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p-adic singular integrals and their commutators in generalized Morrey spaces

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Abstract

For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers. In this paper, we establish the boundedness of a class of p-adic singular integral operators on the p-adic generalized Morrey spaces. We also consider the corresponding boundedness for the commutators generalized by the p-adic singular integral operators and p-adic Lipschitz functions or p-adic generalized Campanato functions.

MSC: 42B20; 42B25

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1 Introduction

Let p be a prime number, and let $x \in \mathbb{Q}$. Then the non-Archimedean p-adic norm $|x|_p$ is defined as follows: if $x = 0$, then $|0|_p = 0$; if $x \neq 0$ is an arbitrary rational number with unique representation $x = p^\gamma \frac{m}{n}$, where m, n are not divisible by p, and $\gamma = \gamma(x) \in \mathbb{Z}$, then $|x|_p = p^{-\gamma}$. This norm has the following properties: $|xy|_p = |x|_p|y|_p$, $|x+y|_p \leq \max\{|x|_p, |y|_p\}$, and $|x|_p = 0$ if and only if $x = 0$. Moreover, when $|x|_p \neq |y|_p$, we have $|x+y|_p = \max\{|x|_p, |y|_p\}$. Let $\mathbb{Q}_p$ be the field of p-adic numbers defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. For $\gamma \in \mathbb{Z}$, we denote the ball $B_\gamma(a)$ with center at $a \in \mathbb{Q}_p$ and radius p^γ and its boundary $S_\gamma(a)$ by

$$B_\gamma(a) = \{x \in \mathbb{Q}_p : |x-a|_p \leq p^\gamma\}, \qquad S_\gamma(a) = \{x \in \mathbb{Q}_p : |x-a|_p = p^\gamma\},$$

respectively. It is easy to see that

$$B_\gamma(a) = \bigcup_{k \leq \gamma} S_k(a).$$

For $n \in \mathbb{N}$, the space $\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ consists of all points $x = (x_1, \ldots, x_n)$ where $x_i \in \mathbb{Q}_p$, $i = 1, \ldots, n$, $n \geq 1$. The p-adic norm of $\mathbb{Q}_p^n$ is defined by

$$|x|_p = \max_{1 \leq i \leq n} |x_i|_p, \quad x \in \mathbb{Q}_p^n.$$

Thus it is easy to see that $|x|_p$ is a non-Archimedean norm on $\mathbb{Q}_p^n$. The balls $B_\gamma(a)$ and the sphere $S_\gamma(a)$ in $\mathbb{Q}_p^n$ for $\gamma \in \mathbb{Z}$ are defined similarly to the case $n = 1$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, by the standard analysis there exists the Haar measure dx on the additive group $\mathbb{Q}_p^n$ normalized by $\int_{B_0} dx = |B_0|_H = 1$, where $|E|_H$ denotes the Haar measure of a measurable set $E \subset \mathbb{Q}_p^n$. Then by a simple calculation the Haar measures of any balls and spheres can be obtained. From the integral theory it is easy to see that $|B_\gamma(a)|_H = p^{n\gamma}$ and $|S_\gamma(a)|_H = p^{n\gamma}(1 - p^{-n})$ for any $a \in \mathbb{Q}_p^n$. For a more complete introduction to the p-adic analysis, we refer to [1–8] and the references therein.

The p-adic numbers have been applied in the string theory, turbulence theory, statistical mechanics, quantum mechanics, and so forth (see [1, 9, 10] for detail). In the past few years, there is an increasing interest in the study of harmonic analysis on p-adic field (see [5–8] for detail).

Let $\Omega \in L^\infty(\mathbb{Q}_p^n)$ be such that $\Omega(p^j x) = \Omega(x)$ for all $j \in \mathbb{Z}$ and $\int_{|x|_p=1} \Omega(x)\,dx = 0$. Then the p-adic singular integral operators defined by Taibleson [5] are as follows:

$$T_k(f)(x) = \int_{|y|_p > p^k} f(x-y)\frac{\Omega(y)}{|y|_p^n}\,dz \quad \text{for } k \in \mathbb{Z}.$$

The p-adic singular integral operator T is defined as the limit of T_k as k goes to $-\infty$.

Moreover, let $\overrightarrow{b} = (b_1, b_2, \ldots, b_m)$, where $b_i \in L_{\mathrm{loc}}(\mathbb{Q}_p^n)$ for $1 \le i \le m$. Then the higher commutator generated by $\vec{b}$ and T_k can be defined as

$$T_k^{\vec{b}} f(x) = \int_{|y|_p > p^k} \prod_{i=1}^m \bigl(b_i(x) - b_i(x-y)\bigr) f(x-y)\frac{\Omega(y)}{|y|_p^n}\,dz \quad \text{for } k \in \mathbb{Z},$$

and the commutator generated by $\overrightarrow{b} = (b_1, b_2, \ldots, b_m)$ and the p-adic singular integral operator T is defined as the limit of $T_k^{\vec{b}}$ as k goes to $-\infty$.

Under some conditions, the authors in [5, 11] showed that T_k were of type (q,q) for $1 < q < \infty$ and of weak type $(1,1)$ on local fields. Wu et al. [12] established the boundedness of T_k on p-adic central Morrey spaces. Furthermore, the λ-central BMO estimates for commutators of these singular integral operators on p-adic central Morrey spaces were obtained in [12]. Moreover, in the p-adic linear space $\mathbb{Q}_p^n$, Volosivets [13] gave sufficient conditions for the boundedness of the maximal function and Riesz potential in p-adic generalized Morrey spaces. Mo et al. [14] established the boundedness of the commutators generated by the p-adic Riesz potential and p-adic generalized Campanato functions in p-adic generalized Morrey spaces.

