Kwun et al. *Journal of Inequalities and Applications* (2019) 2019:44
https://doi.org/10.1186/s13660-019-1993-y

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Hermite–Hadamard-type inequalities for functions whose derivatives are η-convex via fractional integrals

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Abstract

In the present research, we develop some integral inequalities of Hermite–Hadamard type for differentiable η-convex functions. Moreover, our results include several new and known results as particular cases.

Keywords: Convex function; η-convex function; Hermite–Hadamard-type inequality; Fractional integral

1 Introduction

Throughout this paper, let I be an interval in $\mathbb{R}$. Also consider $\eta : A \times A \to B$ for appropriate $A, B \subseteq \mathbb{R}$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, and let $a_1, a_2 \in I$ with $a_1 < a_2$. The following double inequality

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)\, dx \leq \frac{f(a_1) + f(a_2)}{2} \tag{1}$$

is known in the literature as the Hadamard inequality for convex functions. Fejer [1] gave a generalization of (1) as follows. If $f : [a_1, a_2] \to \mathbb{R}$ is a convex function and $g : [a_1, a_2] \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $\frac{a_1+a_2}{2}$, then

$$f\left(\frac{a_1 + a_2}{2}\right) \int_{a_1}^{a_2} g(x)\, dx \leq \int_{a_1}^{a_2} f(x)g(x)\, dx \leq \frac{f(a_1) + f(a_2)}{2} \int_{a_1}^{a_2} g(x)\, dx. \tag{2}$$

Since the Hermite–Hadamard inequality and fractional integrals have a wide range of applications, many researchers extend their studies to Hermite–Hadamard-type inequalities involving fractional integrals.

In 2015, Iscan [2] obtained Hermite–Hadamard–Fejér-type inequalities for convex functions via fractional integrals. In 2017, Farid and Tariq [3] developed fractional integral inequalities for m-convex functions. Also, Farid and Abbas [4] established Hermite–Hadamard–Fejér-type inequalities for p-convex functions via generalized fractional integrals. For recent generalizations, we refer to [5–7], and [8].

Xi and Qi [9], Ozdemir et al. [10], and Sarikaya et al. [5] established Hermite–Hadamard-type inequalities for convex functions. Gordji et al. [11] introduced an important generalization of convexity known as η-convexity.

Definition 1.1 ([11]) A function $f : I \to \mathbb{R}$ is called η-convex if

$$f\big(\alpha x + (1-\alpha)y\big) \le f(y) + \alpha\eta\big(f(x), f(y)\big) \tag{3}$$

for all $x, y \in I$ and $\alpha \in [0,1]$.

Theorem 1.1 ([10]) *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on the interior I° of I such that $f'' \in L^1[a_1,a_2]$, where $a_1, a_2 \in I$ with $a_1 < a_2$. If $|f|$ is convex on $[a_1,a_2]$, then*

$$\begin{aligned}&\left|f\left(\frac{a_1+a_2}{2}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\,dx\right| \\ &\quad\le \frac{(a_2-a_1)^2}{192}\left\{\left|f''(a_1)\right| + 6\left|f''\left(\frac{a_1+a_2}{2}\right)\right| + \left|f''(a_2)\right|\right\}.\end{aligned} \tag{4}$$

Theorem 1.2 ([10]) *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f'' \in L^1[a_1,a_2]$, where $a_1, a_2 \in I$ with $a_1 < a_2$. If $|f''|^q$ for $q \ge 1$ is convex on $[a_1,a_2]$, then*

$$\begin{aligned}&\left|f\left(\frac{a_1+a_2}{2}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\,dx\right| \\ &\quad\le \frac{(a_2-a_1)^2}{48}\left(\frac{3}{4}\right)^{\frac{1}{q}}\left\{\left(\frac{|f''(a_1)|^q}{3} + \left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\ &\qquad\left. + \left(\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q + \frac{|f''(a_2)|^q}{3}\right)^{\frac{1}{q}}\right\}.\end{aligned} \tag{5}$$

