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Blow-up criteria for the 3D Bénard system in Besov spaces

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Abstract

This paper studies the regularity criteria of weak solutions for Bénard fluid flows in three dimensions. When the velocity field and the temperature belong to some critical Besov spaces on $(0, T)$, then the weak solution (u, θ) can be extended smoothly beyond T .

MSC: 75B03; 35B45; 35B65

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1 Introduction

In this paper, we consider the following three-dimensional (3D) Bénard system:

$$\partial_t u + (u \cdot \nabla)u + \nabla \pi = \mu \Delta u + \theta e_3, \quad (1.1)$$

$$\partial_t \theta + (u \cdot \nabla)\theta = \kappa \Delta \theta + u \cdot e_3, \quad (1.2)$$

$$\nabla \cdot u = 0, \quad \text{in } \mathbb{R}^3 \times (0, \infty), \quad (1.3)$$

$$u(0) = u_0, \quad \theta(0) = \theta_0, \quad \text{in } \mathbb{R}^3, \quad (1.4)$$

where $u = u(x, t) \in \mathbb{R}^3$, $\theta = \theta(x, t) \in \mathbb{R}$, and $\pi = \pi(x, t) \in \mathbb{R}$ represent the unknown velocity field, the temperature, and the pressure, respectively, the nonnegative constants μ and κ are the coefficients of dissipation and thermal diffusivity, and $e_3 = (0, 0, 1)^T$ denotes the vertical unit vector; the forcing term θe_3 in the momentum equation (1.1) describes the action of the buoyancy force on fluid motion, and $u \cdot e_3$ models the Rayleigh–Bénard convection in a heated inviscid fluid. We prescribe the initial data to satisfy the condition

$$\nabla \cdot u_0 = 0. \quad (1.5)$$

The Bénard system describes the Rayleigh–Bénard convective motion in a heated 3D inviscid incompressible fluid under thermal effects (see, e.g., [1, 4, 8, 9, 21, 26]). We can easily check that if we ignore the Rayleigh–Bénard convection term $u \cdot e_3$, then system (1.1)–(1.4) becomes the corresponding Boussinesq system, which describes the evolution of the velocity field u for a 3D incompressible fluid moving under the gravity and the earth rotation, which come from atmospheric or oceanographic turbulence, where rotation and

stratification play an important role (see, e.g., [3, 6, 7, 22, 25]). In fact, the Boussinesq system is usually referred to as the Bénard convection problem.

Recently, the Bénard system has attracted considerable attention due to their physical applications and mathematical significance. Let us say some words about system (1.1)–(1.4). Nakamura [24] established the smoothness and the time analyticity of strong solutions to the magnetic Bénard system in \mathbb{R}^n , $n = 2, 3$. Ma and Zhang [17] considered the global weak solution of the 2D Bénard system with partial dissipation and established some regularity criteria for the corresponding Bénard system. Later, they generalized and extended these results in [13]. It is currently unknown whether the solutions of the 3D magnetic Bénard system is globally regular (in time). Ma [14] dealt with the Cauchy problem to the 3D system of incompressible magnetic Bénard fluids and proved that as the initial data satisfy $\|u_0\|_{H^1(\mathbb{R}^3)}^2 + \|\theta_0\|_{H^1(\mathbb{R}^3)}^2 + \|b_0\|_{H^1(\mathbb{R}^3)}^2 \leq \varepsilon$, where ε is a suitably small positive number, the 3D magnetic Bénard system with mixed partial dissipation, magnetic diffusion, and thermal diffusivity admits a global smooth solution. Ma [15] investigated the blow-up criteria of strong solutions and regularity criterion of weak solutions for the magnetic Bénard system in \mathbb{R}^3 in a sense of scaling invariant by employing a different decomposition for nonlinear terms. More precisely, the strong solution (u, θ, b) of the magnetic Bénard system is proved to be smooth on $(0, T]$, provided that the velocity field u satisfies

$$u \in L^{\frac{2}{1-r}}(0, T; \dot{X}_r(\mathbb{R}^3)) \quad \text{with } 0 \leq r < 1$$

or the gradient field of velocity ∇u satisfies

$$\nabla u \in L^{\frac{2}{2-\beta}}(0, T; \dot{X}_\beta(\mathbb{R}^3)) \quad \text{with } 0 \leq \beta \leq 1.$$

