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A generalized nonlinear Picone identity for the p-biharmonic operator and its applications

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Abstract

A generalized nonlinear Picone identity for the p-biharmonic operator is established in this paper. As applications, a Sturmian comparison principle to the p-biharmonic equation with singular term, a Liouville's theorem to the p-biharmonic system, and a generalized Hardy–Rellich type inequality are obtained.

MSC: Primary 26D10; secondary 26D15

Keywords: p-biharmonic operator; Generalized nonlinear Picone identity; Sturmian comparison principle; Liouville's theorem; Hardy–Rellich type inequality

1 Introduction and results

In 1971, Dunninger [1] established a Picone identity

$$\begin{aligned}
&\operatorname{div}\left[u\nabla(a\Delta u) - a\Delta u\nabla u - \frac{u^2}{v}\nabla(A\Delta v) + A\Delta v\nabla\left(\frac{u^2}{v}\right)\right] \\
&= -\frac{u^2}{v}\Delta(A\Delta v) + u\Delta(a\Delta u) + (A-a)(\Delta u)^2 \\
&\quad - A\left(\Delta u - \frac{u}{v}\Delta v\right)^2 + A\frac{2\Delta v}{v}\left(\nabla u - \frac{u}{v}\nabla v\right)^2,
\end{aligned} \tag{1.1}$$

where u, v, $a\Delta u$, $A\Delta v$ are twice continuously differentiable functions with $v \neq 0$ and a and A are positive weights. In [1], the integral form of (1.1) was used to study qualitative results for the fourth order elliptic system

$$\begin{aligned}
&\Delta\big(a(x)\Delta u\big) - c(x)u = 0, \\
&\Delta\big(A(x)\Delta v\big) - C(x)v = 0.
\end{aligned}$$

A Sturmian comparison principle, an integral inequality of Wirtinger type, and lower bound for eigenvalue were obtained. Jaroš [6] extended (1.1) to the case where $\Delta(a(x)\Delta u)$ and $\Delta(A(x)\Delta v)$ were replaced by the weighted p-biharmonic operators $\Delta(a(x)|\Delta u|^{p-2}\Delta u)$ and $\Delta(A(x)|\Delta v|^{p-2}\Delta v)$, respectively, and showed some results similar to [1] for the fourth order elliptic system

$$\Delta\big(a(x)|\Delta u|^{p-2}\Delta u\big) - c(x)|u|^{p-2}u = 0,$$
$$\Delta\big(A(x)|\Delta v|^{p-2}\Delta v\big) - C(x)|v|^{p-2}v = 0.$$

With some simplifications in (1.1), recently, Dwivedi and Tyagi [3] have obtained the following linear Picone identity (see Theorem 1.1) for the biharmonic operator $\Delta^2 u = \Delta(\Delta u)$ and gave several remarks on the qualitative questions such as Morse index and Hardy–Rellich type inequality.

Theorem 1.1 ([3]) *Let u and v be differentiable functions in $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) such that $u \geq 0$, $v > 0$, and $-\Delta v > 0$*

$$L(u,v) = \left(\Delta u - \frac{u}{v}\Delta v\right)^2 - \frac{2\Delta v}{v}\left(\nabla u - \frac{u}{v}\nabla v\right)^2,$$
$$R(u,v) = |\Delta u|^2 - \Delta\left(\frac{u^2}{v}\right)\Delta v.$$

Then $R(u,v) = L(u,v)$. Moreover, $L(u,v) \geq 0$, and $L(u,v) = 0$ if and only if $u = \alpha v$ for $\alpha \in \mathbb{R}$.

It is noteworthy that Dwivedi and Tyagi [4] established a Caccioppoli-type inequality by an application of Theorem 1.1. Moreover, Dwivedi and Tyagi [5] extended the result of Theorem 1.1 on Heisenberg group and obtained its applications.

Recently, Dwivedi [2] has extended the linear Picone identity in Theorem 1.1. He obtained the following linear Picone identity (see Theorem 1.2) for the p-biharmonic operator: $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$, $p > 1$.