Lemma 1.1 ([9]) *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L^1[a_1,a_2]$, where $a_1, a_2 \in I$ with $a_1 < a_2$. If $\alpha, \beta \in \mathbb{R}$, then*

$$\begin{aligned}&\frac{\alpha f(a_1) + \beta f(a_2)}{2} + \frac{2-\alpha-\beta}{2} f\left(\frac{a_1+a_2}{2}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\,dx \\ &\quad= \frac{a_2-a_1}{4}\int_0^1 \left[(1-\alpha-t)f'\left(ta_1 + (1-t)\frac{a_1+a_2}{2}\right)\right. \\ &\qquad\left. + (\beta-t)f'\left(t\frac{a_1+a_2}{2} + (1-t)a_2\right)\right]dt.\end{aligned} \tag{6}$$

Lemma 1.2 ([9]) *For $s > 0$ and $0 \le \epsilon \le 1$, we have*

$$\begin{aligned}&\int_0^1 |\epsilon - t|^s\,dt = \frac{\epsilon^{s+1} + (1-\epsilon)^{s+1}}{s+1}, \\ &\int_0^1 t|\epsilon - t|^s\,dt = \frac{\epsilon^{s+2} + (s+1+\epsilon)(1-\epsilon)^{s+1}}{(s+1)(s+2)}.\end{aligned} \tag{7}$$

The paper is organized as follows. In Sect. 2, we establish Hermite–Hadamard- and Fejer-type inequalities for η-convex functions. In the last section, we derive Fractional integral inequalities for η-convex functions.

2 Hermite–Hadamard- and Fejer-type inequalities

Theorem 2.1 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be an η-convex function with $f \in L^1[a_1, a_2]$, where a_1, $a_2 \in I$ with $a_1 < a_2$,Then*

$$f\left(\frac{a_1+a_2}{2}\right) - \frac{1}{2(a_2-a_1)}\int_{a_1}^{a_2} \eta\big(f(a_1+a_2-x), f(x)\big)\,dx \le \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\,dx \le f(a_2) + \frac{1}{2}\eta\big(f(a_1), f(a_2)\big). \tag{8}$$

Proof According to (3), with $x = ta_1 + (1-t)a_2$, $y = (1-t)a_1 + ta_2$, and $\alpha = \frac{1}{2}$, where $t \in [0,1]$, we find that

$$f\left(\frac{a_1+a_2}{2}\right) \le f\big((1-t)a_1 + ta_2\big) + \frac{1}{2}\eta\big(f\big(ta_1+(1-t)a_2\big), f\big((1-t)a_1+ta_2\big)\big).$$

Thus by integrating we obtain

$$\begin{aligned} f\left(\frac{a_1+a_2}{2}\right) &\le \int_0^1 f\big((1-t)a_1+ta_2\big)\,dt \\ &\quad + \frac{1}{2}\int_0^1 \eta\big(f\big(ta_1+(1-t)a_2\big), f\big((1-t)a_1+ta_2\big)\big)\,dt \\ &\le \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\,dx + \frac{1}{2(a_2-a_1)}\int_{a_1}^{a_2} \eta\big(f(a_1+a_2-x), f(x)\big)\,dx, \end{aligned}$$

so that

$$f\left(\frac{a_1+a_2}{2}\right) - \frac{1}{2(a_2-a_1)}\int_{a_1}^{a_2} \eta\big(f(a_1+a_2-x), f(x)\big)\,dx \le \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\,dx, \tag{9}$$

and the first inequality is proved. Taking $x = a_1$ and $y = a_2$ in (3), we get

$$f\big(\alpha a_1 + (1-\alpha)a_2\big) \le f(a_2) + \alpha\eta\big(f(a_1), f(a_2)\big).$$

Integrating this inequality with respect to α over $[0,1]$, we get

$$\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\,dx \le f(a_2) + \frac{1}{2}\eta\big(f(a_1), f(a_2)\big). \tag{10}$$

Clearly, (9) and (10) yield (8). □

Remark 2.1 Taking $\eta(x,y) = x - y$, we reduce (8) to inequality (1).

