2 + \|x_{n+1}-p\|^2\bigr) + 2\alpha_n\bigl\langle f(p) - \mu F(p), x_{n+1}-p\bigr\rangle \\
&= (1-2\alpha_n\tau + \alpha_n\theta)\|x_n-p\|^2 + \alpha_n^2\tau^2\|x_n-p\|^2 + \alpha_n\theta\|x_{n+1}-p\|^2 \\
&\quad - (1-\alpha_n\tau)^2\beta_n(1-\beta_n)\|Tx_n - x_n\|^2 + 2\alpha_n\bigl\langle f(p) - \mu F(p), x_{n+1}-p\bigr\rangle.
\end{aligned}$$

Summarizing, we get that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \left(1 - \frac{2\alpha_n(\tau-\theta)}{1-\alpha_n\theta}\right)\|x_n-p\|^2 + \frac{\alpha_n^2\tau^2}{1-\alpha_n\theta}\|x_n-p\|^2 \\
&\quad - \frac{(1-\alpha_n\tau)^2\beta_n(1-\beta_n)}{1-\alpha_n\theta}\|Tx_n - x_n\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\theta}\bigl\langle f(p) - \mu F(p), x_{n+1}-p\bigr\rangle.
\end{aligned} \tag{3.4}$$

Set $\Gamma_n = \|x_n - p\|^2$, neglecting the negative term $-\frac{(1-\alpha_n\tau)^2\beta_n(1-\beta_n)}{1-\alpha_n\theta}\|Tx_n - x_n\|^2$ in the obtained relation, we get the following inequality:

$$\begin{aligned}
\Gamma_{n+1} &\le \left(1 - \frac{2\alpha_n(\tau-\theta)}{1-\alpha_n\theta}\right)\Gamma_n + \frac{\alpha_n^2\tau^2}{1-\alpha_n\theta}\Gamma_n \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\theta}\bigl\langle f(p) - \mu F(p), x_{n+1}-p\bigr\rangle \\
&= \left(1 - \frac{2\alpha_n(\tau-\theta)}{1-\alpha_n\theta}\right)\Gamma_n + \frac{2\alpha_n(\tau-\theta)}{1-\alpha_n\theta}\frac{\frac{1}{2}\alpha_n\tau^2\Gamma_n}{\tau-\theta} \\
&\quad + \frac{2\alpha_n(\tau-\theta)}{1-\alpha_n\theta}\left[\frac{1}{\tau-\theta}\bigl\langle f(p) - \mu F(p), x_{n+1}-p\bigr\rangle\right].
\end{aligned} \tag{3.5}$$

From relation (3.4) we also obtain

$$\frac{(1-\alpha_n\tau)^2\beta_n(1-\beta_n)}{1-\alpha_n\theta}\|Tx_n-x_n\|^2 \le (\Gamma_n-\Gamma_{n+1})+\alpha_n M, \tag{3.6}$$

where $M=\sup_{n\in\mathbb{N}}\{\frac{\alpha_n\tau^2}{1-\alpha_n\theta}\Gamma_n+\frac{2}{1-\alpha_n\theta}\|\mu F(p)-f(p)\|\cdot\|x_{n+1}-p\|\}$.

Case 1: Suppose that there exists n_0 such that $\{\Gamma_n\}_{n\ge n_0}$ is nonincreasing, it is equal to $\Gamma_{n+1}\le\Gamma_n$ for all $n\ge n_0$. It follows that $\lim_{n\to\infty}\Gamma_n$ exists, so we conclude that

$$\lim_{n\to\infty}(\Gamma_n-\Gamma_{n+1})=0. \tag{3.7}$$

Since $\lim_{n\to\infty}\alpha_n=0$, and from (3.6), (3.7) we deduce that

$$\begin{aligned} 0 &\le \limsup_{n\to\infty}\frac{(1-\alpha_n\tau)^2\beta_n(1-\beta_n)}{1-\alpha_n\theta}\|Tx_n-x_n\|^2 \\ &\le \limsup_{n\to\infty}\bigl((\Gamma_n-\Gamma_{n+1})+\alpha_n M\bigr)=0. \end{aligned}$$

From assumption (iii), we obtain that

$$\limsup_{n\to\infty}\frac{(1-\alpha_n\tau)^2\beta_n(1-\beta_n)}{1-\alpha_n\theta}>0.$$

Therefore we get

$$\lim_{n\to\infty}\|Tx_n-x_n\|=0. \tag{3.8}$$

In order to apply Xu's Lemma 2.8 to the sequence $a_n=\Gamma_n$, we show that

$$\limsup_{n\to\infty}\bigl\langle f(p)-\mu F(p),x_n-p\bigr\rangle\le 0,$$

where p is the unique solution of the variational inequality (3.2).

Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, there exists $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ and $x\in H$ such that $x_{n_k}\rightharpoonup x$ and

$$\begin{aligned} \limsup_{n\to\infty}\bigl\langle f(p)-\mu F(p),x_n-p\bigr\rangle &= \lim_{n\to\infty}\bigl\langle f(p)-\mu F(p),x_{n_k}-p\bigr\rangle \\ &= \bigl\langle f(p)-\mu F(p),x-p\bigr\rangle. \end{aligned}$$

From (3.8) and $I-T$ demiclosed at 0, we know that $x\in\mathrm{Fix}(T)$. Since $p=P_{\mathrm{Fix}(T)}(I-\lambda(\mu F-f))p$ and $x\in\mathrm{Fix}(T)$, we have

$$\limsup_{n\to\infty}\bigl\langle f(p)-\mu F(p),x_n-p\bigr\rangle=\bigl\langle f(p)-\mu F(p),x-p\bigr\rangle\le 0. \tag{3.9}$$

Set $\sigma_n=\frac{\frac{1}{2}\alpha_n\tau^2\Gamma_n}{\tau-\theta}+\frac{1}{\tau-\theta}\langle f(p)-\mu F(p),x_{n+1}-p\rangle$. Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, so is $\{\Gamma_n\}_{n\in\mathbb{N}}$. Equation (3.9) and assumption (i) imply $\limsup_{n\to\infty}\sigma_n\le 0$. It follows from assumptions

(i) and (ii) that

$$\lim_{n\to\infty} \frac{2\alpha_n(\tau-\theta)}{1-\alpha_n\theta} = 0, \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{2\alpha_n(\tau-\theta)}{1-\alpha_n\theta} = \infty.$$

Then $\lim_{n\to\infty} \|x_n - p\| = 0$ by Lemma 2.8.

Case 2: Suppose that there exists $\{n_k\}_{k\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that $\|x_{n_k} - p\| < \|x_{n_k+1} - p\|$ for all $k \in \mathbb{N}$.

From Maingé's Lemma 2.9, there exists a nondecreasing sequence of integers $\{\delta(n)\}_{n\in\mathbb{N}}$ satisfying, for all $n \geq n_0$, the following:

(a) $\lim_{n\to\infty} \delta(n) = \infty$;

(b) $\|x_{\delta(n)} - p\| < \|x_{\delta(n)+1} - p\|$;

(c) $\|x_n - p\| < \|x_{\delta(n)+1} - p\|$.

From the last and relation (3.5), we have

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty} (\Gamma_{\delta(n)+1} - \Gamma_{\delta(n)}) \\ &\leq \limsup_{n\to\infty} (\Gamma_{\delta(n)+1} - \Gamma_{\delta(n)}) \\ &\leq \limsup_{n\to\infty} (\Gamma_{n+1} - \Gamma_n) \\ &\leq \limsup_{n\to\infty} \biggl\{ \frac{2\alpha_n(\tau-\theta)}{1-\alpha_n\theta} \frac{\frac{1}{2}\alpha_n\tau^2\Gamma_n}{\tau-\theta} \\ &\quad + \frac{2\alpha_n(\tau-\theta)}{1-\alpha_n\theta} \biggl[\frac{1}{\tau-\theta} \bigl\langle f(p) - \mu F(p), x_{n+1} - p \bigr\rangle \biggr] \biggr\} = 0. \end{aligned}$$

Hence it turns out that

$$\lim_{n\to\infty} (\Gamma_{\delta(n)+1} - \Gamma_{\delta(n)}) = 0. \tag{3.10}$$

By (3.10), (3.6), and arguing as in case 1, we get

$$\lim_{n\to\infty} \|Tx_{\delta(n)} - x_{\delta(n)}\| = 0.$$

Similar to Case 1, we have

$$\limsup_{n\to\infty} \bigl\langle f(p) - \mu F(p), x_{\delta(n)+1} - p \bigr\rangle \leq 0. \tag{3.11}$$

If we replace n with $\delta(n)$ in (3.5), by condition (b), we obtain

$$\begin{aligned} \Gamma_{\delta(n)+1} &\leq \biggl(1 - \frac{2\alpha_{\tau(n)}(\tau-\theta)}{1-\alpha_{\delta(n)}\theta}\biggr)\Gamma_{\delta(n)} + \frac{\alpha_{\delta(n)}^2\tau^2}{1-\alpha_{\delta(n)}\theta}\Gamma_{\delta(n)} \\ &\quad + \frac{2\alpha_{\delta(n)}}{1-\alpha_{\delta(n)}\theta} \bigl\langle f(p) - \mu F(p), x_{\delta(n)+1} - p \bigr\rangle \\ &\leq \biggl(1 - \frac{2\alpha_{\delta(n)}(\tau-\theta)}{1-\alpha_{\delta(n)}\theta}\biggr)\Gamma_{\delta(n)+1} + \frac{\alpha_{\delta(n)}^2\tau^2}{1-\alpha_{\delta(n)}\theta}\Gamma_{\delta(n)} \\ &\quad + \frac{2\alpha_{\delta(n)}}{1-\alpha_{\delta(n)}\theta} \bigl\langle f(p) - \mu F(p), x_{\delta(n)+1} - p \bigr\rangle, \end{aligned}$$