Theorem 1.2 ([2]) *Let u and v be differentiable functions in $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) such that $u \geq 0$, $v > 0$, and $-\Delta v > 0$. Denote*

$$L(u,v) = |\Delta u|^p + \frac{(p-1)u^p}{v^p}|\Delta v|^p - \frac{pu^{p-1}}{v^{p-1}}|\Delta v|^{p-2}\Delta v\Delta u - \frac{p(p-1)u^{p-2}}{v^{p-1}}|\Delta v|^{p-2}\Delta v\left(\nabla u - \frac{u}{v}\nabla v\right)^2,$$
$$R(u,v) = |\Delta u|^p - \Delta\left(\frac{u^p}{v^{p-1}}\right)|\Delta v|^{p-2}\Delta v.$$

Then $R(u,v) = L(u,v)$. Moreover, $L(u,v) \geq 0$, and $L(u,v) = 0$ if and only if $u = \alpha v$ for $\alpha \in \mathbb{R}$.

Dwivedi and Tyagi [3] established a nonlinear Picone identity (see Theorem 1.3) for the biharmonic operator and also discussed some qualitative results for biharmonic equation (system).

Theorem 1.3 ([3]) *Let u and v be differentiable functions in $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) such that $u \geq 0$, $v > 0$, and $-\Delta v > 0$. Suppose that $f : \mathbb{R} \to (0,\infty)$ is a C^2 function such that $f'(y) \geq 1$*

and $f''(y) \le 0$, $\forall 0 < y \in \mathbb{R}$. *Denote*

$$L(u,v) = |\Delta u|^2 - \frac{|\Delta u|^2}{f'(v)} + \left(\frac{\Delta u}{\sqrt{f'(v)}} - \frac{u}{f(v)}\sqrt{f'(v)}\Delta v\right)^2 - \frac{2\Delta v}{f(v)}\left(\nabla u - \frac{uf'(v)}{f(v)}\nabla v\right)^2 + \frac{u^2 f''(v)}{f(v)}|\nabla v|^2\Delta v,$$

$$R(u,v) = |\Delta u|^2 - \Delta\left(\frac{u^2}{f(v)}\right)\Delta v.$$

Then $R(u,v) = L(u,v)$. *Moreover*, $L(u,v) \ge 0$, *and* $L(u,v) = 0$ *if and only if* $u = cv + d$ *for* $c, d \in \mathbb{R}$.

From the biharmonic operator to the p-biharmonic operator, Dwivedi [2] developed a nonlinear Picone identity of Dwivedi and Tyagi [3] in the following Theorem 1.4 and obtained some qualitative results for p-biharmonic equation (system).

Theorem 1.4 ([2]) *Let* u *and* v *be differentiable functions in* $\Omega \subset \mathbb{R}^n$ $(n \ge 3)$ *such that* $u \ge 0$, $v > 0$, *and* $-\Delta v > 0$. *Suppose that* $f : \mathbb{R} \to (0,\infty)$ *is a* C^2 *function such that* $f'(y) \ge (p-1)[f(y)]^{\frac{p-2}{p-1}}$, $p > 1$, *and* $f''(y) \le 0$, $\forall 0 < y \in \mathbb{R}$. *Denote*

$$L(u,v) = |\Delta u|^p + \frac{f'(v)u^p}{[f(v)]^2}|\Delta v|^p - \frac{pu^{p-1}}{f(v)}|\Delta v|^{p-2}\Delta v\Delta u + \frac{u^p f''(v)|\nabla v|^2}{[f(v)]^2}|\Delta v|^{p-2}\Delta v + \frac{u^p f''(v)|\nabla v|^2}{[f(v)]^2}|\Delta v|^{p-2}\Delta v,$$

$$R(u,v) = |\Delta u|^p - \Delta\left(\frac{u^p}{f(v)}\right)|\Delta v|^{p-2}\Delta v.$$

Then $R(u,v) = L(u,v)$. *Moreover*, $L(u,v) \ge 0$, *and* $L(u,v) = 0$ *if and only if* $u = cv + d$ *for* $c, d \in \mathbb{R}$.

The purpose of this paper is to present a generalized nonlinear Picone identity for the p-biharmonic operator, which extends the results of Dwivedi and Tyagi [3] and Dwivedi [2]. As applications, a Sturmian comparison principle to the p-biharmonic equation with singular term, a Liouville's theorem to the p-biharmonic system, and a generalized Hardy–Rellich type inequality are obtained. Our main result is described as follows.