Motivated by the works of [12–14], we consider the boundedness of T_k on the p-adic generalized Morrey type spaces, as well as the boundedness of the commutators generated by T_k and p-adic generalized Campanato functions.

Throughout this paper, the letter C will be used to denote constants varying from line to line. The relation $A \lesssim B$ means that $A \le CB$ with some positive constant C independent of appropriate quantities.

2 Some notation and lemmas

Definition 2.1 ([13]) Let $1 \le q < \infty$, and let $\omega(x)$ be a nonnegative measurable function in $\mathbb{Q}_p^n$. A function $f \in L_{\mathrm{loc}}^q(\mathbb{Q}_p^n)$ is said to belong to the generalized Morrey space $GM_{q,\omega}(\mathbb{Q}_p^n)$

if

$$\|f\|_{GM_{q,\omega}} = \sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}} \frac{1}{\omega(B_\gamma(a))}\left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)}\left|f(y)\right|^q dy\right)^{1/q} < \infty,$$

where $\omega(B_\gamma(a)) = \int_{B_\gamma(a)} \omega(x)\,dx$.

Let $\lambda \in \mathbb{R}$. If $\omega(B_\gamma(a)) = |B_\gamma(a)|^\lambda$, then $GM_{q,\omega}(\mathbb{Q}_p^n)$ is the classical Morrey space $M_{q,\lambda}(\mathbb{Q}_p^n)$. About the generalized Morrey space, see [15], and for the classical Morrey spaces, see [16] and so on.

Moreover, let $\lambda \in \mathbb{R}$ and $1 \le q < \infty$. The p-adic central Morrey space $CM_{q,\lambda}(\mathbb{Q}_p^n)$ (see [8]) is defined by

$$\|f\|_{CM_{q,\lambda}} = \sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma(0)|_H^{1+\lambda q}}\int_{B_\gamma(0)}\left|f(y)\right|^q dy\right)^{1/q} < \infty.$$

Definition 2.2 ([17]) For $0 < \beta < 1$, the the p-adic Lipschitz space $\Lambda_\beta(\mathbb{Q}_p^n)$ is defined as the set of all functions $f : \mathbb{Q}_p^n \mapsto \mathbb{C}$ such that

$$\|f\|_{\Lambda_\beta(\mathbb{Q}_p^n)} = \sup_{x,h\in\mathbb{Q}_p^n,h\neq 0}\frac{|f(x+h)-f(x)|}{|h|_p^\beta} < \infty.$$

Definition 2.3 ([13]) Let B be a ball in $\mathbb{Q}_p^n$, $1 \le q < \infty$, and let $\omega(x)$ be a nonnegative measurable function in $\mathbb{Q}_p^n$. A function $f \in L_{\mathrm{loc}}^q(\mathbb{Q}_p^n)$ is said to belong to the generalized Campanato space $GC_{q,\omega}(\mathbb{Q}_p^n)$ if

$$\|f\|_{GC_{q,\omega}} = \sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}} \frac{1}{\omega(B_\gamma(a))}\left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)}\left|f(y)-f_{B_\gamma(a)}\right|^q dy\right)^{1/q} < \infty,$$

where $f_{B_\gamma(a)} = \frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)} f(x)\,dx$ and $\omega(B_\gamma(a)) = \int_{B_\gamma(a)}\omega(x)\,dx$.

The classical Campanato spaces can be found in [18, 19], and so on. The important particular case of $GC_{q,\omega}(\mathbb{Q}_p^n)$ is $BMO_{q,\lambda}(\mathbb{Q}_p^n)$, where $1 < q < \infty$ and $0 < \lambda < 1/n$. The central BMO space $CBMO_{q,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$\|f\|_{CBMO^{q,\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}}\frac{1}{|B_\gamma(0)|_H^\lambda}\left(\frac{1}{|B_\gamma(0)|_H}\int_{B_\gamma(0)}\left|f(y)-f_{B_\gamma(0)}\right|^q dy\right)^{1/q} < \infty. \tag{2.1}$$

Lemma 2.1 ([14]) *Let $1 \le q < \infty$, and let ω be a nonnegative measurable function. Let $b \in GC_{q,\omega}(\mathbb{Q}_p^n)$. Then*

$$|b_{B_k(a)} - b_{B_j(a)}| \le \|b\|_{GC_{q,\omega}}|j-k|\max\{\omega(B_k(a)),\omega(B_j(a))\}$$

for $j,k \in \mathbb{Z}$ and any fixed $a \in \mathbb{Q}_p^n$.

Thus, for $j > k$, from Lemma 2.1 we deduce that

$$\left(\int_{B_j(a)}\left|b(y)-b_{B_k(a)}\right|^q dy\right)^{1/q} \le (j+1-k)\left|B_j(a)\right|_H^{1/q}\omega(B_j(a))\|b\|_{GC_{q,\omega}}. \tag{2.2}$$

Lemma 2.2 ([5]) *Let $\Omega \in L^\infty(\mathbb{Q}_p^n)$ be such that $\Omega(p^j x) = \Omega(x)$ for all $j \in \mathbb{Z}$ and $\int_{|x|_p=1} \Omega(x)\,dx = 0$. If*

$$\sup_{|y|_p=1} \sum_{j=1}^{\infty} \int_{|x|_p=1} \left|\Omega\left(x + p^j y\right) - \Omega(x)\right| dx < \infty,$$

then for $1 < p < \infty$, there is a constant $C > 0$ such that

$$\left\| T_k(f)\right\|_{L^p(\mathbb{Q}_p^n)} \leq C\|f\|_{L^p(\mathbb{Q}_p^n)}$$

for $k \in \mathbb{Z}$, where C is independent of f and $k \in \mathbb{Z}$.

