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Notes on the uniqueness of meromorphic functions concerning differential polynomials

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Abstract

This paper is devoted to the uniqueness problem on meromorphic functions whose differential polynomials share one nonzero finite constant. We improve some previous results and answer two open problems posed by Dyavanal.

MSC: 30D35

Keywords: Differential polynomial; Meromorphic function; Sharing value; Uniqueness

1 Introduction

We assume that the reader is familiar with the usual notation and basic results of the Nevanlinna theory [4, 10]. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let a be a complex number. We say that $f(z)$ and $g(z)$ share a CM (IM) provided that $f(z) - a$ and $g(z) - a$ have the same zeros counting multiplicity (ignoring multiplicity). In addition, f and g sharing ∞ CM (IM) means that f and g have the same poles counting multiplicity (ignoring multiplicity).

The uniqueness theory of meromorphic functions mainly studies conditions under which there is a unique function satisfying the given hypothesis. A great deal of classical results in this field can be seen in [10], where Chap. 9 introduces many works dealing with the relation between two meromorphic functions while their derivatives share values. Over past two decades, the research on the derivatives of polynomials of meromorphic functions sharing values has been ongoing. In 1996, Fang and Hua [3] investigated the relation between two transcendental entire functions f and g when f^nf' and g^ng' share 1 CM. Clearly, $(f^{n+1})' = (n+1)f^nf'$. Later, Yang and Hua [9] considered this problem for meromorphic functions f and g, and they proved the following theorem.

Theorem A *Let f and g be two nonconstant meromorphic functions, let $n \geq 11$ be an integer, and let $a \in \mathbb{C} \setminus \{0\}$. If f^nf' and g^ng' share a CM, then $f(z) \equiv dg(z)$ for some $(n+1)$th roots of unity d, or $g(z) = c_1e^{cz}$ and $f(z) = c_2e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -a^2$.*

Without loss of generality, in Theorem A the complex number a can be replaced by 1. Noting that $(\frac{1}{n+2}f^{n+2} - \frac{1}{n+1}f^{n+1})' = f^n(f-1)f'$, Fang and Hong [2] obtained the following result.

Theorem B *Let f and g be two transcendental entire functions, let $n \geq 11$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f(z) \equiv g(z)$.*

Three years later, Lin and Yi [6] improved their result to $n \geq 7$ and also studied the case that f and g are meromorphic functions. Moreover, they discussed the other polynomial $\frac{1}{n+3}f^{n+3} - \frac{2}{n+2}f^{n+2} + \frac{1}{n+1}f^{n+1}$ of f with its derivative as $f^n(f-1)^2f'$. In fact, Lin and Yi proved the following two theorems.

Theorem C *Let f and g be two nonconstant meromorphic functions, and let $n \geq 12$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then*

$$f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \qquad g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \tag{1.1}$$

where h is a nonconstant meromorphic function.

Theorem D *Let f and g be two nonconstant meromorphic functions, and let $n \geq 13$ be an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share 1 CM, then $f(z) \equiv g(z)$.*

Recently, by introducing the notion of multiplicity, Dyavanal [1] deeply investigated such a uniqueness problem and improved Theorems A, C, and D as follows.

Theorem E *Let f and g be two nonconstant meromorphic functions with zeros and poles of multiplicities at least s, where s is a positive integer. Let $n \geq 2$ be an integer satisfying $(n+1)s \geq 12$. If f^nf' and g^ng' share 1 CM, then $f(z) \equiv dg(z)$ for some $(n+1)$th roots of unity d, or $g(z) = c_1e^{cz}$ and $f(z) = c_2e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.*

Theorem F *Let f and g be two nonconstant meromorphic functions with zeros and poles of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n-2)s \geq 10$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then (1.1) holds.*

Theorem G *Let f and g be two nonconstant meromorphic functions with zeros and poles of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n-3)s \geq 10$. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share 1 CM, then $f(z) \equiv g(z)$.*

In Theorem F, if $f(z) \not\equiv g(z)$, then f, g must satisfy (1.1), so that

$$f = \frac{(n+2)h(h-\beta_1)(h-\beta_2)\cdots(h-\beta_n)}{(n+1)(h-\alpha_1)(h-\alpha_2)\cdots(h-\alpha_{n+1})}, \tag{1.2}$$

where α_i ($\neq 1$) ($i = 1,2,\ldots,n+1$) and β_j ($\neq 1$) ($j = 1,2,\ldots,n$) are distinct roots of $w^{n+2} = 1$ and $w^{n+1} = 1$, respectively. Thus by Valiron–Mokhon'ko theorem (see [10, Thm. 1.13]) $T(r,f) = (n+1)T(r,h) + S(r,h)$. From (1.2) it follows that the poles of h are not poles of f and

$$\overline{N}(r,f) = \sum_{i=1}^{n+1}\overline{N}\left(r, \frac{1}{h-\alpha_i}\right), \qquad \overline{N}\left(r, \frac{1}{f}\right) = \sum_{j=1}^{n}\overline{N}\left(r, \frac{1}{h-\beta_j}\right) + \overline{N}\left(r, \frac{1}{h}\right).$$

By the second main theorem we have

$$
\begin{aligned}
2nT(r,h) &\le \sum_{i=1}^{n+1} \overline{N}\left(r, \frac{1}{h-\alpha_i}\right) + \sum_{j=1}^{n} \overline{N}\left(r, \frac{1}{h-\beta_j}\right) + \overline{N}\left(r, \frac{1}{h}\right) + S(r,h) \\
&\le \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + S(r,h) \\
&\le \frac{1}{s} N(r,f) + \frac{1}{s} N\left(r, \frac{1}{f}\right) + S(r,h) \\
&\le \frac{2}{s} T(r,f) + S(r,h),
\end{aligned}
$$

which leads to $n \le (n+1)/s$. From $(n-2)s \ge 10$ we have $n \ge 3$. According to the above argument, we can deduce a contradiction for $s \ge 2$. Therefore, in Theorem F, if $s \ge 2$, then we must have $f \equiv g$.