Theorem 1.5 *Let* u *and* v *be differentiable functions in* $\Omega \subset \mathbb{R}^n$ $(n \ge 3)$ *such that* $u \ge 0$, $v > 0$, *and* $-\Delta v > 0$. *Suppose that* $f : \mathbb{R} \to (0,\infty)$ *and* $g : \mathbb{R} \to (0,\infty)$ *are* C^2 *functions with*

$$\begin{cases} g(u) > 0, & g'(u) > 0, \quad g''(u) > 0, \quad u > 0, \quad \text{if } x \in \Omega, \\ g(u) = 0, & g'(u) = 0, \quad g''(u) = 0, \quad u = 0, \quad \text{if } x \in \partial\Omega, \end{cases}$$

and $f(v) > 0, f'(v) > 1, f''(v) \le 0$ *in* Ω *such that* f *and* g *satisfy*

$$\frac{g(u)f'(v)}{[f(v)]^2}|\Delta v|^p \ge (p-1)\left[\frac{g'(u)|\Delta v|^{p-1}}{pf(v)}\right]^{\frac{p}{p-1}} - \frac{g(u)f''(v)|\nabla v|^2}{[f(v)]^2}|\Delta v|^{p-2}\Delta v \tag{1.2}$$

and

$$\sqrt{2g''(u)g(u)} \geq g'(u), \tag{1.3}$$

respectively. Denote

$$\begin{aligned} L(u,v) = |\Delta u|^p - \biggl(& \frac{g''(u)|\nabla u|^2}{f(v)} + \frac{g'(u)\Delta u}{f(v)} \\ & - \frac{2g'(u)f'(v)\nabla u \cdot \nabla v}{[f(v)]^2} - \frac{g(u)f''(v)|\nabla v|^2}{[f(v)]^2} - \frac{g(u)f'(v)\Delta v}{[f(v)]^2} \\ & + \frac{2g(u)[f'(v)]^2|\nabla v|^2}{[f(v)]^3} \biggr) |\Delta v|^{p-2}\Delta v \end{aligned} \tag{1.4}$$

and

$$R(u,v) = |\Delta u|^p - \Delta \left(\frac{g(u)}{f(v)} \right) |\Delta v|^{p-2}\Delta v, \tag{1.5}$$

respectively. Then $R(u,v) = L(u,v)$. *Moreover,* $L(u,v) \geq 0$, *and* $L(u,v) = 0$ *if and only if*

$$u = cv, \quad c \in \mathbb{R}, \tag{1.6}$$

$$|\Delta u|^p = \left[\frac{g'(u)|\Delta v|^{p-1}}{pf(v)} \right]^{\frac{p}{p-1}}, \tag{1.7}$$

$$\frac{g(u)f'(v)}{[f(v)]^2}|\Delta v|^p = (p-1)\left[\frac{g'(u)|\nabla v|^{p-1}}{pf(v)} \right]^{\frac{p}{p-1}} - \frac{g(u)f''(v)|\nabla v|^2}{[f(v)]^2}|\Delta v|^{p-2}\Delta v, \tag{1.8}$$

$$\sqrt{g''(u)}\nabla u = \frac{\sqrt{2g(u)}f'(v)\nabla v}{f(v)} \quad \textit{and} \quad \sqrt{2g''(u)g(u)} = g'(u). \tag{1.9}$$

Remark 1.6 If $p = 2$, $g(u) = u^2$ and $f(v) = v$ in (1.4) and (1.5), which is the result of Dwivedi and Tyagi [3] (see Theorem 1.1).

Remark 1.7 If $p = 2$, $g(u) = u^2$ and $f'(v) \geq 1$ and $f''(v) \leq 0$, $\forall 0 < v \in \mathbb{R}$ in (1.4) and (1.5), which is the result of Dwivedi and Tyagi [3] (see Theorem 1.3).

Remark 1.8 If $p > 2$, $g(u) = u^p$ and $f(v) = v^{p-1}$ in (1.4) and (1.5), which is the result of Dwivedi [2] (see Theorem 1.2).

Remark 1.9 If $p > 2$, $g(u) = u^p$ and $f'(v) \geq (p-1)[f(v)]^{\frac{p-2}{p-1}}$, $p > 1$ and $f''(v) \leq 0$, $\forall 0 < v \in \mathbb{R}$ in (1.4) and (1.5), which is the result of Dwivedi [2] (see Theorem 1.4).

We give the proof of Theorem 1.5 in the following.