Furthermore, $T(f) = \lim_{k\to-\infty} T_k(f)$ exists in the L^p norm, and

$$\left\| T(f)\right\|_{L^p(\mathbb{Q}_p^n)} \leq C\|f\|_{L^p(\mathbb{Q}_p^n)}.$$

Moreover, on the p-adic field, the Riesz potential I_α^p is defined by

$$I_p^\alpha f(x) = \frac{1}{\Gamma_n(\alpha)} \int_{\mathbb{Q}_p^n} \frac{f(y)}{|x-y|_p^{n-\alpha}}\,dy,$$

where $\Gamma_n(\alpha) = (1 - p^{\alpha-n})/(1 - p^{-\alpha})$ for $\alpha \in \mathbb{C}$, $\alpha \neq 0$.

Lemma 2.3 ([14]) *Let α be a complex number with $0 < \operatorname{Re}\alpha < n$, and let $1 < r < \infty$, $1 < q < n/\operatorname{Re}\alpha$, and $0 < 1/r = 1/q - \operatorname{Re}\alpha/n$. Suppose that both ω and ν are nonnegative measurable functions such that*

$$\sum_{j=\gamma}^{\infty} p^{j\operatorname{Re}\alpha} \frac{\nu(B_j(a))}{\omega(B_\gamma(a))} = C < \infty$$

for any $a \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$. Then the Riesz potential I_p^α is bounded from $GM_{q,\nu}$ to $GM_{r,\omega}$.

3 Main results

In this section, we state the main results of the paper.

Theorem 3.1 *Let $1 < q < \infty$, and let $\Omega(p^j x) = \Omega(x)$ for all $j \in \mathbb{Z}$, $\int_{|x|_p=1} \Omega(x)\,dx = 0$, and*

$$\sup_{|y|_p=1} \sum_{j=1}^{\infty} \int_{|x|_p=1} \left|\Omega\left(x + p^j y\right) - \Omega(x)\right| dx < \infty.$$

Suppose that both ω and ν are nonnegative measurable functions such that

$$\sum_{j=\gamma}^{\infty} \nu\left(B_j(a)\right)/\omega\left(B_\gamma(a)\right) = C < \infty \tag{3.1}$$

for any $\gamma \in \mathbb{Z}$ and $a \in \mathbb{Q}_p^n$. Then the singular integral operators T_k are bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$ for all $k \in \mathbb{Z}$. Moreover, $T(f) = \lim_{k\to-\infty} T_k(f)$ exists in $GM_{q,\omega}$, and the operator T is bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$.

Corollary 3.1 *Let $1 < q < \infty$, $\lambda < 0$, and let $\Omega \in L^\infty(\mathbb{Q}_p^n)$ be such that $\Omega(p^j x) = \Omega(x)$ for all $j \in \mathbb{Z}$, $\int_{|x|_p=1} \Omega(x)\,dx = 0$, and*

$$\sup_{|y|_p=1} \sum_{j=1}^{\infty} \int_{|x|_p=1} \left|\Omega\left(x + p^j y\right) - \Omega(x)\right| dx < \infty.$$

Then the operators T_k and T are bounded on the space $M_{q,\lambda}$ for all $k \in \mathbb{Z}$.

In fact, for $\lambda < 0$, taking $\omega(B) = \nu(B) = |B|_H^\lambda$ in Theorem 3.1, we obtain Corollary 3.1. If the Morrey space $M_{q,\lambda}(\mathbb{Q}_p^n)$ is replaced by the central Morrey space $CM_{q,\lambda}(\mathbb{Q}_p^n)$ in Corollary 3.1, then the conclusion is that of Theorem 4.1 in [12].

Theorem 3.2 *Let $\Omega \in L^\infty(\mathbb{Q}_p^n)$ be such that $\Omega(p^j x) = \Omega(x)$ for all $j \in \mathbb{Z}$, $\int_{|x|_p=1} \Omega(x)\,dx = 0$, and*

$$\sup_{|y|_p=1} \sum_{j=1}^{\infty} \int_{|x|_p=1} \left|\Omega\left(x + p^j y\right) - \Omega(x)\right| dx < \infty.$$

Let $0 < \beta_i < 1$ for $i = 1, 2, \ldots, m$ be such that $0 < \beta = \sum_{i=1}^m \beta_i < n$, and let $1 < r < \infty$ and $1 < q < n/\beta$ be such that $1/r = 1/q - \beta/n$. Suppose that $b_i \in \Lambda_{\beta_i}$, $i = 1, 2, \ldots, m$, and both ω and ν are nonnegative measurable functions such that

$$\sum_{j=\gamma}^{\infty} p^{j\beta} \nu\left(B_j(a)\right)/\omega\left(B_\gamma(a)\right) = C < \infty \tag{3.2}$$

for any $\gamma \in \mathbb{Z}$ and $a \in \mathbb{Q}_p^n$. Then the commutators $T_k^{\vec{b}}$ are bounded from $GM_{q,\nu}$ to $GM_{r,\omega}$ for all $k \in \mathbb{Z}$. Moreover, the commutator $T^{\vec{b}}(f) = \lim_{k\to-\infty} T_k^{\vec{b}}(f)$ exists in the space of $GM_{q,\omega}$, and $T^{\vec{b}}$ is bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$.