and also

$$\frac{2\alpha_{\delta(n)}(\tau-\theta)}{1-\alpha_{\delta(n)}\theta}\Gamma_{\delta(n)+1} \le \frac{\alpha_{\delta(n)}^2\tau^2}{1-\alpha_{\delta(n)}\theta}\Gamma_{\delta(n)} + \frac{2\alpha_{\delta(n)}}{1-\alpha_{\delta(n)}\theta}\langle f(p)-\mu F(p), x_{\delta(n)+1}-p\rangle.$$

Therefore dividing both sides of the obtained inequality by $\alpha_{\delta(n)}$, we have

$$\frac{2(\tau-\theta)}{1-\alpha_{\delta(n)}\theta}\Gamma_{\delta(n)+1} \le \frac{\alpha_{\delta(n)}\tau^2}{1-\alpha_{\delta(n)}\theta}\Gamma_{\delta(n)} + \frac{2}{1-\alpha_{\delta(n)}\theta}\langle f(p)-\mu F(p), x_{\delta(n)+1}-p\rangle. \tag{3.12}$$

Since $\lim_{n\to\infty}\alpha_{\delta(n)}=0$, by (3.11) and (3.12), we get $\lim_{n\to\infty}\|x_{\delta(n)}-p\|=0$. By condition (c), $\lim_{n\to\infty}\|x_n-p\|=0$. □

Corollary 3.2 *Let $T_\rho : H \to H$ be a ρ-strongly quasi-nonexpansive mapping such that $\mathrm{Fix}(T_\rho)\neq\emptyset$, with $\rho>0$ and $I-T_\rho$ demiclosed at 0. For any given $x_0\in H$, the iterative sequence $\{x_n\}_{n\in\mathbb{N}}\subset H$ is generated by*

$$x_{n+1}=\alpha_n f(x_n)+(I-\alpha_n\mu F)T_\rho x_n, \tag{3.13}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ are two sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n=0$;

(ii) $\sum_{n=0}^{\infty}\alpha_n=\infty$.

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$, defined by (3.13), converges strongly to $p\in\mathrm{Fix}(T_\rho)$, which is the unique solution in $\mathrm{Fix}(T_\rho)$ of the variational inequality (VI)

$$\langle\mu F(p)-f(p), x-p\rangle\ge 0, \quad \forall x\in\mathrm{Fix}(T_\rho).$$

Proof By Lemma 2.1, there exists a quasi-nonexpansive mapping T such that $T_\rho=(1-\frac{1}{1+\rho})I+\frac{1}{1+\rho}T$ and $\mathrm{Fix}(T_\rho)=\mathrm{Fix}(T)$. $I-T_\rho$ is demiclosed at 0, so is $I-T$. Take $\beta_n=1-\frac{1}{1+\rho}$ for all $n\in\mathbb{N}$, since $\rho>0$, then $\beta_n(1-\beta_n)=\frac{\rho}{(1+\rho)^2}>0$. Then Corollary 3.2 follows from Theorem 3.1. □

4 Split equality fixed point problems

Let H_1 and H_2 be two real Hilbert spaces, the product $H=H_1\times H_2$ is a Hilbert space with inner product and norm given by

$$\langle x,y\rangle=\langle x_1,y_1\rangle+\langle x_2,y_2\rangle, \quad \text{and} \quad \|x\|^2=\|x_1\|^2+\|x_2\|^2$$

for any $x=\begin{bmatrix}x_1\\x_2\end{bmatrix}, y=\begin{bmatrix}y_1\\y_2\end{bmatrix}\in H$.

In this section we always assume that

(1) H_1, H_2, H_3 are three real Hilbert spaces and $H=H_1\times H_2$;

(2) $T=\begin{bmatrix}T_1\\T_2\end{bmatrix}$, where T_i, $i=1,2$, is a one-to-one and quasi-nonexpansive mapping;

(3) $G = [A_1\ -A_2]$ and $G^*G = \begin{bmatrix} A_1^*A_2 & -A_2^*A_1 \\ -A_1^*A_2 & A_2^*A_2 \end{bmatrix}$, where A_i, $i = 1,2$ is a bounded linear operator from H_i into H_3 and A_i^* is the adjoint of A_i;
(4) $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, where f_i, $i = 1,2$ is a θ-contraction on H_i with $\theta \in (0,1)$;
(5) $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$, where F_i, $i = 1,2$, is L-Lipschitz continuous and η-strongly monotone on H_i with $L > 0$, and $\eta > 0$.