In the end of his paper, Dyavanal posed four open problems. Two of them, which we are interested in, are as follows.

Problem 1 Can a CM shared value be replaced by an IM shared value in Theorems E–G?

Problem 2 Are the conditions $(n+1)s \ge 12$ in Theorem E, $(n-2)s \ge 10$ in Theorem F, and $(n-3)s \ge 10$ in Theorem G sharp?

In this paper, we try to answer these two questions. We obtain five theorems, which replace CM by IM in Theorems E–G and reduce n for $s \ge 7$ in Theorems F–G in Sect. 3.

2 Preliminary lemmas

We denote by $\overline{N}_{(k}(r, \frac{1}{f-a})$ the reduced counting function for zeros of $f-a$ with multiplicity no less than k. Define

$$
N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).
$$

Lemma 2.1 (see [12, Lemma 2.1]) *Let $f(z)$ be a nonconstant meromorphic function, and let p and k be positive integers. Then*

$$
N_p\left(r, \frac{1}{f^{(k)}}\right) \le N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f). \tag{2.1}
$$

This lemma can be proved in the same way as [5, Lemma 2.3] in the particular case $p = 2$.

Lemma 2.2 (see [9, 11]) *Let f and g be two nonconstant meromorphic functions sharing* 1 *CM. Then we have one of the following three cases*:

(i) $T(r,f) \le N_2(r,1/f) + N_2(r,1/g) + N_2(r,f) + N_2(r,g) + S(r,f) + S(r,g)$;

(ii) $f(z) \equiv g(z)$;

(iii) $f(z)g(z) \equiv 1$.

Lemma 2.3 *Let f and g be two nonconstant meromorphic functions. If f and g share* 1 *IM, then we have one of the following three cases*:

(i) $T(r,f) \le N_2(r,1/f) + N_2(r,1/g) + N_2(r,f) + N_2(r,g) + 2\overline{N}(r,f) + \overline{N}(r,g) + 2\overline{N}(r,1/f) + \overline{N}(r,1/g) + S(r,f) + S(r,g)$;
(ii) $f(z) \equiv g(z)$;
(iii) $f(z)g(z) \equiv 1$.

Proof We first introduce some new notation. Let z_0 be a zero of $f-1$ with multiplicity p and a zero of $g-1$ with multiplicity q. We denote by $N_E^{1)}(r,\frac{1}{f-1})$ the counting function of the zeros of $f-1$ with $p=q=1$, by $\overline{N}_E^{(2}(r,\frac{1}{f-1})$ the counting function of the zeros of $f-1$ satisfying $p=q\ge 2$, and by $\overline{N}_L(r,\frac{1}{f-1})$ the counting function of the zeros of $f-1$ with $p>q\ge 1$, where each point in these counting functions is counted only once.

We set

$$H(z) = \left(\frac{f''}{f'} - 2\frac{f'}{f-1}\right) - \left(\frac{g''}{g'} - 2\frac{g'}{g-1}\right). \tag{2.2}$$

Suppose that $H(z) \not\equiv 0$. Clearly, $m(r,H) = S(r,f) + S(r,g)$. If z_0 is a common simple zero of $f-1$ and $g-1$, then a simple computation on local expansions shows that $H(z_0) = 0$, and then

$$N_E^{1)}\left(r,\frac{1}{f-1}\right) \le N\left(r,\frac{1}{H}\right) \le N(r,H) + S(r,f) + S(r,g). \tag{2.3}$$

The poles of $H(z)$ only come from the zeros of f' and g', the multiple poles of f and g, and the zeros of $f-1$ and $g-1$ with different multiplicity. By analysis we can deduce that

$$\begin{aligned} N(r,H) \le{}& \overline{N}_{(2}(r,f) + \overline{N}_{(2}(r,g) + \overline{N}_{(2}\left(r,\frac{1}{f}\right) + \overline{N}_{(2}\left(r,\frac{1}{g}\right) \\ &+ \overline{N}_L\left(r,\frac{1}{f-1}\right) + \overline{N}_L\left(r,\frac{1}{g-1}\right) \\ &+ \overline{N}_0\left(r,\frac{1}{f'}\right) + \overline{N}_0\left(r,\frac{1}{g'}\right) + S(r,f) + S(r,g), \end{aligned} \tag{2.4}$$

where $N_0(r,\frac{1}{f'})$ denotes the counting function of the zeros of f' but not that of $f(f-1)$, $\overline{N}_0(r,\frac{1}{f'})$ denotes the corresponding reduced counting function, and $N_0(r,\frac{1}{g'})$ and $\overline{N}_0(r,\frac{1}{g'})$ are defined similarly. At the same time, obviously,

$$\overline{N}\left(r,\frac{1}{f-1}\right) = N_E^{1)}\left(r,\frac{1}{f-1}\right) + \overline{N}_E^{(2}\left(r,\frac{1}{f-1}\right) + \overline{N}_L\left(r,\frac{1}{f-1}\right) + \overline{N}_L\left(r,\frac{1}{g-1}\right).$$