Theorem 3.3 *Let $\Omega \in L^\infty(\mathbb{Q}_p^n)$ be such that $\Omega(p^j x) = \Omega(x)$ for all $j \in \mathbb{Z}$, $\int_{|x|_p=1} \Omega(x)\,dx = 0$, and*

$$\sup_{|y|_p=1} \sum_{j=1}^{\infty} \int_{|x|_p=1} \left|\Omega\left(x + p^j y\right) - \Omega(x)\right| dx < \infty.$$

Let $1 < q, r, q_1, \ldots, q_m < \infty$ be such that $1/r = 1/q + 1/q_1 + 1/q_2 + \cdots + 1/q_m$. Suppose that ω, ν, and ν_i $(i = 1, 2, \ldots, m)$ are nonnegative measurable functions. Suppose that $b_i \in GC_{q_i,\nu_i}(\mathbb{Q}_p^n)$, $i = 1, 2, \ldots, m$, and the functions ω, ν, and ν_i $(i = 1, 2, \ldots, m)$ satisfy the following conditions:

(i) $\prod_{i=1}^m \nu_i(B_\gamma(a))\nu(B_\gamma(a))/\omega(B_\gamma(a)) = C < \infty$,

(ii) $\sum_{j=\gamma+1}^{\infty} \prod_{i=1}^m \nu_i(B_j(a))(j + 1 - \gamma)^m \nu(B_j(a))/\omega(B_\gamma(a)) = C < \infty$

for any $\gamma \in \mathbb{Z}$ and $a \in \mathbb{Q}_p^n$. Then the commutators $T_k^{\vec{b}}$ are bounded from $GM_{q,\nu}$ to $GM_{r,\omega}$ for all $k \in \mathbb{Z}$. The commutator $T^{\vec{b}} = \lim_{k\to-\infty} T_k^{\vec{b}}$ exists in the space of $GM_{q,\omega}$, and $T^{\vec{b}}$ is bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$.

Corollary 3.2 *Let $\Omega \in L^\infty(\mathbb{Q}_p^n)$ be such that $\Omega(p^j x) = \Omega(x)$ for all $j \in \mathbb{Z}$, $\int_{|x|_p=1} \Omega(x)\,dx = 0$, and*

$$\sup_{|y|_p=1} \sum_{j=1}^{\infty} \int_{|x|_p=1} \left|\Omega\left(x + p^j y\right) - \Omega(x)\right| dx < \infty.$$

Let $1 < q, r, q_1, \ldots, q_m < \infty$ be such that $1/r = 1/q + 1/q_1 + 1/q_2 + \cdots + 1/q_m$. Let $0 \le \lambda_1, \ldots, \lambda_m < 1/n$, $\lambda < -\sum_{i=1}^m \lambda_i$, and $\tilde{\lambda} = \sum_{i=1}^m \lambda_i + \lambda$. If $b_i \in BMO_{q_i,\lambda_i}(\mathbb{Q}_p^n)$, then the commutators $T_k^{\vec{b}}$ and $T^{\vec{b}}$ are bounded from $M_{q,\lambda}$ to $M_{r,\tilde{\lambda}}$ for all $k \in \mathbb{Z}$.

Moreover, let $1 < r, q, q_1 < \infty$ be such that $1/r = 1/q + 1/q_1$. Let $0 \le \lambda_1 < 1/n$, $\lambda < -\lambda_1$, and $\tilde{\lambda} = \lambda_1 + \lambda$. If $b \in CBMO_{q_1,\lambda_1}(\mathbb{Q}_p^n)$, then from Corollary 3.1 it follows that the commutators $T_k^b = [T_k, b]$ and $T^b = [T, b]$ are bounded from $CM_{q,\lambda}$ to $CM_{r,\tilde{\lambda}}$ for all $k \in \mathbb{Z}$. These results are those of Theorem 4.2 in [12].

4 Proof of Theorems 3.1–3.3

Let us first give the proof of Theorem 3.1.

For any fixed $\gamma \in \mathbb{Z}$ and $a \in \mathbb{Q}_p^n$, it is easy to see that

$$\begin{aligned}
&\frac{1}{\omega(B_\gamma(a))}\left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)} \left|T_k(f)(x)\right|^q dx\right)^{1/q} \\
&\quad\le \frac{1}{\omega(B_\gamma(a))}\left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)} \left|T_k(f)(f\chi_{B_\gamma(a)})(x)\right|^q dx\right)^{1/q} \\
&\qquad + \frac{1}{\omega(B_\gamma(a))}\left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)} \left|T_k(f\chi_{B_\gamma^c(a)})(x)\right|^q dx\right)^{1/q} \\
&\quad := I + II, \qquad (4.1)
\end{aligned}$$

where $B_\gamma^c(a)$ is the complement to $B_\gamma(a)$ in $\mathbb{Q}_p^n$.

Using Lemma 2.2 and (3.1), it follows that

$$\begin{aligned}
I &\lesssim \frac{1}{\omega(B_\gamma(a))}\frac{1}{|B_\gamma(a)|_H^{1/q}}\left(\int_{B_\gamma(a)} |f(x)|^q dx\right)^{1/q} \\
&= \frac{\nu(B_\gamma(a))}{\omega(B_\gamma(a))}\frac{1}{\nu(B_\gamma(a))}\left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)} |f(x)|^q dx\right)^{1/q} \\
&\lesssim \|f\|_{GM_{q,\nu}}. \qquad (4.2)
\end{aligned}$$

For II, let us first estimate $|T_k(f\chi_{B_\gamma^c(a)})(x)|$.