Lemma 4.1 ([17]) *Let $U = I - \lambda G^*G$, where $0 < \lambda < 2/\rho(G^*G)$ with $\rho(G^*G)$ being the spectral radius of the self-adjoint operator G^*G on H. Then we have the following result*:
(1) $\|U\| \leq 1$ (*i.e., U is nonexpansive*) *and averaged*;
(2) $\mathrm{Fix}(U) = \{x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in H : A_1x_1 = A_2x_2\}$, $\mathrm{Fix}(P_CU) = \mathrm{Fix}(P_C) \cap \mathrm{Fix}(U)$.

Theorem 4.2 *Let H_1, H_2, H_3, H, T_1, T_2, T, A_1, A_2, G, G^*G, f_1, f_2, f, F_1, F_2, F satisfy the above conditions* (1)–(5). *For any given $x_0 \in H$, the iterative sequence $\{x_n\}_{n\in\mathbb{N}} = \begin{bmatrix} x_{1,n} \\ x_{2,n} \end{bmatrix}_{n\in\mathbb{N}} \subset H$ is generated by*

$$\begin{aligned} v_n &= \beta_n x_n + (1-\beta_n)T\bigl(I - \lambda G^*G\bigr)x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (I - \alpha_n\mu F)v_n, \end{aligned} \tag{4.1}$$

or its equivalent form

$$\begin{aligned} v_{1,n} &= \beta_n x_{1,n} + (1-\beta_n)T_1\bigl(x_{1,n} - \lambda A_1^*(A_1x_{1,n} - A_2x_{2,n})\bigr), \\ v_{2,n} &= \beta_n x_{2,n} + (1-\beta_n)T_2\bigl(x_{2,n} + \lambda A_2^*(A_1x_{1,n} - A_2x_{2,n})\bigr), \\ x_{1,n+1} &= \alpha_n f_1(x_{1,n}) + (I_1 - \alpha_n\mu F_1)v_{1,n}, \\ x_{2,n+1} &= \alpha_n f_2(x_{2,n}) + (I_2 - \alpha_n\mu F_2)v_{2,n}, \end{aligned}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ are two sequences in $(0,1)$. *If the solution set $\Gamma = \{x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in H : x_1 \in \mathrm{Fix}(T_1), x_2 \in \mathrm{Fix}(T_2)$ such that $A_1x_1 = A_2x_2\}$ of* (*SEFP*) *is nonempty and the following conditions are satisfied*:
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\limsup_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(iv) $\lambda \in (0, \frac{2}{R})$, *where* $R = \|G\|^2$;
(v) *for each $i = 1,2$, T_i is demiclosed*;
(vi) $\mu \in (0, \frac{2\eta}{L^2})$ *and $\theta \in (0,\tau)$, where* $\tau = \mu(\eta - \frac{1}{2}\mu L^2)$.

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$, defined by (4.1), *converges strongly to $p \in \Gamma$, which is the unique solution in Γ of the variational inequality* (*VI*)

$$\bigl\langle \mu F(p) - f(p), x - p\bigr\rangle \geq 0, \quad \forall x \in \Gamma. \tag{4.2}$$

Proof For each $i = 1,2$, since f_i is θ-contraction, F_i is L-Lipschitz continuous and η-strongly monotone on H_i with $L > 0$ and $\eta > 0$, and T_i is quasi-nonexpansive, we have

$$\begin{aligned} \bigl\|f(x) - f(y)\bigr\|^2 &= \bigl\|f_1(x_1) - f_1(y_1)\bigr\|^2 + \bigl\|f_2(x_2) - f_2(y_2)\bigr\|^2 \\ &\leq \theta^2\bigl(\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2\bigr) \\ &= \theta^2\|x - y\|^2, \end{aligned}$$

i.e.,

$$\left\|f(x)-f(y)\right\| \leq \theta\|x-y\|$$

for all $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in H$. This shows that f is a θ-contraction. Similarly, F is L-Lipschitz continuous and η-strongly monotone, and T is a quasi-nonexpansive mapping.

Let $\{x_n\}_{n\in\mathbb{N}} = \begin{bmatrix} x_{1,n} \\ x_{2,n} \end{bmatrix}_{n\in\mathbb{N}}$ be a sequence in $H = H_1 \times H_2$ such that $x_n \rightharpoonup x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Then, for each $i = 1, 2$, we have $\lim_{n\to\infty}\|T_i x_{i,n} - x_{i,n}\| = 0$, and

$$\langle x_n - x, y\rangle = \langle x_{1,n} - x_1, y_1\rangle + \langle x_{2,n} - x_2, y_2\rangle \to 0$$

for each $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in H$. For each $i = 1, 2$, let $y_i \in H_i$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in H$ with $y_j = 0$ $(j \neq i)$. Then $\lim_{n\to\infty}\langle x_n - x, y\rangle = 0$ implies $\lim_{n\to\infty}\langle x_{i,n} - x_i, y_i\rangle = 0$ and $x_{i,n} \rightharpoonup x_i$.