Proof We first prove that $R(u,v) = L(u,v)$ by expanding $R(u,v)$:

$$\begin{aligned} R(u,v) &= |\Delta u|^p - \Delta \left(\frac{g(u)}{f(v)} \right) |\Delta v|^{p-2}\Delta v \\ &= |\Delta u|^p - \biggl(\frac{g''(u)|\nabla u|^2}{f(v)} + \frac{g'(u)\Delta u}{f(v)} - \frac{2g'(u)f'(v)\nabla u \cdot \nabla v}{[f(v)]^2} - \frac{g(u)f''(v)|\nabla v|^2}{[f(v)]^2} \end{aligned}$$

$$
\begin{aligned}
&-\frac{g(u)f'(v)\Delta v}{[f(v)]^2}+\frac{2g(u)[f'(v)]^2|\nabla v|^2}{[f(v)]^3}\Bigg)|\Delta v|^{p-2}\Delta v\\
&=|\Delta u|^p-\frac{g'(u)\Delta u}{f(v)}|\Delta v|^{p-2}\Delta v+\frac{g(u)f'(v)}{[f(v)]^2}|\Delta v|^p\\
&-\frac{|\Delta v|^{p-2}\Delta v}{f(v)}\left(g''(u)|\nabla u|^2-\frac{2g'(u)f'(v)\nabla u\cdot\nabla v}{f(v)}+\frac{2g(u)[f'(v)]^2|\nabla v|^2}{[f(v)]^2}\right)\\
&+\frac{g(u)f''(v)|\nabla v|^2}{[f(v)]^2}|\Delta v|^{p-2}\Delta v\\
&=L(u,v).
\end{aligned}
$$

Next we verify $L(u,v)\geq 0$, we can rewrite $L(u,v)$ as

$$
\begin{aligned}
L(u,v)\ =\ &p\left(\frac{1}{p}|\Delta u|^p+\frac{p-1}{p}\left[\frac{g'(u)|\Delta v|^{p-1}}{pf(v)}\right]^{\frac{p}{p-1}}\right)-\frac{g'(u)|\Delta u|}{f(v)}|\Delta v|^{p-1}\\
&+\frac{g(u)f'(v)}{[f(v)]^2}|\Delta v|^p-(p-1)\left[\frac{g'(u)|\Delta v|^{p-1}}{pf(v)}\right]^{\frac{p}{p-1}}+\frac{g(u)f''(v)|\nabla v|^2}{[f(v)]^2}|\Delta v|^{p-2}\Delta v\\
&+\frac{g'(u)|\Delta v|^{p-2}}{f(v)}\big(|\Delta u||\Delta v|-\Delta u\Delta v\big)\\
&-\frac{|\Delta v|^{p-2}\Delta v}{f(v)}(\left(\sqrt{g''(u)}\nabla u-\frac{\sqrt{2g(u)}f'(v)\nabla v}{f(v)}\right)^2\\
&+\frac{2(\sqrt{2g''(u)g(u)}-g'(u))f'(v)\nabla u\cdot\nabla v}{f(v)}\\
:=\ &F_1+F_2+F_3+F_4,
\end{aligned}
\tag{1.10}
$$

where

$$
\begin{aligned}
F_1&=p\left(\frac{1}{p}|\Delta u|^p+\frac{p-1}{p}\left[\frac{g'(u)|\Delta v|^{p-1}}{pf(v)}\right]^{\frac{p}{p-1}}\right)-\frac{g'(u)|\Delta u|}{f(v)}|\Delta v|^{p-1},\\
F_2&=\frac{g(u)f'(v)}{[f(v)]^2}|\Delta v|^p-(p-1)\left[\frac{g'(u)|\Delta v|^{p-1}}{pf(v)}\right]^{\frac{p}{p-1}}+\frac{g(u)f''(v)|\nabla v|^2}{[f(v)]^2}|\Delta v|^{p-2}\Delta v,\\
F_3&=\frac{g'(u)|\Delta v|^{p-2}}{f(v)}\big(|\Delta u||\Delta v|-\Delta u\Delta v\big),\\
F_4&=-\frac{|\Delta v|^{p-2}\Delta v}{f(v)}\Bigg(\left(\sqrt{g''(u)}\nabla u-\frac{\sqrt{2g(u)}f'(v)\nabla v}{f(v)}\right)^2\\
&\quad+\frac{2(\sqrt{2g''(u)g(u)}-g'(u))f'(v)\nabla u\cdot\nabla v}{f(v)}\Bigg).
\end{aligned}
$$