Combining this with (2.3) and (2.4) yields

$$\begin{aligned} \overline{N}\left(r,\frac{1}{f-1}\right) \le{}& \overline{N}_{(2}(r,f) + \overline{N}_{(2}(r,g) + \overline{N}_{(2}\left(r,\frac{1}{f}\right) + \overline{N}_{(2}\left(r,\frac{1}{g}\right) \\ &+ 2\overline{N}_L\left(r,\frac{1}{f-1}\right) + 2\overline{N}_L\left(r,\frac{1}{g-1}\right) + \overline{N}_0\left(r,\frac{1}{f'}\right) \\ &+ \overline{N}_0\left(r,\frac{1}{g'}\right) + \overline{N}_E^{(2}\left(r,\frac{1}{f-1}\right) + S(r,f) + S(r,g). \end{aligned} \tag{2.5}$$

Since

$$\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}_L\left(r,\frac{1}{g-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{f-1}\right)\le N\left(r,\frac{1}{g-1}\right)\le T(r,g)+S(r,g),$$

combining this with (2.5), we have

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)\\
&\quad\le \overline{N}_{(2}(r,f)+\overline{N}_{(2}(r,g)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)+\overline{N}_{(2}\left(r,\frac{1}{g}\right)+2\overline{N}_L\left(r,\frac{1}{f-1}\right)\\
&\qquad+\overline{N}_L\left(r,\frac{1}{g-1}\right)+\overline{N}_0\left(r,\frac{1}{f'}\right)+\overline{N}_0\left(r,\frac{1}{g'}\right)+T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

We apply the second fundamental theorem to f and g and consider the above inequality. Then

$$\begin{aligned}
T(r,f)+T(r,g)&\le \overline{N}(r,f)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)\\
&\quad+\overline{N}\left(r,\frac{1}{g-1}\right)-N_0\left(r,\frac{1}{f'}\right)-N_0\left(r,\frac{1}{g'}\right)+S(r,f)+S(r,g)\\
&\le N_2(r,f)+N_2(r,g)+N_2\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{g}\right)+2\overline{N}_L\left(r,\frac{1}{f-1}\right)\\
&\quad+T(r,g)+\overline{N}_L\left(r,\frac{1}{g-1}\right)+S(r,f)+S(r,g).
\end{aligned}$$

Clearly, this leads to

$$\begin{aligned}
T(r,f)&\le N_2(r,f)+N_2(r,g)+N_2\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{g}\right)+2\overline{N}_L\left(r,\frac{1}{f-1}\right)\\
&\quad+\overline{N}_L\left(r,\frac{1}{g-1}\right)+S(r,f)+S(r,g).
\end{aligned}\tag{2.6}$$

By Lemma 2.1 we have

$$\overline{N}_L\left(r,\frac{1}{f-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)\le \overline{N}\left(r,\frac{1}{f'}\right)\le N_2\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f).$$

Then, using this inequality, we get

$$\begin{aligned}
2\overline{N}_L\left(r,\frac{1}{f-1}\right)+N_2\left(r,\frac{1}{f}\right)&\le 2N_2\left(r,\frac{1}{f}\right)+N_{1)}\left(r,\frac{1}{f}\right)+2\overline{N}(r,f)+S(r,f)\\
&\le N_2\left(r,\frac{1}{f}\right)+2\overline{N}\left(r,\frac{1}{f}\right)+2\overline{N}(r,f)+S(r,f),
\end{aligned}\tag{2.7}$$

where $N_{1)}(r,\frac{1}{f})$ denotes the counting function of simple zeros of f. Similarly, we obtain

$$\overline{N}_L\left(r,\frac{1}{g-1}\right)+N_2\left(r,\frac{1}{g}\right)\le N_2\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,g)+S(r,g).\tag{2.8}$$

Substituting (2.7) and (2.8) into (2.6), this yields Case (i).

It remains to treat the case $H(z) \equiv 0$. Integrating twice results in

$$\frac{1}{f-1} = A\frac{1}{g-1} + B, \tag{2.9}$$

where $A \neq 0$ and B are two constants. If now $B \neq 0, -1$, then we rewrite (2.9) as

$$A\frac{1}{g-1} = -\frac{B(f - \frac{1+B}{B})}{f-1},$$

and then

$$\overline{N}\left(r, \frac{1}{f - \frac{1+B}{B}}\right) = \overline{N}(r,g).$$

By the second fundamental theorem we obtain

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f - \frac{1+B}{B}}\right) + S(r,f) \\ &= \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r,g) + S(r,f), \end{aligned}$$

which leads to Case (i). A similar reasoning results in Case (i) again, unless either $A = 1$ and $B = 0$ or $A = -1$ and $B = -1$. Hence, if $A = 1$ and $B = 0$, then $f \equiv g$, that is, Case (ii). If $A = -1$ and $B = -1$, then $f \cdot g \equiv 1$, which is Case (iii). □