For each $i = 1, 2$, since $T_i : H_i \to H_i$ is demiclosed, $x_i \in \mathrm{Fix}(T_i)$. It is easy to see that $x \in \mathrm{Fix}(T) \neq \emptyset$. Hence T is demiclosed.

By Lemma 4.1, $U = I - \lambda G^*G$ is a $\frac{1-\alpha}{\alpha}$-strongly quasi-nonexpansive mapping for some $\alpha > 0$, and $\mathrm{Fix}(U) = \left\{x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in H : A_1x_1 = A_2x_2\right\}$. Since $\Gamma \neq \emptyset$, U is a strictly quasi-nonexpansive mapping. By Lemma 2.3, $\mathrm{Fix}(TU) = \mathrm{Fix}(T) \cap \mathrm{Fix}(U) = \Gamma \neq \emptyset$.

Then Theorem 4.2 follows from Theorem 3.1. □

5 Applications

In this section, we always assume that H_1, H_2, H_3, H, T_1, T_2, T, A_1, A_2, G, G^*G, f_1, f_2, f, F_1, F_2, F satisfy conditions (1)–(5) in Sect. 4.

5.1 Split equality common fixed point problems

The split equality common fixed point problem (SECFPP) is the problem of finding

$$\begin{aligned} &x_1 \in H_1, x_2 \in H_2 \text{ such that } x_1 \in \mathrm{Fix}(T_1) \cap \mathrm{Fix}(S_1), \\ &x_2 \in \mathrm{Fix}(T_2) \cap \mathrm{Fix}(S_2), \text{and } A_1x_1 = A_2x_2, \end{aligned} \tag{5.1}$$

where T_i, $i = 1, 2$, is a quasi-nonexpansive mapping on H_i and S_i, $i = 1, 2$, is a firmly nonexpansive mapping on H_i, respectively. Its solution set is denoted by $\Gamma_{\mathrm{CF}} = \left\{x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in H : x_1 \in \mathrm{Fix}(T_1) \cap \mathrm{Fix}(S_1), x_2 \in \mathrm{Fix}(T_2) \cap \mathrm{Fix}(S_2) \text{ such that } A_1x_1 = A_2x_2\right\}$, then we have the following result.

Theorem 5.1 *Let $S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$, where S_i, $i = 1, 2$, is a firmly nonexpansive mapping on H_i. For any given $x_0 \in H$, the iterative sequence $\{x_n\}_{n\in\mathbb{N}} = \begin{bmatrix} x_{1,n} \\ x_{2,n} \end{bmatrix}_{n\in\mathbb{N}} \subset H$ is generated by*

$$\begin{aligned} u_n &= S\left(I - \lambda G^*G\right)x_n, \\ v_n &= \beta_n x_n + (1 - \beta_n)Tu_n, \\ x_{n+1} &= \alpha_n f(x_n) + (I - \alpha_n \mu F)v_n, \end{aligned} \tag{5.2}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ are two sequences in $(0, 1)$. If the solution set Γ_{CF} is nonempty and the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(iii) $\limsup_{n\to\infty}\beta_n(1-\beta_n) > 0$;
(iv) $\lambda \in (0, \frac{2}{R})$, *where* $R = \|G\|^2$;
(v) *for each* $i = 1, 2$, T_i *is demiclosed*;
(vi) $\mu \in (0, \frac{2\eta}{L^2})$ *and* $\theta \in (0, \tau)$, *where* $\tau = \mu(\eta - \frac{1}{2}\mu L^2)$.

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$, *defined by* (5.2), *converges strongly to* $p \in \Gamma_{\mathrm{CF}}$, *which is the unique solution in* Γ_{CF} *of the variational inequality* (*VI*)

$$\langle \mu F(p) - f(p), x - p\rangle \geq 0, \quad \forall x \in \Gamma_{\mathrm{CF}}.$$

Proof For each $i = 1, 2$, S_i is a firmly nonexpansive mapping on H_i, it is easy to verify that S is a firmly nonexpansive mapping on H. Since $\Gamma_{\mathrm{CF}} \neq \emptyset$, S is a strictly quasi-nonexpansive mapping on H. By Lemma 2.3, $\mathrm{Fix}(TS) = \mathrm{Fix}(T) \cap \mathrm{Fix}(S)$ and TS is a quasi-nonexpansive mapping. Then Theorem 5.1 follows from Theorem 4.2. □

5.2 Split equality null point problems and split equality fixed point problems

Let $\mathcal{M}$ be a set-valued mapping of H into 2^H. The effective domain of $\mathcal{M}$ is denoted by $\mathrm{dom}(\mathcal{M})$; that is, $\mathrm{dom}(\mathcal{M}) = \{x \in H : \mathcal{M}x \neq \emptyset\}$. A set-valued mapping $\mathcal{M}$ is said to be a monotone operator on H if $\langle u - v, x - y\rangle \geq 0$ for all $x, y \in \mathrm{dom}(M)$, $u \in \mathcal{M}x$, and $v \in \mathcal{M}y$. A monotone operator $\mathcal{M}$ on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H.