We now recall Young's inequality

$$
a_0b_0\geq\frac{{a_0}^p}{p}+\frac{{b_0}^q}{q},
\tag{1.11}
$$

where $a_0 \geq 0$, $b_0 \geq 0$, $p > 1$, $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, the equality holds if and only if ${a_0}^p = {b_0}^q = {b_0}^{\frac{p}{p-1}}$. Setting $a_0 = |\Delta u|$, $b_0 = \frac{g'(u)|\Delta v|^{p-1}}{pf(v)}$ in (1.11), we obtain

$$\frac{g'(u)|\Delta u|}{f(v)}|\Delta v|^{p-1} \leq p\left(\frac{1}{p}|\Delta u|^p - \frac{p-1}{p}\left[\frac{g'(u)|\Delta v|^{p-1}}{pf(v)}\right]^{\frac{p}{p-1}}\right),$$

which implies $F_1 \geq 0$. Clearly $F_2 \geq 0$ by (1.2). Since $|\Delta u||\Delta v| - \Delta u \Delta v \geq 0$, the equality holds if and only if $u = cv$, $c \in \mathbb{R}$, and combining with $\frac{g'(u)|\Delta v|^{p-2}}{f(v)} \geq 0$, we obtain $F_3 \geq 0$. By $-\Delta v > 0$, $f(v) > 0$, and (1.3), we have $F_4 \geq 0$. Hence $L(u,v) \geq 0$ from (1.10).

We now verify $L(u,v) = 0$ by (1.6)–(1.9). It follows from (1.6) that there exists a positive constant c such that $u = cv$, namely we have

$$|\Delta v||\Delta u| - \Delta v \cdot \Delta u = c|\Delta v||\Delta v| - c\Delta v \cdot \Delta v = c|\Delta v|^2 - c|\Delta v|^2 = 0,$$

which implies $F_3 = 0$. By $|\Delta u|^p = [\frac{g'(u)|\Delta v|^{p-1}}{pf(v)}]^{\frac{p}{p-1}}$ in (1.7), we obtain

$$\frac{g'(u)|\Delta v|^{p-1}}{f(v)} = p|\Delta u|^{p-1}. \tag{1.12}$$

It follows from (1.12) that

$$\begin{aligned} I &= p\left(\frac{1}{p}|\Delta u|^p + \frac{p-1}{p}\left[\frac{g'(u)|\Delta v|^{p-1}}{pf(v)}\right]^{\frac{p}{p-1}}\right) - \frac{g(u)|\Delta u||\Delta v|^{p-1}}{f(v)} \\ &= p\left(\frac{1}{p}|\Delta u|^p + \frac{p-1}{p}|\Delta u|^p\right) - |\Delta u|p|\Delta u|^{p-1} \\ &= |\Delta u|^p + (p-1)|\Delta u|^p - p|\Delta u|^p \\ &= 0. \end{aligned}$$

We can prove $F_2 = 0$ by (1.8). A direct calculation shows

$$\left(\sqrt{g''(u)}\nabla u - \frac{\sqrt{2g(u)}f'(v)\nabla v}{f(v)}\right)^2 = 0$$

by $\sqrt{g''(u)}\nabla u = \frac{\sqrt{2g(u)}f'(v)\nabla v}{f(v)}$ in (1.9), we can also show

$$\frac{2(\sqrt{2g''(u)g(u)} - g'(u))f'(v)\nabla u \cdot \nabla v}{f(v)} = 0$$

by $\sqrt{2g''(u)g(u)} = g'(u)$ in (1.9), hence $F_4 = 0$ by (1.9). Summing up these, it follows $L(u,v) = F_1 + F_2 + F_3 + F_4 = 0$. Hence we can conclude that $L(u,v) = 0$ if and only if (1.6)–(1.9) hold. In fact, if $u = 0$, it clearly follows. If $u \neq 0$, the conclusion holds from the above process of proof. □

2 Applications

Throughout this section, we always assume that f and g are $C^2(\Omega)$ functions and satisfy the conditions in Theorem 1.5, unless otherwise stated, and give applications for the generalized nonlinear Picone identity. We first show a Sturmian comparison principle to the p-biharmonic equation with singular term by Theorem 1.5 as follows.

Proposition 2.1 *Let $k_1(x)$ and $k_2(x)$ be two continuous weighted functions with $k_1(x) < k_2(x)$. Assume that there exists a positive solution satisfying*

$$\begin{cases} \Delta_p^2 u = \frac{k_1(x)g(u)}{u}, & x \in \Omega, \\ g(u) > 0, \quad u > 0, & x \in \Omega, \\ g(u) = 0, \quad u = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Then any nontrivial solution v of the following p-biharmonic equation

$$\Delta_p^2 v = k_2(x) f(v), \quad x \in \Omega, \tag{2.2}$$

must change sign.