Moreover, he proved that if

$$u \in L^\infty(0, T; \dot{X}_1(\mathbb{R}^3)) \quad \text{and} \quad \|u\|_{L^\infty(0, T; \dot{X}_1(\mathbb{R}^3))} < \varepsilon,$$

where $\varepsilon > 0$ is a suitable small constant, then the strong solution (u, θ, b) of magnetic Bénard system can also be extended beyond $t = T$. Finally, he showed that if some partial derivatives of the velocity components, magnetic components, and temperature components (i.e., $\tilde{\nabla} \tilde{u}$, $\tilde{\nabla} \theta$, and $\tilde{\nabla} \tilde{b}$) belong to the multiplier space, then the solution (u, θ, b) is in fact smooth on $(0, T)$. Very recently, Ma [16] studied the global regularity for the $2\frac{1}{2}$ D magnetic Bénard system with mixed partial viscosity. More precisely, he showed not only the global regularity for the $2\frac{1}{2}$ D magnetic Bénard system with zero thermal diffusivity by a well-known property of the Hardy space and BMO, but also obtained the global regularity for the $2\frac{1}{2}$ D magnetic Bénard system with zero thermal diffusivity and horizontal magnetic diffusion as well as vertical magnetic diffusion resorting to the method of the local-in-time analysis. Ma and Zhang [18] obtained a blow-up criterion of smooth solutions to the magnetic Bénard system without zero magnetic diffusion and thermal diffusivity in \mathbb{R}^2 , derived a blow-up criterion of smooth solutions to the magnetic Bénard system without dissipation in \mathbb{R}^n ($n = 2, 3$), and in what follows, they improved this result. In [19], they studied the Cauchy problem for the $2\frac{1}{2}$ D incompressible magnetic system with partial viscosity and derived some blow-up criteria of smooth solutions. In [20], the

authors concentrated on the global regularity of classical solution to the $2\frac{1}{2}$ D magnetic Bénard system with partial dissipation, magnetic diffusion, and thermal diffusivity (i.e., horizontal dissipation, horizontal magnetic diffusion, and horizontal thermal diffusivity; vertical dissipation, vertical magnetic diffusion, and vertical thermal diffusivity).

Here it is worth particularly mentioning some regularity criteria for the fluid dynamic systems. Lin and Du [12] gave some global regularity criteria for 3D incompressible magnetohydrodynamics (MHD) equations. Mechdene et al. [23] established a sufficient condition for the regularity criterion of the Boussinesq equation in terms of the derivative of the pressure in one direction. Zhang [28] considered the 3D Hall-magnetohydrodynamic system, and showed that if the velocity and the magnetic fields belong to some critical Besov spaces on $(0, T)$, then the solution can be extended smoothly beyond T . Xu [27] considered the 3D magneto-micropolar fluid equations in Besov spaces.

Motivated by these results on the regularity criteria of fluid dynamics system, our aim in this study is to investigate the regularity criteria for the 3D Bénard system. To aid the introduction of our main results, let us first recall the definition of the first-time blow-up time (see [28]).

Definition 1.1 A number T^* is called the first-time blow-up time if it satisfies the following conditions:

$$\|(u, \theta)\|_{H^r}(t) < \infty \quad \text{for all } t < T^*,$$

and

$$\limsup_{t \nearrow T^*} \|(u, \theta)\|_{H^r}(t) = \infty,$$

where $r \geq 3$ is an integer, and (u, θ) is a solution to (1.1)–(1.4).

With the use of the notations described in Sect. 2, our main results now read as follows.

Theorem 1.1 Let $(u_0, \theta_0) \in H^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. If the velocity field u and the gradient of the temperature $\nabla \theta$ satisfy

$$\|u\|_{L^{\frac{2}{1-s}}(0, T^*; \dot{B}_{\infty, \infty}^{-s})} + \|\nabla \theta\|_{L^{\frac{2}{2-\gamma}}(0, T^*; \dot{B}_{\infty, \infty}^{-\gamma})} = \infty, \quad -1 < s < 1, 0 < \gamma < 2,$$

then for the first-time blow-up time $T^* < \infty$ of the solution (u, θ) to (1.1)–(1.4), we have that

$$\limsup_{t \nearrow T^*} \|(u, \theta)\|_{H^3}^2(t) = \infty.$$

Using the Bernstein inequality (see Lemma 2.1), we easily obtain that $\nabla : \dot{B}_{p,q}^r \rightarrow \dot{B}_{p,q}^{r-1}$ is an isometry, and we can deduce from Theorem 1.1 the following:

Corollary 1.2 Let $(u_0, \theta_0) \in H^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. If the gradient field of the velocity ∇u and the gradient of the temperature $\nabla \theta$ satisfy

$$\|\nabla u\|_{L^{\frac{2}{2-s}}(0, T^*; \dot{B}_{\infty, \infty}^{-s})} + \|\nabla \theta\|_{L^{\frac{2}{2-\gamma}}(0, T^*; \dot{B}_{\infty, \infty}^{-\gamma})} = \infty, \quad 0 < s < 2, 0 < \gamma < 2,$$

then for the first-time blow-up time $T^* < \infty$ of the solution (u, θ) to (1.1)–(1.4), we have that

$$\limsup_{t \nearrow T^*} \|(u, \theta)\|_{H^3}^2(t) = \infty.$$

Because of the boundedness of the Riesz transformation in $\dot{B}_{p,q}^r(\mathbb{R}^3)$, we may further replace ∇u in Corollary 1.2 by the vorticity $\omega = \nabla \times u$ and obtain the following result.

Corollary 1.3 *Let $(u_0, \theta_0) \in H^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. If the vorticity $\omega = \nabla \times u$ and the gradient of the temperature $\nabla \theta$ satisfy*

$$\|\omega\|_{L^{\frac{2}{2-s}}(0, T^*; \dot{B}_{\infty, \infty}^{-s})} + \|\nabla \theta\|_{L^{\frac{2}{2-\gamma}}(0, T^*; \dot{B}_{\infty, \infty}^{-\gamma})} = \infty, \quad 0 < s < 2, 0 < \gamma < 2,$$

then for the first-time blow-up time $T^* < \infty$ of the solution (u, θ) to (1.1)–(1.4), we have that

$$\limsup_{t \nearrow T^*} \|(u, \theta)\|_{H^3}^2(t) = \infty.$$

Theorem 1.4 *Let $(u_0, \theta_0) \in H^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. If the velocity field u and the temperature θ satisfy*

$$\|u\|_{L^{\frac{2}{1-s}}(0, T^*; \dot{B}_{\infty, \infty}^{-s})} + \|\theta\|_{L^{\frac{2}{1-\gamma}}(0, T^*; \dot{B}_{\infty, \infty}^{-\gamma})} = \infty, \quad -1 < s < 1, -1 < \gamma < 1,$$

then for the first-time blow-up time $T^* < \infty$ of the solution (u, θ) to (1.1)–(1.4), we have that

$$\limsup_{t \nearrow T^*} \|(u, \theta)\|_{H^3}^2(t) = \infty.$$

As an application of these results, we can easily deduce the following results.

Corollary 1.5 *Let $(u_0, \theta_0) \in H^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. If the gradient of the velocity field ∇u and the temperature θ satisfy*

$$\|\nabla u\|_{L^{\frac{2}{2-s}}(0, T^*; \dot{B}_{\infty, \infty}^{-s})} + \|\theta\|_{L^{\frac{2}{1-\gamma}}(0, T^*; \dot{B}_{\infty, \infty}^{-\gamma})} = \infty, \quad 0 < s < 2, -1 < \gamma < 1,$$

then for the first-time blow-up time $T^* < \infty$ of the solution (u, θ) to (1.1)–(1.4), we have that

$$\limsup_{t \nearrow T^*} \|(u, \theta)\|_{H^3}^2(t) = \infty.$$

Corollary 1.6 *Let $(u_0, \theta_0) \in H^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. If the vorticity $\omega = \nabla \times u$ and the temperature θ satisfy*

$$\|\omega\|_{L^{\frac{2}{2-s}}(0, T^*; \dot{B}_{\infty, \infty}^{-s})} + \|\theta\|_{L^{\frac{2}{1-\gamma}}(0, T^*; \dot{B}_{\infty, \infty}^{-\gamma})} = \infty, \quad 0 < s < 2, -1 < \gamma < 1,$$

then for the first-time blow-up time $T^* < \infty$ of the solution (u, θ) to (1.1)–(1.4), we have that

$$\limsup_{t \nearrow T^*} \|(u, \theta)\|_{H^3}^2(t) = \infty.$$

2 Preliminaries

Throughout this paper, the letter C denotes various positive and finite constants that may vary from line to line. We will use the notation $A \lesssim B$ to denote the relation $A \leq CB$ and the notation $A \approx B$ to denote the relations $A \lesssim B$ and $B \lesssim A$. For reader's convenience, we first recall some basic facts on Littlewood–Paley theory; we refer to [5] for more detail.

Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. The Fourier transform $\mathcal{F}f = \hat{f}$ of $f \in \mathcal{S}(\mathbb{R}^3)$ is defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) dx.$$

Choose two nonnegative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^3)$ supported in $\mathfrak{B} = \{\xi \in \mathbb{R}^3 : |\xi| \leq \frac{4}{3}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^3 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, respectively, such that

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^3.$$

Set $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ and let $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$. Then we define the frequency localization operators as follows:

$$\begin{aligned} \Delta_j f &= \varphi(2^{-j}D)f = 2^{3j} \int_{\mathbb{R}^3} h(2^j y) f(x-y) dy \quad \text{for } j \geq 0, \\ S_j f &= \chi(2^{-j}D)f = \sum_{-1 \leq k \leq j-1} \Delta_k f = 2^{3j} \int_{\mathbb{R}^3} \tilde{h}(2^j y) f(x-y) dy, \\ \Delta_{-1} f &= S_0 f, \quad \Delta_j f = 0 \quad \text{for } j \leq -2. \end{aligned}$$

Formally, $\Delta_j = S_{j+1} - S_j$ is a frequency projection into the annulus $\{|\xi| \approx 2^j\}$, whereas S_j is a frequency projection into the ball $\{|\xi| \lesssim 2^j\}$ for $j \in \mathbb{Z}$. We can easily verify that, with the our choice of φ ,

$$\Delta_j \Delta_k f \equiv 0 \quad \text{if } |j-k| \geq 2,$$

and

$$\Delta_j (S_{k-1} f \Delta_k f) \equiv 0 \quad \text{if } |j-k| \geq 5.$$

We now introduce the following definition of a homogeneous Besov space by means of Littlewood–Paley projection Δ_j and S_j .

Definition 2.1 Let $r \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. The homogeneous Besov space $\dot{B}_{p,q}^r(\mathbb{R}^3)$ is defined by the full dyadic decomposition

$$\dot{B}_{p,q}^r = \{f \in \mathcal{Z}'(\mathbb{R}^3); \|f\|_{\dot{B}_{p,q}^r} < \infty\},$$

where

$$\|f\|_{\dot{B}_{p,q}^r} = \left\| \left\{ 2^{jr} \|\Delta_j f\|_{L^p} \right\}_{j=-\infty}^{\infty} \right\|_{\ell^q}.$$

Let us point out that $\mathcal{Z}'(\mathbb{R}^3)$ is the dual space of

$$\mathcal{Z}(\mathbb{R}^3) = \{f \in \mathcal{S}(\mathbb{R}^3); D^\alpha \widehat{f}(0) = 0 \ \forall \alpha \in \mathbb{N}^3\}.$$

The following Bernstein inequality is due to [5, 11].

Lemma 2.1 *For all $k \in \mathbb{N} \cup \{0\}$, $j \in \mathbb{Z}$, and $1 \leq p \leq q \leq \infty$, we have, for all $f \in \mathcal{S}(\mathbb{R}^3)$,*

$$\begin{aligned} \text{(i)} \quad & \sup_{|\alpha|=k} \|\nabla^\alpha \Delta_j f\|_{L^q} \leq C_1 2^{jk+3j(\frac{1}{p}-\frac{1}{q})} \|\Delta_j f\|_{L^p}, \\ \text{(ii)} \quad & \|\Delta_j f\|_{L^p} \leq C_2 2^{-jk} \sup_{|\alpha|=k} \|\nabla^\alpha \Delta_j f\|_{L^p}, \end{aligned}$$

with positive constants C_1 and C_2 independent of f and j .

We conclude this section by recalling the refined interpolation inequality relating the Lebesgue and Besov spaces (see [2]), which will be frequently used in the proof of Theorem 1.1.