The split equality null point problems and split equality fixed point problems are the problem of finding

$$x_1 \in \mathrm{Fix}(T_1) \cap \mathcal{M}_1^{-1}(0), x_2 \in \mathrm{Fix}(T_2) \cap \mathcal{M}_2^{-1}(0) \text{ such that } A_1x_1 = A_2x_2, \tag{5.3}$$

where T_i, $i = 1, 2$ is a quasi-nonexpansive mapping on H_i, and $\mathcal{M}_i$, $i = 1, 2$, is a set-valued maximal monotone operator of H_i into 2^{H_i} with $r > 0$, respectively. Its solution set is denoted by $\Gamma_{NF} = \{x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in H : x_1 \in \mathrm{Fix}(T_1) \cap \mathcal{M}_1^{-1}(0), x_2 \in \mathrm{Fix}(T_2) \cap \mathcal{M}_2^{-1}(0) \text{ such that } A_1x_1 = A_2x_2\}$.

For a maximal monotone operator $\mathcal{M}$ on H and $r > 0$, we may define a single-valued operator $J_r^{\mathcal{M}} = (I + r\mathcal{M})^{-1} : H \to \mathrm{dom}(\mathcal{M})$, which is called the resolvent of $\mathcal{M}$ for r. From Chuang [18] and Chang [19], we know that the associated resolvent mapping $J_r^{\mathcal{M}}$ is a firmly nonexpansive mapping, and

$$x \in \mathcal{M}^{-1}(0) \quad \Leftrightarrow \quad x \in \mathrm{Fix}(J_r^{\mathcal{M}}).$$

This implies that the split equality null point problems and split equality fixed point problems (5.3) are equivalent to SECFPP (5.1). Then the following theorem can be obtained from Theorem 5.1 immediately.

Theorem 5.2 *Let* $J_r^{\mathcal{M}} = \left[\begin{smallmatrix} J_r^{\mathcal{M}_1} \\ J_r^{\mathcal{M}_2} \end{smallmatrix}\right]$, *where* $\mathcal{M}_i$, $i = 1, 2$, *is a set-valued maximal monotone operator of* H_i *into* 2^{H_i}, $J_r^{\mathcal{M}_i}$ *is the associated resolvent of* $\mathcal{M}_i$ *with* $r > 0$, *respectively. For any* *given $x_0 \in H$, the iterative sequence $\{x_n\}_{n\in\mathbb{N}} = \left[\begin{smallmatrix} x_{1,n} \\ x_{2,n} \end{smallmatrix}\right]_{n\in\mathbb{N}} \subset H$ is generated by*

$$\begin{aligned} u_n &= J_r^{\mathcal{M}}\bigl(I - \lambda G^*G\bigr)x_n, \\ v_n &= \beta_n x_n + (1-\beta_n)Tu_n, \\ x_{n+1} &= \alpha_n f(x_n) + (I - \alpha_n \mu F)v_n, \end{aligned} \tag{5.4}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ are two sequences in $(0,1)$. If the solution set $\varGamma_{NF}$ is nonempty and the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\limsup_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(iv) $\lambda \in (0, \frac{2}{R})$, *where* $R = \|G\|^2$;
(v) *for each* $i = 1,2$, T_i *is demiclosed*;
(vi) $\mu \in (0, \frac{2\eta}{L^2})$ *and* $\theta \in (0,\tau)$, *where* $\tau = \mu(\eta - \frac{1}{2}\mu L^2)$.

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$, defined by (5.4), converges strongly to $p \in \varGamma_{NF}$, which is the unique solution in $\varGamma_{NF}$ of the variational inequality (VI)

$$\bigl\langle \mu F(p) - f(p), x - p\bigr\rangle \geq 0, \quad \forall x \in \varGamma_{NF}.$$

5.3 Split equality optimization problems and split equality fixed point problems

Let H_i, $i = 1,2,3$, be a real Hilbert space. Let $g_i : H_i \to \mathbb{R}$, $i = 1,2$, be a proper, convex, and lower semi-continuous function, and $A_i : H_i \to H_3$, $i = 1,2$, be a bounded linear operator.

The split equality optimization problem (SEOP) is the problem of finding

$$\begin{aligned} &x_1 \in H_1, x_2 \in H_2 \text{ such that } g_1(x_1) = \min_{y\in H_1} g_1(y), \\ &g_2(x_2) = \min_{y\in H_2} g_2(y), \text{and } A_1x_1 = A_2x_2. \end{aligned} \tag{5.5}$$

The split equality optimization problem and split equality fixed point problems are the problem of finding

$$\begin{aligned} &x_1 \in \operatorname{Fix}(T_1), x_2 \in \operatorname{Fix}(T_2) \text{ such that } g_1(x_1) = \min_{y\in H_1} g_1(y), \\ &g_2(x_2) = \min_{y\in H_2} g_2(y) \text{ and } A_1x_1 = A_2x_2, \end{aligned} \tag{5.6}$$

where T_i, $i = 1,2$, is a quasi-nonexpansive mapping on H_i. The solution set of (5.6) is denoted by $\varGamma_{\mathrm{OF}} = \bigl\{x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in H : x_1 \in \operatorname{Fix}(T_1), x_2 \in \operatorname{Fix}(T_2)$ such that $g_1(x_1) = \min_{y\in H_1} g_1(y)$, $g_2(x_2) = \min_{y\in H_2} g_2(y)$, and $A_1x_1 = A_2x_2\bigr\}$.