Theorem 2.2 *Let f and g be nonnegative η-convex functions with $fg \in L^1[a_1, a_2]$, where $a_1, a_2 \in I$, $a_1 < a_2$. Then*

$$\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)g(x)\,dx \le M'(a_1, a_2), \tag{11}$$

where

$$M'(a_1,a_2) = f(a_2)g(a_2) + \frac{1}{2}f(a_2)\eta\big(g(a_1),g(a_2)\big) + \frac{1}{2}g(a_2)\eta\big(f(a_1),f(a_2)\big) + \frac{1}{3}\eta\big(f(a_1),f(a_2)\big)\eta\big(g(a_1),g(a_2)\big).$$

Proof Since f and g are η-convex functions, we have

$$f\big(ta_1 + (1-t)a_2\big) \le f(a_2) + t\eta\big(f(a_1),f(a_2)\big),$$
$$g\big(ta_1 + (1-t)a_2\big) \le g(a_2) + t\eta\big(g(a_1),g(a_2)\big)$$

for all $t \in [0,1]$. Since f and g are nonnegative, we have

$$\begin{aligned} &f\big(ta_1 + (1-t)a_2\big)g\big(ta_1 + (1-t)a_2\big) \\ &\quad\le f(a_2)g(a_2) + tf(a_2)\eta\big(g(a_1),g(a_2)\big) \\ &\qquad + tg(a_2)\eta\big(f(a_1),f(a_2)\big) + t^2\eta\big(f(a_1),f(a_2)\big)\eta\big(g(a_1),g(a_2)\big). \end{aligned}$$

Integrating both sides of the inequality over $[0,1]$, we obtain

$$\begin{aligned} &\int_0^1 f\big(ta_1 + (1-t)a_2\big)g\big(ta_1 + (1-t)a_2\big)\,dt \\ &\quad\le f(a_2)g(a_2) + \frac{1}{2}f(a_2)\eta\big(g(a_1),g(a_2)\big) + \frac{1}{2}g(a_2)\eta\big(f(a_1),f(a_2)\big) \\ &\qquad + \frac{1}{3}\eta\big(f(a_1),f(a_2)\big)\eta\big(g(a_1),g(a_2)\big). \end{aligned}$$

Then

$$\frac{1}{a_2 - a_1}\int_{a_1}^{a_2} f(x)g(x)\,dx \le M'(a_1,a_2).$$

□

Remark 2.2 By taking $\eta(x,y) = x - y$ inequality (11) becomes inequality (1.4) in [5].

Theorem 2.3 *Let f be an η-convex function with $f \in L^1[a_1,a_2]$, where $a_1, a_2 \in I$, $a_1 < a_2$, and let $g : [a_1,a_2] \to \mathbb{R}$ be nonnegative, integrable, and symmetric about $\frac{(a_1+a_2)}{2}$. Then*

$$\int_{a_1}^{a_2} f(y)g(y)\,dy \le \left[f(a_2) + \frac{1}{2}\eta\big(f(a_1),f(a_2)\big)\right]\int_{a_1}^{a_2} g(y)\,dy. \tag{12}$$