Since $x \in B_\gamma(a)$ and $\Omega \in L^\infty(\mathbb{Q}_p^n)$, we have

$$\begin{aligned}
\left|T_k(f\chi_{B_\gamma^c(a)})(x)\right| &= \left|\int_{|y|_p > p^k} (f\chi_{B_\gamma^c(a)})(x-y)\frac{\Omega(y)}{|y|_p^n}\,dy\right| \\
&= \left|\int_{|x-z|_p > p^k} (f\chi_{B_\gamma^c(a)})(z)\frac{\Omega(x-z)}{|x-z|_p^n}\,dz\right| \\
&\lesssim \int_{B_\gamma^c(a)} \frac{|f(z)|}{|x-z|_p^n}\,dz
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{j=\gamma+1}^{\infty} \int_{S_j(a)} p^{-jn} |f(y)|\, dy \\
&\le \sum_{j=\gamma+1}^{\infty} p^{-jn} \left(\int_{B_j(a)} |f(y)|^q \, dy \right)^{1/q} |B_j(a)|_H^{1-1/q} \\
&= \|f\|_{GM_{q,\nu}} \sum_{j=\gamma+1}^{\infty} \nu\big(B_j(a)\big). \qquad (4.3)
\end{aligned}$$

Thus from (3.1) and (4.3) it follows that

$$\begin{aligned}
II &= \frac{1}{\omega(B_\gamma(a))} \left(\frac{1}{|B_\gamma(a)|_H} \int_{B_\gamma(a)} \big| T_k(f\chi_{B_\gamma^c(a)})(x)\big|^q \, dx \right)^{1/q} \\
&\lesssim \|f\|_{GM_{q,\nu}} \sum_{j=\gamma+1}^{\infty} \nu\big(B_j(a)\big)/\omega\big(B_\gamma(a)\big) \\
&\lesssim \|f\|_{GM_{q,\nu}}. \qquad (4.4)
\end{aligned}$$

Combining the estimates of (4.1), (4.2), and (4.4), we have

$$\frac{1}{\omega(B_\gamma(a))} \left(\frac{1}{|B_\gamma(a)|_H} \int_{B_\gamma(a)} \big| T_k(f)(x)\big|^q \, dx \right)^{1/q} \lesssim \|f\|_{GM_{q,\nu}},$$

which means that T_k is bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$ for all $k \in \mathbb{Z}$.

Moreover, from Lemma 2.2 and the definition of $GM_{q,\omega}(\mathbb{Q}_p^n)$ it is obvious that $T(f) = \lim_{k\to-\infty} T_k(f)$ exists in $GM_{q,\omega}$ and the operator T is bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$.

Proof of Theorem 3.2 For any $x \in \mathbb{Q}_p^n$, since $\Omega \in L^\infty(\mathbb{Q}_p^n)$ and $b_i \in \Lambda_{\beta_i}$, $i = 1, 2, \ldots, m$, it is easy to see that

$$\begin{aligned}
&\big| T_k^{\vec{b}} f(x)\big| \\
&\quad \le \int_{|y|_p > p^k} \prod_{i=1}^{m} \big| b_i(x) - b_i(x-y)\big| \big| f(x-y)\big| \frac{|\Omega(y)|}{|y|_p^n} \, dy \\
&\quad \lesssim \int_{\mathbb{Q}_p^n} \frac{|f(z)|}{|x-z|_p^{n-\beta}} \, dz \\
&\quad \lesssim I_p^\beta\big(|f|\big)(x).
\end{aligned}$$

Thus from Lemma 2.3 it is obvious that the commutators $T_k^{\vec{b}}$ are bounded from $GM_{q,\nu}$ to $GM_{r,\omega}$ for all $k \in \mathbb{Z}$.

Moreover, from the definition of $GM_{q,\omega}(\mathbb{Q}_p^n)$ it is obvious that $T^{\vec{b}}(f) = \lim_{k\to-\infty} T_k^{\vec{b}}(f)$ exists in the space of $GM_{q,\omega}$, and the commutator $T^{\vec{b}}$ is bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$. □

Proof of Theorem 3.3 Without loss of generality, we need only to show that the conclusion holds for $m = 2$.

For any fixed $\gamma \in \mathbb{Z}$ and $a \in \mathbb{Q}_p^n$, we write $f^0 = f\chi_{B_\gamma(a)}$ and $f^\infty = f\chi_{B_\gamma^c(a)}$. Then

$$\begin{aligned}
&\frac{1}{\omega(B_\gamma(a))}\left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)}\left|T_k^{(b_1,b_2)}(f)(x)\right|^r dx\right)^{1/r} \\
&\leq \frac{1}{\omega(B_\gamma(a))}\left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)}\left|\left(b_1(x)-(b_1)_{B_\gamma(a)}\right)\left(b_2(x)-(b_2)_{B_\gamma(a)}\right)T_k\left(f^0\right)(x)\right|^r dx\right)^{1/r} \\
&\quad + \frac{1}{\omega(B_\gamma(a))}\left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)}\left|\left(b_1(x)-(b_1)_{B_\gamma(a)}\right)T_k\left(\left(b_2-(b_2)_{B_\gamma(a)}\right)f^0\right)(x)\right|^r dx\right)^{1/r} \\
&\quad + \frac{1}{\omega(B_\gamma(a))}\left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)}\left|\left(b_2(x)-(b_2)_{B_\gamma(a)}\right)T_k\left(\left(b_1-(b_1)_{B_\gamma(a)}\right)f^0\right)(x)\right|^r dx\right)^{1/r} \\
&\quad + \frac{1}{\omega(B_\gamma(a))}\left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)}\left|T_k\left(\left(b_1-(b_1)_{B_\gamma(a)}\right)\left(b_2-(b_2)_{B_\gamma(a)}\right)f^0\right)(x)\right|^r dx\right)^{1/r} \\
&\quad + \frac{1}{\omega(B_\gamma(a))} \\
&\quad \times \left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)}\left|\left(b_1(x)-(b_1)_{B_\gamma(a)}\right)\left(b_2(x)-(b_2)_{B_\gamma(a)}\right)T_k\left(f^\infty\right)(x)\right|^r dx\right)^{1/r} \\
&\quad + \frac{1}{\omega(B_\gamma(a))} \\
&\quad \times \left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)}\left|\left(b_1(x)-(b_1)_{B_\gamma(a)}\right)T_k\left(\left(b_2-(b_2)_{B_\gamma(a)}\right)f^\infty\right)(x)\right|^r dx\right)^{1/r} \\
&\quad + \frac{1}{\omega(B_\gamma(a))} \\
&\quad \times \left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)}\left|\left(b_2(x)-(b_2)_{B_\gamma(a)}\right)T_k\left(\left(b_1-(b_1)_{B_\gamma(a)}\right)f^\infty\right)(x)\right|^r dx\right)^{1/r} \\
&\quad + \frac{1}{\omega(B_\gamma(a))}\left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)}\left|T_k\left(\left(b_1-(b_1)_{B_\gamma(a)}\right)\left(b_2-(b_2)_{B_\gamma(a)}\right)f^\infty\right)(x)\right|^r dx\right)^{1/r} \\
&=: E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8. \qquad (4.5)
\end{aligned}$$

We further estimate every part.