When meromorphic functions f_1 and f_2 share 1 IM, Sun and Xu [8] once obtained a result, whose proof can be also found in [7]. They proved that $f_1 \equiv f_2$ or $f_1 f_2 \equiv 1$ if

$$\limsup_{r\to\infty, r\notin E} \frac{\overline{N}(r,f_j) + \overline{N}(r,\frac{1}{f_j})}{T(r,f_j)} < \frac{1}{7}, \quad j = 1,2,$$

where E is a set of finite linear measure. By Lemma 2.3, when

$$\limsup_{r\to\infty, r\notin E} \frac{2N_2(r,f_j) + 3\overline{N}(r,f_j) + 2N_2(r,\frac{1}{f_j}) + 3\overline{N}(r,\frac{1}{f_j})}{T(r,f_j)} < 1, \quad j = 1,2,$$

Case (i) cannot happen, and thus $f_1 \equiv f_2$ or $f_1 f_2 \equiv 1$. Since $N_2(r,f) \leq 2\overline{N}(r,f)$ and $N_2(r,1/f) \leq 2\overline{N}(r,1/f)$, Lemma 2.3 is an improvement of Sun and Xu's result.

3 Main results

Based on Problems 1 and 2 in Sect. 1, we introduce our main results.

Theorem 3.1 *Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s, where s is a positive integer. Let $n \geq 2$ be an integer satisfying $(n-4)s \geq 19$ for $s = 1,2$ and $ns \geq 28$ for $s \geq 3$. If $f^n f'$ and $g^n g'$ share 1 IM, then $f(z) \equiv dg(z)$ for some $(n+1)$th root d of unity, or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

Proof Let $F = \frac{1}{n+1}f^{n+1}$ and $G = \frac{1}{n+1}g^{n+1}$. Then $T(r,F) = (n+1)T(r,f)$, $T(r,G) = (n+1)T(r,g)$, and F', G' share 1 IM. Suppose first that Case (i) of Lemma 2.3 holds. From this we have

$$\begin{aligned} T(r,F') &\le N_2\left(r,\frac{1}{F'}\right) + N_2\left(r,\frac{1}{G'}\right) + N_2(r,F') + N_2(r,G') + 2\overline{N}(r,F') + \overline{N}(r,G') \\ &\quad + 2\overline{N}\left(r,\frac{1}{F'}\right) + \overline{N}\left(r,\frac{1}{G'}\right) + S(r,f) + S(r,g) \\ &\le 4\overline{N}\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{f'}\right) + 3\overline{N}\left(r,\frac{1}{g}\right) + N_2\left(r,\frac{1}{g'}\right) + 4\overline{N}(r,f) + 3\overline{N}(r,g) \\ &\quad + 2\overline{N}\left(r,\frac{1}{f'}\right) + \overline{N}\left(r,\frac{1}{g'}\right) + S(r,f) + S(r,g). \end{aligned} \tag{3.1}$$

At the same time, we have

$$\begin{aligned} T(r,F) &\le T(r,F') + N\left(r,\frac{1}{F}\right) - N\left(r,\frac{1}{F'}\right) + S(r,f) \\ &\le T(r,F') + N\left(r,\frac{1}{f}\right) - N\left(r,\frac{1}{f'}\right) + S(r,f). \end{aligned}$$

Then from this inequality and (3.1) it follows that

$$\begin{aligned} T(r,F) &\le 4\overline{N}\left(r,\frac{1}{f}\right) + 3\overline{N}\left(r,\frac{1}{g}\right) + N_2\left(r,\frac{1}{g'}\right) + 4\overline{N}(r,f) + 3\overline{N}(r,g) \\ &\quad + 2\overline{N}\left(r,\frac{1}{f'}\right) + \overline{N}\left(r,\frac{1}{g'}\right) + N\left(r,\frac{1}{f}\right) + S(r,f) + S(r,g). \end{aligned} \tag{3.2}$$

Using Lemma 2.1, we get

$$\begin{aligned} N_2\left(r,\frac{1}{g'}\right) + \overline{N}\left(r,\frac{1}{g'}\right) &\le 2N\left(r,\frac{1}{g}\right) + 2\overline{N}(r,g) + S(r,g) \\ &\le 2\left(1+\frac{1}{s}\right)T(r,g) + S(r,g), \end{aligned} \tag{3.3}$$

$$2\overline{N}\left(r,\frac{1}{f'}\right) \le 2N\left(r,\frac{1}{f}\right) + 2\overline{N}(r,f) + S(r,f) \le 2\left(1+\frac{1}{s}\right)T(r,f) + S(r,f). \tag{3.4}$$

Then substituting (3.3) and (3.4) into (3.2) yields

$$T(r,F) \le \left(3+\frac{10}{s}\right)T(r,f) + \left(2+\frac{8}{s}\right)T(r,g) + S(r,f) + S(r,g). \tag{3.5}$$

A similar inequality for G also holds. Therefore we can conclude that

$$\begin{aligned} (n+1)\{T(r,f)+T(r,g)\} &= T(r,F) + T(r,G) \\ &\le \left(5+\frac{18}{s}\right)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g), \end{aligned}$$

which contradicts the condition $(n-4)s \ge 19$ for $s = 1,2$.