Proof Since, f is an η-convex function and g is nonnegative, integrable. and symmetric about $\frac{(a_1+a_2)}{2}$, we find that

$$\begin{aligned} \int_{a_1}^{a_2} f(y)g(y)\,dy &= \frac{1}{2}\left[\int_{a_1}^{a_2} f(y)g(y)\,dy + \int_{a_1}^{a_2} f(a_1+a_2-y)g(a_1+a_2-y)\,dy\right] \\ &= \frac{1}{2}\int_{a_1}^{a_2}\left[\big(f(y) + f(a_1+a_2-y)\big)g(y)\,dy\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{a_1}^{a_2}\left[f\left(\frac{a_2-y}{a_2-a_1}a_1+\frac{y-a_1}{a_2-a_1}a_2\right)\right. \\
&\quad \left.+f\left(\frac{y-a_1}{a_2-a_1}a_1+\frac{a_2-y}{a_2-a_1}a_2\right)\right]g(y)\,dy \\
&\le \frac{1}{2}\int_{a_1}^{a_2}\left[\left(f(a_2)+\frac{a_2-y}{a_2-a_1}\eta\big(f(a_1),f(a_2)\big)\right)\right. \\
&\quad \left.+\left(f(a_2)+\frac{y-a_1}{a_2-a_1}\eta\big(f(a_1),f(a_2)\big)\right)\right]g(y)\,dy \\
&\le \left[f(a_2)+\frac{1}{2}\eta\big(f(a_1),f(a_2)\big)\right]\int_{a_1}^{a_2}g(y)\,dy. \qquad \square
\end{aligned}
$$

Remark 2.3 If we choose $\eta(x,y) = x - y$ and $g(x) = 1$, then (12) reduces to the second inequality in (1), and if we take $\eta(x,y) = x - y$, then (12) reduces to the second inequality in (2).

3 Fractional integral inequalities

Theorem 3.1 *Let $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ be a differentiable mapping on I^0 with $f' \in L^1[a_1,a_2]$, where $a_1, a_2 \in I$, $a_1 < a_2$. If $|f'(x)|^q$ for $q \ge 1$ is η-convex on $[a_1,a_2]$ and $0 \le \alpha, \beta \le 1$, then*

$$
\begin{aligned}
&\left|\frac{\alpha f(a_1)+\beta f(a_2)}{2}+\frac{2-\alpha-\beta}{2}f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)\,dx\right| \\
&\le \frac{a_2-a_1}{8}\left(\frac{1}{6}\right)^{\frac{1}{q}}\Big\{\big(1-2\alpha+2\alpha^2\big)^{1-\frac{1}{q}}\Big[\big(6-12\alpha+12\alpha^2\big)\big|f'(a_2)\big|^q \\
&\quad +\big(4-9\alpha+12\alpha^2-2\alpha^3\big)\eta\big(\big|f'(a_1)\big|^q,\big|f'(a_2)\big|^q\big)\Big]^{\frac{1}{q}} \\
&\quad +\big(1-2\beta+2\beta^2\big)^{1-\frac{1}{q}}\Big[\big(6-12\beta+12\beta^2\big)\big|f'(a_2)\big|^q \\
&\quad +\big(2-3\beta+2\beta^3\big)\eta\big(\big|f'(a_1)\big|^q,\big|f'(a_2)\big|^q\big)\Big]^{\frac{1}{q}}\Big\}. \qquad (13)
\end{aligned}
$$

Proof For $q > 1$, by Lemma 1.1, the η-convexity of $|f'(x)|^q$ on $[a_1,a_2]$, and the Hölder integral inequality, we have

$$
\begin{aligned}
&\left|\frac{\alpha f(a_1)+\beta f(a_2)}{2}+\frac{2-\alpha-\beta}{2}f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)\,dx\right| \\
&\le \frac{a_2-a_1}{4}\left[\int_0^1|1-\alpha-t|\left|f'\left(ta_1+(1-t)\frac{a_1+a_2}{2}\right)\right|dt\right. \\
&\quad \left.+\int_0^1|\beta-t|\left|f'\left(t\frac{a_1+a_2}{2}+(1-t)a_2\right)\right|dt\right] \\
&\le \frac{a_2-a_1}{4}\left\{\left(\int_0^1|1-\alpha-t|\,dt\right)^{1-\frac{1}{q}}\left[\int_0^1|1-\alpha-t|\left(\big|f'(a_2)\big|^q\right.\right.\right. \\
&\quad \left.\left.+\left(\frac{1+t}{2}\right)\eta\big(\big|f'(a_1)\big|^q,\big|f'(a_2)\big|^q\big)\right)dt\right]^{\frac{1}{q}}+\left(\int_0^1|\beta-t|\,dt\right)^{1-\frac{1}{q}} \\
&\quad \left.\times\left[\int_0^1|\beta-t|\left(\big|f'(a_2)\big|^q+\left(\frac{t}{2}\right)\eta\big(\big|f'(a_1)\big|^q,\big|f'(a_2)\big|\big)\right)dt\right]^{\frac{1}{q}}\right\}. \qquad (14)
\end{aligned}
$$