Since $1/r = 1/q + 1/q_1 + 1/q_2$, from Hölder's inequality, Lemma 2.2, and (i) it follows that

$$\begin{aligned}
E_1 &= \frac{1}{\omega(B_\gamma(a))} \\
&\quad \times \left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)}\left|\left(b_1(x)-(b_1)_{B_\gamma(a)}\right)\left(b_2(x)-(b_2)_{B_\gamma(a)}\right)T_k\left(f^0\right)(x)\right|^r dx\right)^{1/r} \\
&\leq \frac{1}{\omega(B_\gamma(a))|B_\gamma(a)|_H^{1/r}}\prod_{i=1}^{2}\left(\int_{B_\gamma(a)}\left|b_i(x)-(b_i)_{B_\gamma(a)}\right|^{q_i} dx\right)^{1/q_i} \\
&\quad \times \left(\int_{B_\gamma(a)}\left|T_k\left(f^0\right)(x)\right|^q dx\right)^{1/q}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \frac{\nu_1(B_\gamma(a))\nu_2(B_\gamma(a))}{\omega(B_\gamma(a))|B_\gamma(a)|_H^{1/q}} \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,\nu_i}} \biggl(\int_{B_\gamma(a)} \bigl|f(x)\bigr|^q\,dx\biggr)^{1/q} \\
&\le \frac{\nu(B_\gamma(a))\nu_1(B_\gamma(a))\nu_2(B_\gamma(a))}{\omega(B_\gamma(a))} \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}} \\
&\lesssim \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}}.
\end{aligned}$$

Let $1/\bar{q} = 1/q + 1/q_2$. Then $1/r = 1/q_1 + 1/\bar{q}$. Thus, from Hölder's inequality, Lemma 2.2, and (i) we obtain

$$\begin{aligned}
E_2 &= \frac{1}{\omega(B_\gamma(a))}\biggl(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)} \bigl|\bigl(b_1(x)-(b_1)_{B_\gamma(a)}\bigr)T_k\bigl(\bigl(b_2-(b_2)_{B_\gamma(a)}\bigr)f^0\bigr)(x)\bigr|^r\,dx\biggr)^{1/r} \\
&\le \frac{1}{\omega(B_\gamma(a))|B_\gamma(a)|_H^{1/r}}\biggl(\int_{B_\gamma(a)} \bigl|b_1(x)-(b_1)_{B_\gamma(a)}\bigr|^{q_1}\,dx\biggr)^{1/q_1} \\
&\quad\times\biggl(\int_{B_\gamma(a)} \bigl|T_k\bigl(\bigl(b_2-(b_2)_{B_\gamma(a)}\bigr)f^0\bigr)(x)\bigr|^{\bar{q}}\,dx\biggr)^{1/\bar{q}} \\
&\lesssim \frac{1}{\omega(B_\gamma(a))|B_\gamma(a)|_H^{1/r}}\biggl(\int_{B_\gamma(a)} \bigl|b_1(x)-(b_1)_{B_\gamma(a)}\bigr|^{q_1}\,dx\biggr)^{1/q_1} \\
&\quad\times\biggl(\int_{B_\gamma(a)} \bigl|\bigl(b_2(x)-(b_2)_{B_\gamma(a)}\bigr)f(x)\bigr|^{\bar{q}}\,dx\biggr)^{1/\bar{q}} \\
&\le \frac{1}{\omega(B_\gamma(a))|B_\gamma(a)|_H^{1/r}}\prod_{i=1}^{2}\biggl(\int_{B_\gamma(a)} \bigl|b_i(x)-(b_i)_{B_\gamma(a)}\bigr|^{q_i}\,dx\biggr)^{1/q_i}\biggl(\int_{B_\gamma(a)} \bigl|f(x)\bigr|^q\,dx\biggr)^{1/q} \\
&\le \frac{\nu(B_\gamma(a))\nu_1(B_\gamma(a))\nu_2(B_\gamma(a))}{\omega(B_\gamma(a))} \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}} \\
&\lesssim \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}}.
\end{aligned}$$

Similarly,

$$E_3 \lesssim \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}}.$$

For E_4, from Lemma 2.2, Hölder's inequality, and (i) we obtain

$$\begin{aligned}
E_4 &= \frac{1}{\omega(B_\gamma(a))}\biggl(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)} \bigl|T_k\bigl(b_1-(b_1)_{B_\gamma(a)}\bigr)\bigl(b_2-(b_2)_{B_\gamma(a)}\bigr)f^0\bigr)(x)\bigr|^r\,dx\biggr)^{1/r} \\
&\lesssim \frac{1}{\omega(B_\gamma(a))|B_\gamma(a)|_H^{1/r}}\biggl(\int_{B_\gamma(a)} \bigl|\bigl(b_1(x)-(b_1)_{B_\gamma(a)}\bigr)\bigl(b_2(x)-(b_2)_{B_\gamma(a)}\bigr)f(x)\bigr|^r\,dx\biggr)^{1/r} \\
&\le \frac{1}{\omega(B_\gamma(a))|B_\gamma(a)|_H^{1/r}}\prod_{i=1}^{2}\biggl(\int_{B_\gamma(a)} \bigl|b_i(x)-(b_i)_{B_\gamma(a)}\bigr|^{q_i}\,dx\biggr)^{1/q_i}\biggl(\int_{B_\gamma(a)} \bigl|f(x)\bigr|^q\,dx\biggr)^{1/q}
\end{aligned}$$