Again using Lemma 2.1, we have

$$N_2\left(r,\frac{1}{g'}\right)+\overline{N}\left(r,\frac{1}{g'}\right)\le N_3\left(r,\frac{1}{g}\right)+N_2\left(r,\frac{1}{g}\right)+2\overline{N}(r,g)+S(r,g)$$
$$\le \frac{7}{s}T(r,g)+S(r,g), \tag{3.6}$$

$$2\overline{N}\left(r,\frac{1}{f'}\right)\le 2N_2\left(r,\frac{1}{f}\right)+2\overline{N}(r,f)+S(r,f)\le \frac{6}{s}T(r,f)+S(r,f). \tag{3.7}$$

Then substituting the two inequalities into (3.2) leads to

$$T(r,F)\le\left(1+\frac{14}{s}\right)T(r,f)+\frac{13}{s}T(r,g)+S(r,f)+S(r,g). \tag{3.8}$$

Similarly, we can get

$$(n+1)\left\{T(r,f)+T(r,g)\right\}\le\left(1+\frac{27}{s}\right)\left\{T(r,f)+T(r,g)\right\}+S(r,f)+S(r,g),$$

which contradicts the condition $ns\ge 28$ for $s\ge 3$.

Thus by Lemma 2.3 there we must have $F'G'\equiv 1$ or $F'\equiv G'$. Consider case $F'G'\equiv 1$, that is $f^nf'g^ng'\equiv 1$. Suppose that f has a pole z_0 with multiplicity p. Then z_0 must be a zero of g of order q satisfying $nq+q-1=np+p+1$. We rewrite it as $(q-p)(n+1)=2$, which is a contradiction since $n\ge 2$. Similarly to [9], we get $g=c_1e^{cz}, f=c_2e^{-cz}$. For the case $F'\equiv G'$, it is easy to see that $F\equiv G+c$, where c is a constant, so that $T(r,f)=T(r,g)+S(r,g)$. If $c\ne 0$, then

$$\overline{N}\left(r,\frac{1}{G-c}\right)=\overline{N}\left(r,\frac{1}{F}\right)=\frac{1}{s}T(r,f)+S(r,f)=\frac{1}{s}T(r,g)+S(r,g).$$

Applying the second main theorem to G, we have

$$T(r,G)\le\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G-c}\right)+S(r,g)\le\frac{3}{s}T(r,g)+S(r,g),$$

which leads to $(n+1)s\le 3$. This contradicts the condition on n and s. Therefore, $c=0$, and thus $F\equiv G$, that is, $f^{n+1}=g^{n+1}$. Hence $f\equiv dg$ for some $(n+1)$th root d of unity. □

Theorem 3.2 *Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s. Suppose that $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 IM, where s and n are positive integers. Then we have one of the following two cases*:

(i) *if $s=1$ and $n\ge 27$, then $f(z)\equiv g(z)$, or we have* (1.1);

(ii) *if $(n-8)s\ge 19$ for $s=2$ and $(n-4)s\ge 28$ for $s\ge 5$, then $f(z)\equiv g(z)$.*

Proof Let $F=\frac{1}{n+2}f^{n+2}-\frac{1}{n+1}f^{n+1}$ and $G=\frac{1}{n+2}g^{n+2}-\frac{1}{n+1}g^{n+1}$. Then F' and G' share 1 IM, and by the Valiron–Mokhon'ko theorem we have

$$T(r,F)=(n+2)T(r,f)+S(r,f),\qquad T(r,G)=(n+2)T(r,g)+S(r,g). \tag{3.9}$$

Suppose now that Case (i) of Lemma 2.3 holds. Then we have

$$\begin{aligned} T(r,F') \le{}& 4\overline{N}\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{f-1}\right) + N_2\left(r,\frac{1}{f'}\right) + 3\overline{N}\left(r,\frac{1}{g}\right) + N_2\left(r,\frac{1}{g-1}\right) \\ &+ N_2\left(r,\frac{1}{g'}\right) + 2\overline{N}\left(r,\frac{1}{f-1}\right) + 2\overline{N}\left(r,\frac{1}{f'}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) + \overline{N}\left(r,\frac{1}{g'}\right) \\ &+ 4\overline{N}(r,f) + 3\overline{N}(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{3.10}$$

Since $T(r,F) \le T(r,F') + N(r,1/F) - N(r,1/F') + S(r,f)$, we get

$$\begin{aligned} T(r,F) \le{}& T(r,F') + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-(n+2)/(n+1)}\right) \\ &- N\left(r,\frac{1}{f-1}\right) - N\left(r,\frac{1}{f'}\right) + S(r,f). \end{aligned}$$

Combining this inequality with (3.10) leads to

$$\begin{aligned} T(r,F) \le{}& 4\overline{N}\left(r,\frac{1}{f}\right) + 3\overline{N}\left(r,\frac{1}{g}\right) + N_2\left(r,\frac{1}{g-1}\right) + N_2\left(r,\frac{1}{g'}\right) + 2\overline{N}\left(r,\frac{1}{f-1}\right) \\ &+ 2\overline{N}\left(r,\frac{1}{f'}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) + \overline{N}\left(r,\frac{1}{g'}\right) + 4\overline{N}(r,f) + 3\overline{N}(r,g) + N\left(r,\frac{1}{f}\right) \\ &+ N\left(r,\frac{1}{f-(n+2)/(n+1)}\right) + S(r,f) + S(r,g). \end{aligned} \tag{3.11}$$

If we use (3.3) and (3.4), then (3.11) means

$$T(r,F) \le \left(6 + \frac{10}{s}\right)T(r,f) + \left(4 + \frac{8}{s}\right)T(r,g) + S(r,f) + S(r,g).$$

Then this yields

$$(n+2)\bigl\{T(r,f) + T(r,g)\bigr\} \le \left(10 + \frac{18}{s}\right)\bigl\{T(r,f) + T(r,g)\bigr\} + S(r,f) + S(r,g),$$

which contradicts to $(n-8)s \ge 19$ for $s = 1,2$. If we use (3.6) and (3.7), then (3.11) implies

$$T(r,F) \le \left(4 + \frac{14}{s}\right)T(r,f) + \left(2 + \frac{13}{s}\right)T(r,g) + S(r,f) + S(r,g).$$

Similarly as before, we conclude that

$$(n+2)\bigl\{T(r,f) + T(r,g)\bigr\} \le \left(6 + \frac{28}{s}\right)\bigl\{T(r,f) + T(r,g)\bigr\} + S(r,f) + S(r,g),$$

which contradicts with $(n-4)s \ge 28$ when $s \ge 3$.