Proof Suppose that v of (2.2) does not change sign. Without loss of generality, we assume that $v > 0$ in Ω. By (2.1), (2.2), and Theorem 1.5, we have

$$\begin{aligned} 0 &\le \int_\Omega L(u,v)\,dx = \int_\Omega R(u,v)\,dx \\ &= \int_\Omega |\Delta u|^p\,dx - \int_\Omega \Delta\left(\frac{g(u)}{f(v)}\right)|\Delta v|^{p-2}\Delta v\,dx \\ &= \int_\Omega |\Delta u|^p\,dx - \int_\Omega \frac{g(u)}{f(v)}\Delta_p^2 v\,dx \\ &= \int_\Omega k_1(x)g(u)\,dx - \int_\Omega k_2(x)g(u)\,dx \\ &= \int_\Omega \big(k_1(x) - k_2(x)\big)g(u)\,dx \\ &< 0, \end{aligned}$$

which is a contradiction. This accomplishes the proof. □

We next show a Liouville's theorem for the p-biharmonic system by Theorem 1.5 as follows.

Proposition 2.2 *Let $(u,v) \in [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)] \times [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]$ be a pair of weak solutions to the p-biharmonic system*

$$\begin{cases} \Delta_p^2 u = f(v), & & & & x \in \Omega, \\ \Delta_p^2 v = \frac{[f(v)]^2 u}{g(u)}, & & & & x \in \Omega, \\ g(u) > 0, & f(v) > 0, & u > 0, & v > 0, & x \in \Omega, \\ g(u) = 0, & f(v) = 0, & u = 0, & v = 0, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

Then $u = cv$ in Ω, where c is a constant.

Proof For any test functions $\phi_1, \phi_2 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, it follows from (2.3) that

$$\int_\Omega |\Delta u|^{p-2} \Delta u \Delta \phi_1 \, dx = \int_\Omega f(v)\phi_1 \, dx, \tag{2.4}$$

$$\int_\Omega |\Delta v|^{p-2} \Delta v \Delta \phi_2 \, dx = \int_\Omega \frac{[f(v)]^2 u}{g(u)} \phi_2 \, dx. \tag{2.5}$$

Taking $\phi_1 = u$ and $\phi_2 = \frac{g(u)}{f(v)}$ in (2.4) and (2.5), respectively, we obtain

$$\int_\Omega |\Delta u|^p \, dx = \int_\Omega f(v) u \, dx = \int_\Omega \Delta\left(\frac{g(u)}{f(v)}\right) |\Delta v|^{p-2} \Delta v \, dx,$$

which implies

$$\int_\Omega L(u,v) \, dx = \int_\Omega R(u,v) \, dx = \int_\Omega |\Delta u|^p \, dx - \int_\Omega \Delta\left(\frac{g(u)}{f(v)}\right) |\Delta v|^{p-2} \Delta v \, dx = 0,$$

hence the conclusion follows by an application of Theorem 1.5. □

Finally, we obtain a generalized Hardy–Rellich type inequality by Theorem 1.5.

Proposition 2.3 *Suppose that a function* $0 < v \in C^2(\Omega)$ *with* $-\Delta v > 0$ *in* Ω, *and it satisfies*

$$\Delta_p^2 v \geq \lambda k(x) f(v), \quad x \in \Omega, \tag{2.6}$$

where $\lambda > 0$ *is a constant*, $k(x)$ *is a positive continuous function. Then there holds*

$$\int_\Omega |\Delta u|^p \, dx \geq \lambda \int_\Omega k(x) g(u) \, dx \tag{2.7}$$

for any $0 \leq u \in C_0^2(\Omega)$.

Proof It follows from (2.6) and Theorem 1.5 that

$$\begin{aligned} 0 &\leq \int_\Omega L(u,v) \, dx = \int_\Omega R(u,v) \, dx \\ &= \int_\Omega |\Delta u|^p \, dx - \int_\Omega \Delta\left(\frac{g(u)}{f(v)}\right) |\Delta v|^{p-2} \Delta v \, dx \\ &= \int_\Omega |\Delta u|^p \, dx - \int_\Omega \frac{g(u)}{f(v)} \Delta_p^2 v \, dx \\ &\leq \int_\Omega |\Delta u|^p \, dx - \int_\Omega \lambda k(x) g(u) \, dx, \end{aligned}$$

which implies (2.7). □

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Authors' contributions
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