$$\begin{aligned}
&\le \frac{\nu(B_\gamma(a))\nu_1(B_\gamma(a))\nu_2(B_\gamma(a))}{\omega(B_\gamma(a))}\prod_{i=1}^{2}\|b_i\|_{GC_{q_i,\nu_i}}\|f\|_{GM_{q,\nu}} \\
&\lesssim \prod_{i=1}^{2}\|b_i\|_{GC_{q_i,\nu_i}}\|f\|_{GM_{q,\nu}}.
\end{aligned}$$

To estimate E_5, we first need to consider $|T_k(f^\infty)(x)|$. In fact, by (4.3) it is easy to see that

$$\bigl|T_k\bigl(f^\infty\bigr)(x)\bigr| \lesssim \|f\|_{GM_{q,\nu}}\sum_{j=\gamma+1}^{\infty}\nu\bigl(B_j(a)\bigr). \tag{4.6}$$

Therefore from Hölder's inequality, (4.6), and (ii) we get

$$\begin{aligned}
E_5 &= \frac{1}{\omega(B_\gamma(a))} \\
&\quad\times\left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)}\bigl|\bigl(b_1(x)-(b_1)_{B_\gamma(a)}\bigr)\bigl(b_2(x)-(b_2)_{B_\gamma(a)}\bigr)T_k\bigl(f^\infty\bigr)(x)\bigr|^r\,dx\right)^{1/r} \\
&\le \frac{1}{\omega(B_\gamma(a))|B_\gamma(a)|_H^{1/r}}\prod_{i=1}^{2}\left(\int_{B_\gamma(a)}\bigl|b_i(x)-(b_i)_{B_\gamma(a)}\bigr|^{q_i}\,dx\right)^{1/q_i} \\
&\quad\times\left(\int_{B_\gamma(a)}\bigl|T_k\bigl(f^\infty\bigr)(x)f(x)\bigr|^q\,dx\right)^{1/q} \\
&\lesssim \sum_{j=\gamma+1}^{\infty}\frac{\nu(B_j(a))\nu_1(B_\gamma(a))\nu_2(B_\gamma(a))}{\omega(B_\gamma(a))}\prod_{i=1}^{2}\|b_i\|_{GC_{q_i,\nu_i}}\|f\|_{GM_{q,\nu}} \\
&\lesssim \prod_{i=1}^{2}\|b_i\|_{GC_{q_i,\nu_i}}\|f\|_{GM_{q,\nu}}.
\end{aligned}$$

It is similar to estimate (4.3) for $x\in B_\gamma(a)$. By $\Omega\in L^\infty(\mathbb{Q}_p^n)$ and (2.2) we can deduce that

$$\begin{aligned}
&\bigl|T_k\bigl(b_2-(b_2)_{B_\gamma(a)}\bigr)f^\infty\bigr)(x)\bigr| \\
&\quad=\left|\int_{|y|_p>p^k}\bigl(b_2(x-y)-(b_2)_{B_\gamma(a)}\bigr)f\chi_{B_\gamma^c(a)}(x-y)\frac{\Omega(y)}{|y|_p^n}\,dy\right| \\
&\quad\le\int_{B_\gamma^c}\bigl|b_2(z)-(b_2)_{B_\gamma(a)}\bigr|\bigl|f(z)\bigr|\frac{|\Omega(x-z)|}{|x-z|_p^n}\,dz \\
&\quad\lesssim\int_{B_\gamma^c}\frac{|b_2(z)-(b_2)_{B_\gamma(a)}||f(z)|}{|x-z|_p^n}\,dz \\
&\quad\lesssim\sum_{j=\gamma+1}^{\infty}\int_{S_j(a)}p^{-jn}\bigl|b_2(z)-(b_2)_{B_\gamma(a)}\bigr|\bigl|f(y)\bigr|\,dy \\
&\quad=\sum_{j=\gamma+1}^{\infty}p^{-jn}\bigl|B_j(a)\bigr|_H^{1-1/q-1/q_2}\left(\int_{S_j(a)}\bigl|f(y)\bigr|^q\,dy\right)^{1/q}\left(\int_{S_j(a)}\bigl|b_2(y)-(b_2)_{B_\gamma(a)}\bigr|^{q_2}\,dy\right)^{1/q_2}
\end{aligned}$$

$$
\begin{aligned}
&\le \|f\|_{GM_{q,\nu}} \sum_{j=\gamma+1}^{\infty} p^{-jn} \left|B_j(a)\right|_H^{1-1/q_2} \nu\big(B_j(a)\big) \left(\int_{B_j(a)} \left|b_2(y) - (b_2)_{B_\gamma(a)}\right|^{q_2} dy \right)^{1/q_2} \\
&\lesssim \|b_2\|_{GC_{q_2,\nu_2}} \|f\|_{GM_{q,\nu}} \sum_{j=\gamma+1}^{\infty} (j+1-\gamma) \nu\big(B_j(a)\big) \nu_2\big(B_j(a)\big).
\end{aligned} \tag{4.7}
$$