Thus, by Lemma 2.3, $F'G' \equiv 1$ or $F' \equiv G'$. Consider the case $F'G' \equiv 1$, that is,

$$f^n(f-1)f'g^n(g-1)g' \equiv 1. \tag{3.12}$$

Let z_0 be a zero of f with multiplicity p_0. Then z_0 must be a pole of g of order q_0 satisfying

$$np_0 + p_0 - 1 = nq_0 + 2q_0 + 1.$$

We rewrite it as $(n+1)(p_0 - q_0) = q_0 + 2$, which implies $p_0 \geq q_0 + 1$ and $q_0 + 2 \geq n + 1$, so that $p_0 \geq t = \max\{n, s+1\}$. Let z_1 be a zero of $f - 1$ with multiplicity p_1. Then by (3.12) z_1 must be a pole of g of order q_1 satisfying

$$2p_1 - 1 = nq_1 + 2q_1 + 1.$$

Rewrite it as $p_1 = 1 + (n+2)q_1/2$, so that $p_1 \geq 1 + (n+2)s/2$. Again from (3.12) we have

$$\begin{aligned}\overline{N}(r,f) &= \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) + \overline{N}_0\left(r,\frac{1}{g'}\right)\\ &\leq \frac{1}{t}N\left(r,\frac{1}{g}\right) + \frac{2}{(n+2)s+2}N\left(r,\frac{1}{g-1}\right) + N_0\left(r,\frac{1}{g'}\right).\end{aligned}$$

By the second main theorem we obtain

$$\begin{aligned}T(r,f) &\leq \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right) - N_0\left(r,\frac{1}{f'}\right) + S(r,f)\\ &\leq \frac{1}{t}N\left(r,\frac{1}{f}\right) + \frac{1}{t}N\left(r,\frac{1}{g}\right) + \frac{2}{(n+2)s+2}N\left(r,\frac{1}{f-1}\right) + S(r,f)\\ &\quad + \frac{2}{(n+2)s+2}N\left(r,\frac{1}{g-1}\right) + N_0\left(r,\frac{1}{g'}\right) - N_0\left(r,\frac{1}{f'}\right)\end{aligned} \tag{3.13}$$

and a similar inequality for $T(r,g)$. Combining the two inequalities, we get

$$T(r,f) + T(r,g) \leq \left(\frac{2}{t} + \frac{4}{(n+2)s+2}\right)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \tag{3.14}$$

Since $(n-8)s \geq 19$ for $s = 1, 2$ and $(n-4)s \geq 28$ for $s \geq 3$, we have

$$\frac{1}{t} \leq \frac{1}{4}, \qquad \frac{1}{(n+2)s+2} \leq \frac{1}{21+6s} \leq \frac{1}{27}.$$

Thus (3.14) leads to a contradiction. Similarly as in the proof of Theorem 3.1, $F' \equiv G'$ means that $F \equiv G$. Let $h \equiv f/g$. If $h \not\equiv 1$, then $F \equiv G$ implies (1.1). As we pointed out in Sect. 1, (1.1) leads to a contradiction for $s \geq 2$. Hence, when $s \geq 2$, we must have $f(z) \equiv g(z)$. For $s = 1$, $f(z) \equiv g(z)$, or (1.1) holds. □

Theorem 3.3 *Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s. Suppose that $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share 1 IM, where s and n are positive integers. If $(n-9)s \geq 19$ for $s = 1, 2$ and $(n-5)s \geq 28$ for $s \geq 3$, then $f(z) \equiv g(z)$.*

Proof Let $F = \frac{1}{n+3}f^{n+3} - \frac{2}{n+2}f^{n+2} + \frac{1}{n+1}f^{n+1}$ and $G = \frac{1}{n+3}g^{n+3} - \frac{2}{n+1}g^{n+2} + \frac{1}{n+1}g^{n+1}$. Then F' and G' share 1 IM, and

$$T(r,F) = (n+3)T(r,f) + S(r,f), \qquad T(r,G) = (n+3)T(r,g) + S(r,g). \tag{3.15}$$

Suppose now that Case (i) of Lemma 2.3 holds. Then we have

$$\begin{aligned} T(r,F') \leq{}& 4\overline{N}\left(r,\frac{1}{f}\right) + 4\overline{N}\left(r,\frac{1}{f-1}\right) + N_2\left(r,\frac{1}{f'}\right) \\ &+ 3\overline{N}\left(r,\frac{1}{g}\right) + 3\overline{N}\left(r,\frac{1}{g-1}\right) + N_2\left(r,\frac{1}{g'}\right) \\ &+ 4\overline{N}(r,f) + 3\overline{N}(r,g) + 2\overline{N}\left(r,\frac{1}{f'}\right) + \overline{N}\left(r,\frac{1}{g'}\right) \\ &+ S(r,f) + S(r,g). \end{aligned} \tag{3.16}$$