The subdifferential of g_i, $i = 1,2$, at x is the set

$$\partial g_i(x) = \bigl\{u \in H_i : g_i(y) \geq g_i(x) + \langle u, y - x\rangle, \forall y \in H_i\bigr\}.$$

Denoted by $\partial g_i = \mathcal{M}_i$, $i = 1,2$, is a maximal monotone mapping, so we can define the resolvent $J_r^{\mathcal{M}_i}$, where $r > 0$. Since x_1 and x_2 are a minimum of g_1 on H_1 and that of g_2 on H_2, respectively, for any given $r > 0$, we have

$$x_1 \in \mathcal{M}_1^{-1}(0) = \operatorname{Fix}\bigl(J_r^{\mathcal{M}_1}\bigr) \quad \text{and} \quad x_2 \in \mathcal{M}_2^{-1}(0) = \operatorname{Fix}\bigl(J_r^{\mathcal{M}_2}\bigr).$$

This implies that the split equality optimization problem and split equality fixed point problems (5.6) are equivalent to the split equality null point problems and split equality fixed point problems (5.3). Then the following theorem can be obtained from Theorem 5.2 immediately.

Theorem 5.3 *Let $g_i : H_i \to \mathbb{R}$, $i = 1,2$, be a proper, convex, and lower semi-continuous function. Let $J_r^{\mathcal{M}} = \begin{bmatrix} J_r^{\mathcal{M}_1} \\ J_r^{\mathcal{M}_2} \end{bmatrix}$, where $\mathcal{M}_i = \partial g_i$, $i = 1,2$, and $J_r^{\mathcal{M}_i}$ is the associated resolvent of $\mathcal{M}_i$ with $r > 0$, respectively. For any given $x_0 \in H$, the iterative sequence $\{x_n\}_{n\in\mathbb{N}} = \begin{bmatrix} x_{1,n} \\ x_{2,n} \end{bmatrix}_{n\in\mathbb{N}} \subset H$ is generated by*

$$\begin{aligned} u_n &= J_r^{\mathcal{M}}\left(I - \lambda G^* G\right)x_n, \\ v_n &= \beta_n x_n + (1-\beta_n)Tu_n, \\ x_{n+1} &= \alpha_n f(x_n) + (I - \alpha_n \mu F)v_n, \end{aligned} \tag{5.7}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ are two sequences in $(0,1)$. If the solution set Γ_{OF} is nonempty and the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\limsup_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(iv) $\lambda \in (0, \frac{2}{R})$, *where* $R = \|G\|^2$;
(v) *for each* $i = 1,2$, T_i *is demiclosed*;
(vi) $\mu \in (0, \frac{2\eta}{L^2})$ *and* $\theta \in (0,\tau)$, *where* $\tau = \mu(\eta - \frac{1}{2}\mu L^2)$.

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$, defined by (5.7), converges strongly to $p \in \Gamma_{\mathrm{OF}}$, which is the unique solution in Γ_{OF} of the variational inequality (VI)

$$\left\langle \mu F(p) - f(p), x - p\right\rangle \geq 0, \quad \forall x \in \Gamma_{\mathrm{OF}}.$$

5.4 Split equality equilibrium problems and split equality fixed point problems

Let C be a nonempty closed and convex subset of a real Hilbert space. A bifunction $\Theta : C \times C \to \mathbb{R}$ is called an equilibrium function if and only if it satisfies the following conditions:

(A1) $\Theta(x,x) = 0$ for all $x \in C$;
(A2) Θ is monotone, i.e., $g(x,y) + g(y,x) \leq 0$ for all $x, y \in C$;
(A3) Θ is upper-hemicontinuous, i.e., for all $x,y,z \in C$,
$\limsup_{t\to 0^+} \Theta(tz + (1-t)x, y) \leq \Theta(x,y)$;
(A4) $\Theta(x,\cdot)$ is convex and lower semicontinuous for all $x \in C$.

The so-called equilibrium problem with respect to the equilibrium function Θ is the problem of finding

$$x^* \in C \text{ such that } \Theta\left(x^*, y\right) \geq 0, \quad \forall y \in C. \tag{5.8}$$

Its solution set is denoted by $\mathrm{EP}(\Theta, C)$. Numerous problems in physics, optimization, and economics are reduced to finding a solution of (5.8) (see [20, 21]).