Let $1/\bar{q} = 1/q + 1/q_2$. Then $1/r = 1/q_1 + 1/\bar{q}$. Thus from Hölder's inequality, (4.7), and (ii) it follows that

$$
\begin{aligned}
E_6 &= \frac{1}{\omega(B_\gamma(a))} \left(\frac{1}{|B_\gamma(a)|_H} \int_{B_\gamma(a)} \left| \big(b_1(x) - (b_1)_{B_\gamma(a)}\big) T_k\big(\big(b_2 - (b_2)_{B_\gamma(a)}\big) f^\infty\big)(x) \right|^r dx \right)^{1/r} \\
&\le \frac{1}{\omega(B_\gamma(a)) |B_\gamma(a)|_H^{1/r}} \left(\int_{B_\gamma(a)} \left|b_1(x) - (b_1)_{B_\gamma(a)}\right|^{q_1} dx \right)^{1/q_1} \\
&\quad \times \left(\int_{B_\gamma(a)} \left| T_k\big(\big(b_2 - (b_2)_{B_\gamma(a)}\big) f^\infty\big)(x) \right|^{\bar{q}} dx \right)^{1/\bar{q}} \\
&\le \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}} \frac{1}{\omega(B_\gamma(a))} \sum_{j=\gamma+1}^{\infty} (j+1-\gamma) \nu\big(B_j(a)\big) \nu_2\big(B_j(a)\big) \nu_1\big(B_\gamma(a)\big) \\
&\lesssim \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}}.
\end{aligned}
$$

Similarly estimating E_6, we obtain

$$
E_7 \lesssim \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}}.
$$

Moreover, since $\Omega \in L^\infty(\mathbb{Q}_p^n)$, by (2.2) it is easy to see that

$$
\begin{aligned}
&\left| T_k\big(\big(b_1 - (b_1)_{B_\gamma(a)}\big)\big(b_2 - (b_2)_{B_\gamma(a)}\big) f^\infty\big)(x) \right| \\
&\quad = \left| \int_{|x-z|_p > p^k} \big(b_1(z) - (b_1)_{B_\gamma(a)}\big)\big(b_2(z) - (b_2)_{B_\gamma(a)}\big) f \chi_{B_\gamma^c(a)}(z) \frac{\Omega(x-z)}{|x-z|_p^n} dz \right| \\
&\quad \le \int_{B_\gamma^c} \left|b_1(z) - (b_1)_{B_\gamma(a)}\right| \left|b_2(z) - (b_2)_{B_\gamma(a)}\right| \left|f(z)\right| \frac{|\Omega(x-z)|}{|x-z|_p^n} dz \\
&\quad \lesssim \sum_{j=\gamma+1}^{\infty} \int_{S_j(a)} p^{-jn} \left|b_1(z) - (b_1)_{B_\gamma(a)}\right| \left|b_2(z) - (b_2)_{B_\gamma(a)}\right| \left|f(y)\right| dy \\
&\quad = \sum_{j=\gamma+1}^{\infty} p^{-jn} \left|B_j(a)\right|_H^{1-1/q-1/q_1-1/q_2} \left(\int_{S_j(a)} \left|f(y)\right|^q dy \right)^{1/q} \\
&\qquad \times \left(\int_{S_j(a)} \left|b_1(y) - (b_1)_{B_\gamma(a)}\right|^{q_1} dy \right)^{1/q_1} \\
&\qquad \times \left(\int_{S_j(a)} \left|b_2(y) - (b_2)_{B_\gamma(a)}\right|^{q_2} dy \right)^{1/q_2} \\
&\quad \lesssim \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}} \sum_{j=\gamma+1}^{\infty} (j+1-\gamma)^2 \nu\big(B_j(a)\big) \nu_1\big(B_j(a)\big) \nu_2\big(B_j(a)\big).
\end{aligned} \tag{4.8}
$$

Therefore from (4.8) and (ii) we get that

$$\begin{aligned} E_8 &= \frac{1}{\omega(B_\gamma(a))}\left(\frac{1}{B_\gamma(a)|_H}\int_B \left|T_k\big((b_1-(b_1)_{B_\gamma(a)})(b_2-(b_2)_{B_\gamma(a)})f^\infty\big)(x)\right|^r dx\right)^{1/r} \\ &\le \prod_{i=1}^{2}\|b_i\|_{GC_{q_i,\nu_i}}\|f\|_{GM_{q,\nu}}\frac{1}{\omega(B_\gamma(a))}\sum_{j=\gamma+1}^{\infty}(j+1-\gamma)^2\nu\big(B_j(a)\big)\nu_1\big(B_j(a)\big)\nu_2\big(B_j(a)\big) \\ &\lesssim \prod_{i=1}^{2}\|b_i\|_{GC_{q_i,\nu_i}}\|f\|_{GM_{q,\nu}}. \end{aligned}$$

Combining (4.5) and the estimates of $E_1, E_2, \dots, E_8$, we have

$$\frac{1}{\omega(B_\gamma(a))}\left(\frac{1}{|B_\gamma(a)|_H}\int_{B_\gamma(a)}\left|T_k^{(b_1,b_2)}(f)(x)\right|^r dx\right)^{1/r} \le \prod_{i=1}^{2}\|b_i\|_{GC_{q_i,\nu_i}}\|f\|_{GM_{q,\nu}},$$

which means that the commutator $T_k^{(b_1,b_2)}$ is bounded from $GM_{q,\nu}$ to $GM_{r,\omega}$.

Moreover, by Lemma 2.2 and the definition of $GM_{q,\omega}(\mathbb{Q}_p^n)$ it is obvious that the commutator $T^{\vec{b}}(f) = \lim_{k\to-\infty} T_k^{\vec{b}}(f)$ exists in the space of $GM_{q,\omega}$, and $T^{\vec{b}}$ is bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$.

Therefore the proof of Theorem 3.3 is complete. □

5 Conclusion

In this paper, we established the boundedness of a class of p-adic singular integral operators on the p-adic generalized Morrey spaces. We also considered the corresponding boundedness for the commutators generalized by the p-adic singular integral operators and p-adic Lipschitz functions or p-adic generalized Campanato functions.

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