Lemma 2.2 *Let $1 \leq q < p < \infty$, and let α be a positive real number. There exists a constant C such that*

$$\|f\|_{L^p} \leq C \|f\|_{\dot{B}_{\infty,\infty}^{-\alpha}}^{1-\gamma} \|f\|_{\dot{B}_{q,q}^{\beta}}^{\gamma} \quad \text{with } \beta = \alpha \left(\frac{p}{q} - 1 \right) \text{ and } \gamma = \frac{q}{p}.$$

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We will complete the proof through three steps. First, we will get the basic energy inequality. Second, we will establish lower-order estimates of u and θ . Finally, we will obtain higher-order estimates of u and θ . To prove Theorem 1.1, we will apply a contradiction argument method. More explicitly, we need to show that if

$$\|u\|_{L^{\frac{2}{1-s}}(0,T^*;\dot{B}_{\infty,\infty}^{-s})} + \|\nabla \theta\|_{L^{\frac{2}{2-\gamma}}(0,T^*;\dot{B}_{\infty,\infty}^{-\gamma})} < \infty$$

for some $-1 < s < 1$ and $0 < \gamma < 2$, then we could conduct uniform bounds for $\|(u, \theta)\|_{H^m}(t)$ for $0 \leq t < T^*$, $m \geq 3$. Then we give a proof of Theorem 1.1. First, we will establish the following basic energy estimate for system (1.1)–(1.4).

Lemma 3.1 *Let (u_0, θ_0) satisfy the conditions of Theorem 1.1. Then for any corresponding solution (u, θ) of system (1.1)–(1.4), there exists a constant C such that, for almost every $t \geq 0$,*

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau + \kappa \int_0^t \|\nabla \theta(\tau)\|_{L^2}^2 d\tau \leq C, \quad (3.1)$$

where C is a constant depending on time t and the initial data.

Proof Multiplying (1.1) and (1.2) by u and θ , respectively, integrating them over \mathbb{R}^3 with respect to x , and then adding the resulting equations, we get

$$\frac{1}{2} \frac{d}{dt} (\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2) + \mu \|\nabla u\|_{L^2}^2 + \kappa \|\nabla \theta\|_{L^2}^2 \leq C(\|u\|_{L^2}^2 + \|\theta\|_{L^2}^2). \quad (3.2)$$

Integration of (3.1) with respect to time variable over $(0, t)$ directly implies the desired estimate (3.1). \square

Next, we will establish the global H^1 -estimation, which plays an essential role in proving our main theorem.

Lemma 3.2 *Let (u_0, θ_0) satisfy the conditions of Theorem 1.1. Then for any corresponding solution (u, θ) of the system (1.1)–(1.4), there exists a constant C such that, for any $0 \leq t < T^*$,*

$$\|\nabla u(t)\|_{L^2}^2 + \|\nabla \theta(t)\|_{L^2}^2 + \mu \int_0^t \|\Delta u(\tau)\|_{L^2}^2 d\tau + \kappa \int_0^t \|\Delta \theta(\tau)\|_{L^2}^2 d\tau \leq C < \infty, \quad (3.3)$$

where C is a constant depending on time t and the initial data.

Proof Taking the L^2 inner product to Eqs. (1.1) and (1.2) with $-\Delta u$ and $-\Delta \theta$, respectively, and summing up the resultants, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla u(t)\|_{L^2}^2 + \|\nabla \theta(t)\|_{L^2}^2) + \mu \|\Delta u\|_{L^2}^2 + \kappa \|\Delta \theta\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} [(u \cdot \nabla)u] \cdot \Delta u \, dx - \int_{\mathbb{R}^3} \theta e_3 \cdot \Delta u \, dx \\ & \quad + \int_{\mathbb{R}^3} [(u \cdot \nabla)\theta] \cdot \Delta \theta \, dx - \int_{\mathbb{R}^3} (u \cdot e_3) \cdot \Delta \theta \, dx \\ &:= I + II + III + IV. \end{aligned} \quad (3.4)$$