The split equality equilibrium problems and split equality fixed point problems are the problem of finding

$$x_1 \in \mathrm{Fix}(T_1) \cap \mathrm{EP}(\Theta_1, C_1), x_2 \in \mathrm{Fix}(T_2) \cap \mathrm{EP}(\Theta_2, C_2) \text{ such that } A_1 x_1 = A_2 x_2, \tag{5.9}$$

where T_i, $i = 1,2$, is a quasi-nonexpansive mapping on H_i and Θ_i is an equilibrium function of $C_i \times C_i$ into $\mathbb{R}$, respectively. Its solution set is denoted by $\Gamma_{EF} = \{x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in H : x_1 \in \operatorname{Fix}(T_1) \cap \operatorname{EP}(\Theta_1, C_1), x_2 \in \operatorname{Fix}(T_2) \cap \operatorname{EP}(\Theta_2, C_2)$ such that $A_1 x_1 = A_2 x_2\}$.

For given $r > 0$ and $x \in H$, the resolvent of the equilibrium function Θ is the operator defined by

$$T_r^{\Theta}(x) = \left\{ z \in C : \Theta(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}.$$

Proposition 5.4 ([22]) *The resolvent operator T_r^{Θ} of the equilibrium function Θ has the following properties*:

(i) T_r^{Θ} *is single-valued*;

(ii) $\operatorname{Fix}(T_r^{\Theta}) = \operatorname{EP}(\Theta, C)$, *and* $\operatorname{EP}(\Theta, C)$ *is a nonempty closed and convex subset of* C;

(iii) T_r^{Θ} *is a firmly nonexpansive mapping.*

Using the above lemma, Takahashi et al. [23] obtained the following lemma. See Aoyama et al. [24] for a more general result.

Lemma 5.5 ([23, 24]) *Let C be a nonempty closed convex subset of H, and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1) *to* (A4). *Let $\mathcal{M}_{\Theta}$ be a set-valued mapping of H into itself defined by*

$$\mathcal{M}_{\Theta}(x) = \begin{cases} \{z \in H : \Theta(x,y) + \langle y - x, z\rangle \geq 0, \forall y \in C\}, & \forall x \in C, \\ \emptyset, & \forall x \notin C. \end{cases}$$

Then $\operatorname{EP}(\Theta, C) = \mathcal{M}_{\Theta}^{-1}(0)$ *and* $\mathcal{M}_{\Theta}$ *is a maximal monotone operator with* $\operatorname{dom}(\mathcal{M}_{\Theta}) \subset C$. *Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r^{Θ} of Θ coincides with the resolvent of $\mathcal{M}_{\Theta}$, i.e.,*

$$T_r^{\Theta}(x) = (I + r\mathcal{M}_{\Theta})^{-1}(x).$$

This implies that the split equality equilibrium problems and split equality fixed point problems (5.9) are equivalent to the split equality null point problems and split equality fixed point problems (5.3). Then we have the following result.

Theorem 5.6 *Let C_i, $i = 1,2$, be a nonempty closed convex subset of H_i. Let $T_r^{\Theta} = \left[\begin{smallmatrix} T_r^{\Theta_1} \\ T_r^{\Theta_2} \end{smallmatrix}\right]$, where Θ_i, $i = 1,2$, is an equilibrium function of $C_i \times C_i$ into $\mathbb{R}$ with $r > 0$. For any given $x_0 \in H$, the iterative sequence $\{x_n\}_{n\in\mathbb{N}} = \left[\begin{smallmatrix} x_{1,n} \\ x_{2,n} \end{smallmatrix}\right]_{n\in\mathbb{N}} \subset H$ is generated by*

$$\begin{aligned} u_n &= T_r^{\Theta}\big(I - \lambda G^* G\big)x_n, \\ v_n &= \beta_n x_n + (1 - \beta_n) T u_n, \\ x_{n+1} &= \alpha_n f(x_n) + (I - \alpha_n \mu F) v_n, \end{aligned} \tag{5.10}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ are two sequences in $(0,1)$. If the solution set Γ_{OF} is nonempty and the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\limsup_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(iv) $\lambda \in (0, \frac{2}{R})$, *where* $R = \|G\|^2$;
(v) *for each* $i = 1, 2$, T_i *is demiclosed*;
(vi) $\mu \in (0, \frac{2\eta}{L^2})$ *and* $\theta \in (0, \tau)$, *where* $\tau = \mu(\eta - \frac{1}{2}\mu L^2)$.

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$, *defined by* (5.10), *converges strongly to* $p \in \Gamma_{\mathrm{OF}}$, *which is the unique solution in* Γ_{OF} *of the variational inequality* (*VI*)

$$\langle \mu F(p) - f(p), x - p \rangle \geq 0, \quad \forall x \in \Gamma_{\mathrm{OF}}.$$

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Authors' contributions
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