Consider $T(r,F) \leq T(r,F') + N(r,1/F) - N(r,1/F') + S(r,f)$. Then we obtain

$$\begin{aligned} T(r,F) \leq{}& T(r,F') + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-a_1}\right) + N\left(r,\frac{1}{f-a_2}\right) \\ &- 2N\left(r,\frac{1}{f-1}\right) - N\left(r,\frac{1}{f'}\right) + S(r,f), \end{aligned}$$

where a_1 and a_2 are distinct solutions of the equation $\frac{1}{n+3}w^2 - \frac{2}{n+2}w + \frac{1}{n+1} = 0$. Combining this with (3.16), we get

$$\begin{aligned} T(r,F) \leq{}& 4\overline{N}\left(r,\frac{1}{f}\right) + 2\overline{N}\left(r,\frac{1}{f-1}\right) + 3\overline{N}\left(r,\frac{1}{g}\right) + 3\overline{N}\left(r,\frac{1}{g-1}\right) \\ &+ N_2\left(r,\frac{1}{g'}\right) + 4\overline{N}(r,f) + 3\overline{N}(r,g) \\ &+ 2\overline{N}\left(r,\frac{1}{f'}\right) + \overline{N}\left(r,\frac{1}{g'}\right) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-a_1}\right) \\ &+ N\left(r,\frac{1}{f-a_2}\right) + S(r,f) + S(r,g). \end{aligned} \tag{3.17}$$

By (3.3) and (3.4) from (3.17) it follows that

$$T(r,F) \leq \left(7 + \frac{10}{s}\right)T(r,f) + \left(5 + \frac{8}{s}\right)T(r,g) + S(r,f) + S(r,g).$$

This implies

$$(n+3)\bigl\{T(r,f) + T(r,g)\bigr\} \leq \left(12 + \frac{18}{s}\right)\bigl\{T(r,f) + T(r,g)\bigr\} + S(r,f) + S(r,g),$$

which is a contradiction unless $(n-9)s \geq 19$. For $s \geq 3$, we use (3.6) and (3.7), and (3.17) leads to

$$T(r,F) \leq \left(5 + \frac{15}{s}\right)T(r,f) + \left(3 + \frac{12}{s}\right)T(r,g) + S(r,f) + S(r,g).$$

Similarly as before, we can conclude that

$$(n+3)\bigl\{T(r,f) + T(r,g)\bigr\} \leq \left(8 + \frac{27}{s}\right)\bigl\{T(r,f) + T(r,g)\bigr\} + S(r,f) + S(r,g),$$

which contradicts to $(n-5)s \geq 28$. Thus, by Lemma 2.3, $F'G' \equiv 1$ or $F' \equiv G'$. As in the proof Theorem 3.2, the case $F'G' \equiv 1$ leads to a contradiction, so we obtain that $F \equiv G$. Let $h \equiv f/g$. Then, similarly as in the proof of Theorem G, we only get $h \equiv 1$. Hence $f(z) \equiv g(z)$. □

Given specific values of s in Theorems 3.1–3.3, we can compare n in the two conditions of n and s and see that the second condition is always better than the first one for $s \geq 3$. For example, we consider $(n-4)s \geq 19$: if $s = 3$, then $n \geq 11$; if $s = 4$, then $n \geq 9$; if $s = 5, 6$, then $n \geq 8$; if $s = 7, 8, 9$, then $n \geq 7$; if $10 \leq s \leq 18$, then $n \geq 6$; and if $s \geq 19$, then $n \geq 5$. For the condition $ns \geq 28$, if $s = 3$, then $n \geq 10$; if $s = 4$, then $n \geq 7$; if $s = 5, 6$, then $n \geq 5$; if $s = 7, 8$, then $n \geq 4$; if $s = 9, 10$, then $n \geq 3$; and if $11 \leq s \leq 18$, then $n \geq 2$.

Theorem 3.4 *Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s, where s (≥ 7) is a positive integer. Let n be an integer satisfying $(n-1)s \geq 13$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f(z) \equiv g(z)$.*

Proof Let $F = \frac{1}{n+2}f^{n+2} - \frac{1}{n+1}f^{n+1}$ and $G = \frac{1}{n+2}g^{n+2} - \frac{1}{n+1}g^{n+1}$. Then F' and G' share 1 CM, and (3.9) holds. Suppose now that Case (i) of Lemma 2.2 holds. Then

$$\begin{aligned} T(r,F') &\leq N_2\left(r,\frac{1}{F'}\right) + N_2\left(r,\frac{1}{G'}\right) + N_2(r,F') + N_2(r,G')) + S(r,f) + S(r,g) \\ &\leq 2\overline{N}\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{f-1}\right) + N_2\left(r,\frac{1}{f'}\right) + 2\overline{N}\left(r,\frac{1}{g}\right) + N_2\left(r,\frac{1}{g-1}\right) \\ &\quad + N_2\left(r,\frac{1}{g'}\right) + 2\overline{N}(r,f) + 2\overline{N}(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{3.18}$$