Using the incompressible condition, the Hölder inequality, the interpolation inequality, and Lemma 2.2, we estimate I and III as follows:

$$\begin{aligned} I &= - \sum_{i=1}^3 \int_{\mathbb{R}^3} [(\partial_i u \cdot \nabla)u] \cdot \partial_i u \, dx \leq C \|\nabla u\|_{L^3}^3 \\ &\leq C \|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1-s}} \|\nabla u\|_{\dot{H}^{\frac{s+1}{2}}}^2 \leq C \|u\|_{\dot{B}_{\infty,\infty}^{-s}} \|\nabla u\|_{L^2}^{1-s} \|\nabla u\|_{\dot{H}^1}^{1+s} \\ &\leq \frac{\mu}{4} \|\Delta u\|_{L^2}^2 + C \|u\|_{\dot{B}_{\infty,\infty}^{-s}}^{\frac{2}{1-s}} \|\nabla u\|_{L^2}^2, \\ III &= - \sum_{i=1}^3 \int_{\mathbb{R}^3} [(\partial_i u \cdot \nabla)\theta] \cdot \partial_i \theta \, dx \leq C \int_{\mathbb{R}^3} |\nabla u| \cdot |\nabla \theta|^2 \, dx \\ &\leq C \|\nabla u\|_{L^3} \|\nabla \theta\|_{L^3}^2 \leq C \|\nabla u\|_{L^3}^3 + C \|\nabla \theta\|_{L^3}^3 \\ &\leq C \|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1-s}} \|\nabla u\|_{\dot{H}^{\frac{s+1}{2}}}^2 + C \|\nabla \theta\|_{\dot{B}_{\infty,\infty}^{-\gamma}} \|\nabla \theta\|_{\dot{H}^{\frac{\gamma}{2}}}^2 \end{aligned} \quad (3.5)$$

$$\begin{aligned}
&\leq C\|u\|_{\dot{B}_{\infty,\infty}^{-s}}\|\nabla u\|_{L^2}^{1-s}\|\nabla u\|_{H^1}^{1+s} + C\|\nabla\theta\|_{\dot{B}_{\infty,\infty}^{-\gamma}}\|\nabla\theta\|_{L^2}^{2-\gamma}\|\nabla\theta\|_{H^1}^{\gamma} \\
&\leq \frac{\mu}{4}\|\Delta u\|_{L^2}^2 + \frac{\kappa}{2}\|\Delta\theta\|_{L^2}^2 + C\left(\|u\|_{\dot{B}_{\infty,\infty}^{-s}}^{\frac{2}{1-s}} + \|\nabla\theta\|_{\dot{B}_{\infty,\infty}^{-\gamma}}^{\frac{2}{2-\gamma}}\right)(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2). \quad (3.6)
\end{aligned}$$

By the integration-by-parts formula and Hölder inequality we easily observe that

$$II + IV \leq C(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2). \quad (3.7)$$

Putting estimates I–IV into (3.4) and absorbing the diffusive terms ensure that

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2) + \mu\|\Delta u\|_{L^2}^2 + \kappa\|\Delta\theta\|_{L^2}^2 \\
&\leq C\left(1 + \|u\|_{\dot{B}_{\infty,\infty}^{-s}}^{\frac{2}{1-s}} + \|\nabla\theta\|_{\dot{B}_{\infty,\infty}^{-\gamma}}^{\frac{2}{2-\gamma}}\right) \cdot (\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2). \quad (3.8)
\end{aligned}$$

Thanks to the Gronwall inequality, we show that, for any $0 \leq t < T^*$,

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2 + \mu \int_0^t \|\Delta u(\tau)\|_{L^2}^2 d\tau + \kappa \int_0^t \|\Delta\theta(\tau)\|_{L^2}^2 d\tau \\
&\leq (\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2) \exp\left\{CT^* + \int_0^{T^*} \left(\|u(\tau)\|_{\dot{B}_{\infty,\infty}^{-s}}^{\frac{2}{1-s}} + \|\nabla\theta(\tau)\|_{\dot{B}_{\infty,\infty}^{-\gamma}}^{\frac{2}{2-\gamma}}\right) d\tau\right\} \\
&\leq C < \infty. \quad (3.9)
\end{aligned}$$

Therefore we finally conclude the proof of Lemma 3.2. \square

With the help of Lemma 3.2, we establish the global H^3 -estimation.

Lemma 3.3 *Let (u_0, θ_0) satisfies the conditions of Theorem 1.1. Then for any corresponding solution (u, θ) of system (1.1)–(1.4), there exists a constant C such that, for any $0 \leq t < T^*$,*

$$\|u(t)\|_{\dot{H}^3}^2 + \|\theta(t)\|_{\dot{H}^3}^2 + \mu \int_0^t \|\nabla u(\tau)\|_{\dot{H}^3}^2 d\tau + \kappa \int_0^t \|\nabla\theta(\tau)\|_{\dot{H}^3}^2 d\tau \leq C < \infty, \quad (3.10)$$

where C is a constant depending on time t and the initial data.