Using Lemma 1.2, by a direct calculation we get

$$\begin{aligned}
&\int_0^1 |1-\alpha-t|\left(\left|f'(a_2)\right|^q+\left(\frac{1+t}{2}\right)\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\right)dt\\
&=\left(\left|f'(a_2)\right|^q+\frac{1}{2}\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\right)\int_0^1|1-\alpha-t|\,dt\\
&\quad+\frac{1}{2}\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\int_0^1 t|1-\alpha-t|\,dt\\
&=\left(\left|f'(a_2)\right|^q+\frac{1}{2}\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\right)\left(\frac{1}{2}-\alpha+\alpha^2\right)\\
&\quad+\frac{1}{12}\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\left[(1-\alpha)^3+\alpha^2(3-\alpha)\right]\\
&=\frac{1}{2}\left(1-2\alpha+2\alpha^2\right)\left|f'(a_2)\right|^q+\frac{1}{12}\left(4-9\alpha+12\alpha^2-2\alpha^3\right)\\
&\quad\times\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 |\beta-t|\left(\left|f'(a_2)\right|^q+\left(\frac{t}{2}\right)\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\right)dt\\
&=\left|f'(a_2)\right|^q\int_0^1|\beta-t|\,dt+\frac{1}{2}\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\int_0^1 t|\beta-t|\,dt\\
&=\left|f'(a_2)\right|^q\left(\frac{1}{2}-\beta-\beta^2\right)+\frac{1}{12}\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\left(\beta^3+(2+\beta)(1-\beta)^2\right)\\
&=\frac{1}{2}\left(1-2\beta+2\beta^2\right)\left|f'(a_2)\right|^q+\frac{1}{12}\left(2-3\beta+2\beta^3\right)\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right).
\end{aligned}$$

Substituting these two inequalities into inequality (14) and using Lemma 1.2 result in inequality (13) for $q > 1$.

For $q = 1$, from Lemmas 1.1 and 1.2 it follows that

$$\begin{aligned}
&\left|\frac{\alpha f(a_1)+\beta f(a_2)}{2}+\frac{2-\alpha-\beta}{2}f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)\,dx\right|\\
&\le\frac{a_2-a_1}{4}\left\{\int_0^1|1-\alpha-t|\left(\left|f'(a_2)\right|+\left(\frac{1+t}{2}\right)\eta\left(\left|f'(a_1)\right|,\left|f'(a_2)\right|\right)\right)dt\right.\\
&\quad\left.+\int_0^1|\beta-t|\left(\left|f'(a_2)\right|+\frac{t}{2}\eta\left(\left|f'(a_1)\right|,\left|f'(a_2)\right|\right)\right)dt\right\}\\
&=\frac{a_2-a_1}{48}\left\{\left(6-12\alpha+12\alpha^2\right)\left|f'(a_2)\right|+\left(4-9\alpha+12\alpha^2-2\alpha^3\right)\right.\\
&\quad\times\eta\left(\left|f'(a_1)\right|,\left|f'(a_2)\right|\right)+\left(6-12\beta+12\beta^2\right)\left|f'(a_2)\right|\\
&\quad\left.+\left(2-3\beta+2\beta^3\right)\eta\left(\left|f'(a_1)\right|,\left|f'(a_2)\right|\right)\right\}.
\end{aligned}\tag{15}$$

□

Remark 3.1 If we take $\eta(x,y) = x - y$, then inequality (13) reduces to inequality (3.1) in [9].