From (3.18) we get

$$\begin{aligned} T(r,F) &\leq T(r,F') + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-(n+2)/(n+1)}\right) \\ &\quad - N\left(r,\frac{1}{f-1}\right) - N\left(r,\frac{1}{f'}\right) + S(r,f) \\ &\leq 2\overline{N}\left(r,\frac{1}{f}\right) + 2\overline{N}\left(r,\frac{1}{g}\right) + N_2\left(r,\frac{1}{g-1}\right) + N_2\left(r,\frac{1}{g'}\right) \\ &\quad + 2\overline{N}(r,f) + 2\overline{N}(r,g) + N\left(r,\frac{1}{f}\right) \\ &\quad + N\left(r,\frac{1}{f-(n+2)/(n+1)}\right) + S(r,f) + S(r,g) \\ &\leq \left(\frac{4}{s}+2\right)T(r,f) + \left(\frac{8}{s}+1\right)T(r,g) + S(r,f) + S(r,g), \end{aligned} \tag{3.19}$$

where by Lemma 2.1 for $N_2(r,1/g')$, we use

$$N_2\left(r,\frac{1}{g'}\right) \leq N_3\left(r,\frac{1}{g}\right) + \overline{N}(r,g) + S(r,g) \leq \frac{4}{s}T(r,g) + S(r,g).$$

There also exists a similar inequality for $T(r,G)$. Therefore we have

$$(n+2)\{T(r,f)+T(r,g)\} \le \left(\frac{12}{s}+3\right)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

which contradicts to $(n-1)s \ge 13$. Thus, by Lemma 2.2, $F'G' \equiv 1$ or $F' \equiv G'$. Then, as in the proof of Theorem 3.2, we can deduce $f(z) \equiv g(z)$. □

Theorem 3.5 *Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s, where s (≥ 7) is a positive integer. Let n be an integer satisfying $(n-2)s \ge 13$. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share 1 CM, then $f(z) \equiv g(z)$.*

Proof Let $F = \frac{1}{n+3}f^{n+3} - \frac{2}{n+2}f^{n+2} + \frac{1}{n+1}f^{n+1}$ and $G = \frac{1}{n+3}g^{n+3} - \frac{2}{n+1}g^{n+2} + \frac{1}{n+1}g^{n+1}$. Then F' and G' share 1 CM, and (3.15) holds. Suppose now that Case (i) of Lemma 2.2 holds. Proceeding as in the proof of Theorem 3.4, we have

$$\begin{aligned} T(r,F') \le{}& 2\overline{N}\left(r,\frac{1}{f}\right)+2\overline{N}\left(r,\frac{1}{f-1}\right)+N_2\left(r,\frac{1}{f'}\right)+2\overline{N}\left(r,\frac{1}{g}\right)+2\overline{N}\left(r,\frac{1}{g-1}\right) \\ &+N_2\left(r,\frac{1}{g'}\right)+2\overline{N}(r,f)+2\overline{N}(r,g)+S(r,f)+S(r,g). \end{aligned} \tag{3.20}$$

Then we obtain

$$\begin{aligned} T(r,F) \le{}& T(r,F')+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-a_1}\right)+N\left(r,\frac{1}{f-a_2}\right) \\ &-2N\left(r,\frac{1}{f-1}\right)-N\left(r,\frac{1}{f'}\right)+S(r,f) \\ \le{}& 2\overline{N}\left(r,\frac{1}{f}\right)+2\overline{N}\left(r,\frac{1}{g}\right)+2\overline{N}\left(r,\frac{1}{g-1}\right)+N_2\left(r,\frac{1}{g'}\right) \\ &+2\overline{N}(r,f)+2\overline{N}(r,g)+N\left(r,\frac{1}{f}\right) \\ &+N\left(r,\frac{1}{f-a_1}\right)+N\left(r,\frac{1}{f-a_1}\right)+S(r,f)+S(r,g) \\ \le{}& \left(\frac{4}{s}+3\right)T(r,f)+\left(\frac{8}{s}+2\right)T(r,g)+S(r,f)+S(r,g), \end{aligned} \tag{3.21}$$

where we use the inequality $N_2(r,1/g') \le (4/s)T(r,g)+S(r,g)$. Similarly as before, we get

$$(n+3)\{T(r,f)+T(r,g)\} \le \left(\frac{12}{s}+5\right)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

which contradicts to $(n-2)s \ge 13$. Thus by Lemma 2.2, $F'G' \equiv 1$ or $F' \equiv G'$. As in the proof Theorem 3.3, we must have $f(z) \equiv g(z)$. □

By giving specific values for $s \ge 7$ it is easy to see that the condition $(n-1)s \ge 13$ in Theorem 3.4 and $(n-2)s \ge 13$ in Theorem 3.5 are sharper than the condition $(n-2)s \ge 10$ in Theorem F and $(n-3)s \ge 10$ in Theorem G, respectively.

For further study of related problems, we would like to pose the following question.

Open question Let n, k be positive integers, and let m be a nonnegative integer. Suppose that $f^n(f-1)^m f^{(k)}$ and $g^n(g-1)^m g^{(k)}$ share a CM (or IM), where a ($\not\equiv 0, \infty$) is a small function of f and g. Under what conditions can we get $f \equiv g$?

4 Conclusions

Using the notion of multiplicity, in this paper, we provide five results, which extend the main results that were derived in the paper [1] and answer two open problems posed by Dyavanal in the same paper. Obtaining our results from more general hypotheses without complicated calculations is probably the most interesting feature of this paper. Finally, in this paper, we pose one more general open question for further studies.

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