Proof Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multiindex with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = 3$. Applying the operator $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}$, where $0 \leq \alpha \leq 3$, to equations (1.1) and (1.2) and then taking the L^2 inner product of the resulting equations with $D^\alpha u$ and $D^\alpha \theta$, after adding them, we obtain that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} (\|u(t)\|_{\dot{H}^3}^2 + \|\theta(t)\|_{\dot{H}^3}^2) + \mu \|\nabla u\|_{\dot{H}^3}^2 + \kappa \|\nabla\theta\|_{\dot{H}^3}^2 \\
&= - \sum_{|\alpha|=3} \int_{\mathbb{R}^3} D^\alpha [(u \cdot \nabla)u] \cdot D^\alpha u dx + \sum_{|\alpha|=3} \int_{\mathbb{R}^3} D^\alpha (\theta e_3) \cdot D^\alpha u dx \\
&\quad - \sum_{|\alpha|=3} \int_{\mathbb{R}^3} D^\alpha [(u \cdot \nabla)\theta] \cdot D^\alpha \theta dx + \sum_{|\alpha|=3} \int_{\mathbb{R}^3} D^\alpha (u \cdot e_3) \cdot D^\alpha \theta dx. \quad (3.11)
\end{aligned}$$

Due to the divergence-free condition $\nabla \cdot u = 0$, we can rewrite (3.11) as

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u(t)\|_{\dot{H}^3}^2 + \|\theta(t)\|_{\dot{H}^3}^2) + \mu \|\nabla u\|_{\dot{H}^3}^2 + \kappa \|\nabla \theta\|_{\dot{H}^3}^2 \\ &= \sum_{i=1}^3 \sum_{|\alpha|=3} \int_{\mathbb{R}^3} D^\alpha (u_i u) \cdot D^\alpha \partial_i u \, dx + \sum_{|\alpha|=3} \int_{\mathbb{R}^3} D^\alpha (\theta e_3) \cdot D^\alpha u \, dx \\ & \quad + \sum_{i=1}^3 \sum_{|\alpha|=3} \int_{\mathbb{R}^3} D^\alpha (u_i \theta) \cdot D^\alpha \partial_i \theta \, dx + \sum_{|\alpha|=3} \int_{\mathbb{R}^3} D^\alpha (u \cdot e_3) \cdot D^\alpha \theta \, dx \\ &:= I_1 + II_1 + III_1 + IV_1. \end{aligned} \quad (3.12)$$

Before bounding terms in (3.12), we first recalling the following useful lemma (see [10]).

Lemma 3.4 *If $r > 0$ and $1 < p < \infty$, then $W^{r,p} \cap L^\infty$ is an algebra. Moreover,*

$$\|fg\|_{W^{r,p}} \leq C(\|f\|_{L^\infty} \|g\|_{L^p} + \|f\|_{L^p} \|g\|_{L^\infty}).$$

With Lemma 3.4 at our disposal, we are now ready to bound terms in (3.12) as follows:

$$\begin{aligned} I_1 &\leq C \|u\|_{L^\infty} \|u\|_{\dot{H}^3} \|\nabla u\|_{\dot{H}^3} \leq C \|u\|_{L^\infty} (\|u\|_{L^2} + \|u\|_{\dot{H}^3}) \|\nabla u\|_{\dot{H}^3} \\ &\leq \frac{\mu}{4} \|\nabla u\|_{\dot{H}^3}^2 + C \|u\|_{L^\infty}^2 (1 + \|u\|_{\dot{H}^3}^2), \end{aligned} \quad (3.13)$$

$$\begin{aligned} III_1 &\leq C \|(u, \theta)\|_{L^\infty} \|(u, \theta)\|_{\dot{H}^3} \|(\nabla u, \nabla \theta)\|_{\dot{H}^3} \\ &\leq C \|(u, \theta)\|_{L^\infty} (\|(u, \theta)\|_{L^2} + \|(u, \theta)\|_{\dot{H}^3}) \|(\nabla u, \nabla \theta)\|_{\dot{H}^3} \\ &\leq \frac{\mu}{4} \|\nabla u\|_{\dot{H}^3}^2 + \frac{\kappa}{2} \|\nabla \theta\|_{\dot{H}^3}^2 + C \|(u, \theta)\|_{L^\infty}^2 (1 + \|(u, \theta)\|_{\dot{H}^3}^2), \end{aligned} \quad (3.14)$$

$$II_1 + IV_1 \leq C (\|u\|_{\dot{H}^3}^2 + \|\theta\|_{\dot{H}^3}^2). \quad (3.15)$$