Taking $\alpha = \beta$ in Theorem 3.1, we derive the following corollary.

Corollary 3.1 *Let $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ be a differentiable mapping on I^0 with $f' \in L^1[a_1,a_2]$, where $a_1,a_2 \in I$, $a_1 < a_2$. If $|f'(x)|^q$ for $q \geq 1$ is η-convex on $[a_1,a_2]$ and $0 \leq \alpha \leq 1$, then*

$$\begin{aligned}
&\left|\frac{\alpha}{2}\bigl[f(a_1)+f(a_2)\bigr]+(1-\alpha)f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)\,dx\right| \\
&\quad\leq \frac{a_2-a_1}{8}\left(\frac{1}{6}\right)^{\frac{1}{q}}\bigl(1-2\alpha+2\alpha^2\bigr)^{1-\frac{1}{q}}\Bigl[\bigl(6-12\alpha+12\alpha^2\bigr)\bigl|f'(a_2)\bigr|^q \\
&\qquad+\bigl(4-9\alpha+12\alpha^2-2\alpha^3\bigr)\eta\bigl(\bigl|f'(a_1)\bigr|^q,\bigl|f'(a_2)\bigr|^q\bigr)\Bigr]^{\frac{1}{q}} \\
&\qquad+\Bigl[\bigl(6-12\alpha+12\alpha^2\bigr)\bigl|f'(a_2)\bigr|^q+\bigl(2-3\alpha+2\alpha^3\bigr)\eta\bigl(\bigl|f'(a_1)\bigr|^q,\bigl|f'(a_2)\bigr|^q\bigr)\Bigr]^{\frac{1}{q}}.
\end{aligned} \tag{16}$$

Remark 3.2 If we take $\eta(x,y) = x - y$, then inequality (16) reduces to inequality (3.5) in [9].

By choosing $\alpha = \beta = \frac{1}{2}, \frac{1}{3}$, respectively, in Theorem 3.1 we can deduce the following inequalities.

Corollary 3.2 *Let $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, be a differentiable mapping on I^0 with $f' \in L^1[a_1,a_2]$, where $a_1,a_2 \in I$, $a_1 < a_2$. If $|f'(x)|^q$ for $q \geq 1$ is η-convex on $[a_1,a_2]$ and $0 \leq \alpha,\beta \leq 1$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a_1)+f(a_2)}{2}+f\left(\frac{a_1+a_2}{2}\right)\right]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)\,dx\right| \\
&\quad\leq \frac{a_2-a_1}{16}\left(\frac{1}{12}\right)^{\frac{1}{q}}\Bigl\{\Bigl[12\bigl|f'(a_2)\bigr|^q+9\eta\bigl(\bigl|f'(a_1)\bigr|^q,\bigl|f'(a_2)\bigr|^q\bigr)\Bigr]^{\frac{1}{q}} \\
&\qquad+\Bigl[12\bigl|f'(a_2)\bigr|^q+3\eta\bigl(\bigl|f'(a_1)\bigr|^q,\bigl|f'(a_2)\bigr|^q\bigr)\Bigr]^{\frac{1}{q}}\Bigr\}, \\
&\left|\frac{1}{6}\left[f(a_1)+f(a_2)+4f\left(\frac{a_1+a_2}{2}\right)\right]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)\,dx\right| \\
&\quad\leq \frac{5(a_2-a_1)}{72}\left(\frac{1}{90}\right)^{\frac{1}{q}}\Bigl\{\Bigl[90\bigl|f'(a_2)\bigr|^q+61\eta\bigl(\bigl|f'(a_1)\bigr|^q,\bigl|f'(a_2)\bigr|^q\bigr)\Bigr]^{\frac{1}{q}} \\
&\qquad+\Bigl[90\bigl|f'(a_2)\bigr|^q+29\eta\bigl(\bigl|f'(a_1)\bigr|^q,\bigl|f'(a_2)\bigr|^q\bigr)\Bigr]^{\frac{1}{q}}\Bigr\}.
\end{aligned} \tag{17}$$