Substituting all the preceding estimates into (3.12) directly gives

$$\begin{aligned} & \frac{d}{dt} (\|u(t)\|_{\dot{H}^3}^2 + \|\theta(t)\|_{\dot{H}^3}^2) + \mu \|\nabla u\|_{\dot{H}^3}^2 + \kappa \|\nabla \theta\|_{\dot{H}^3}^2 \\ & \leq C (1 + \|(u, \theta)\|_{L^\infty}^2) (1 + \|(u, \theta)\|_{\dot{H}^3}^2). \end{aligned} \quad (3.16)$$

A combination of this fact with the Gronwall inequality leads to the desired global bound in Lemma 3.3. We can also obtain

$$\|(u, \theta)\|_{H^m}(t) \leq C, \quad t \in [0, T^*), m \geq 3.$$

This completes the proof of Theorem 1.1. \square

4 The proof of Theorem 1.4

Due to the similarity to the proof in the previous section, the proof of Theorem 1.4 is much simpler. In fact, it is sufficient to change estimate (3.6) as follows:

$$\begin{aligned}
 III &\leq C\|\nabla u\|_{L^3}^3 + C\|\nabla \theta\|_{L^3}^3 \\
 &\leq C\|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1-s}}^2 \|\nabla u\|_{\dot{H}^{\frac{s+1}{2}}}^2 + C\|\nabla \theta\|_{\dot{B}_{\infty,\infty}^{-1-\gamma}}^2 \|\nabla \theta\|_{\dot{H}^{\frac{\gamma+1}{2}}}^2 \\
 &\leq C\|u\|_{\dot{B}_{\infty,\infty}^{-s}} \|\nabla u\|_{L^2}^{1-s} \|\nabla u\|_{\dot{H}^1}^{1+s} + C\|\theta\|_{\dot{B}_{\infty,\infty}^{-\gamma}} \|\nabla \theta\|_{L^2}^{1-\gamma} \|\nabla \theta\|_{\dot{H}^1}^{1+\gamma} \\
 &\leq \frac{\mu}{4} \|\Delta u\|_{L^2}^2 + \frac{\kappa}{2} \|\Delta \theta\|_{L^2}^2 + C\left(\|u\|_{\dot{B}_{\infty,\infty}^{-s}}^{\frac{2}{1-s}} + \|\theta\|_{\dot{B}_{\infty,\infty}^{-\gamma}}^{\frac{2}{1-\gamma}}\right) (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2). \quad (4.1)
 \end{aligned}$$

5 Results and discussion

In this paper, we consider the 3D Bénard system and investigate the regularity criteria for it. We prove that if the velocity field u and the temperature θ belong to some critical Besov space, then the solution (u, θ) can be extended smoothly beyond T .

6 Conclusions

In this paper, we show that if the velocity field u and the gradient of the temperature $\nabla \theta$ satisfy

$$\|u\|_{L^{\frac{2}{1-s}}(0,T^*; \dot{B}_{\infty,\infty}^{-s})} + \|\nabla \theta\|_{L^{\frac{2}{2-\gamma}}(0,T^*; \dot{B}_{\infty,\infty}^{-\gamma})} = \infty, \quad -1 < s < 1, 0 < \gamma < 2,$$

or the velocity field u and the temperature θ satisfy

$$\|u\|_{L^{\frac{2}{1-s}}(0,T^*; \dot{B}_{\infty,\infty}^{-s})} + \|\theta\|_{L^{\frac{2}{1-\gamma}}(0,T^*; \dot{B}_{\infty,\infty}^{-\gamma})} = \infty, \quad -1 < s < 1, -1 < \gamma < 1,$$

then for the first-time blow-up time $T^* < \infty$ of the solution (u, θ) , we have that

$$\limsup_{t \nearrow T^*} \|(u, \theta)\|_{H^3}^2(t) = \infty.$$

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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