Setting $q = 1$ in Corollary 3.2, we have the following:

Corollary 3.3 *Let $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, be a differentiable mapping on I^0 with $f' \in L^1[a_1,a_2]$, where $a_1,a_2 \in I$, $a_1 < a_2$. If $|f'(x)|$ is η-convex on $[a_1,a_2]$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a_1)+f(a_2)}{2}+f\left(\frac{a_1+a_2}{2}\right)\right]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)\,dx\right| \\
&\quad\leq \frac{a_2-a_1}{16}\bigl[2\bigl|f'(a_2)\bigr|+\eta\bigl(\bigl|f'(a_1)\bigr|,\bigl|f'(a_2)\bigr|\bigr)\bigr], \\
&\left|\frac{1}{6}\left[f(a_1)+f(a_2)+4f\left(\frac{a_1+a_2}{2}\right)\right]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)\,dx\right| \\
&\quad\leq \frac{5(a_2-a_1)}{72}\bigl[2\bigl|f'(a_2)\bigr|+\eta\bigl(\bigl|f'(a_1)\bigr|,\bigl|f'(a_2)\bigr|\bigr)\bigr].
\end{aligned} \tag{18}$$

Remark 3.3 If we take $\eta(x,y) = x - y$, then inequalities (17) and (18) reduce to inequalities (3.6) and (3.7) in [9].

Theorem 3.2 *Let $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, be a differentiable mapping on I^0 with $f' \in L^1[a_1,a_2]$, where $a_1, a_2 \in I$, $a_1 < a_2$. If $|f'(x)|^q$ for $q \geq 1$ is η-convex on $[a_1,a_2]$ and $0 \leq \alpha, \beta \leq 1$, then*

$$\begin{aligned}
&\left|\frac{\alpha f(a_1)+\beta f(a_2)}{2}+\frac{2-\alpha-\beta}{2}f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)\,dx\right| \\
&\leq \frac{a_2-a_1}{4}\left[\frac{1}{2(q+1)(q+2)}\right]^{\frac{1}{q}} \\
&\quad\times\Big\{\Big[\big(\big[2(q+2)(1-\alpha)^{q+1}+2(q+2)\alpha^{q+1}\big]\big)\big|f'(a_2)\big|^q \\
&\quad+\big[(q+3-\alpha)(1-\alpha)^{q+1}+(2q+4-\alpha)\alpha^{q+1}\big]\eta\big(\big|f'(a_1)\big|^q,\big|f'(a_2)\big|^q\big)\Big]^{\frac{1}{q}} \\
&\quad+\Big[\big(2(q+2)(1-\beta)^{q+1}+2(q+2)\beta^{q+1}\big)\big|f'(a_2)\big|^q \\
&\quad+\big(\beta^{q+2}+(q+1+\beta)(1-\beta)^{q+1}\big)\eta\big(\big|f'(a_1)\big|^q,\big|f'(a_2)\big|^q\big)\Big]^{\frac{1}{q}}\Big\}.
\end{aligned} \tag{19}$$

Proof For $q > 1$, by the η-convexity of $|f'(x)|^q$ on $[a_1,a_2]$ and Hölder's integral inequality it follows that

$$\begin{aligned}
&\left|\frac{\alpha f(a_1)+\beta f(a_2)}{2}+\frac{2-\alpha-\beta}{2}f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)\,dx\right| \\
&\leq \frac{a_2-a_1}{4}\left[\int_0^1|1-\alpha-t|\left|f'\left(ta_1+(1-t)\frac{a_1+a_2}{2}\right)\right|dt\right. \\
&\quad\left.+\int_0^1|\beta-t|\left|f'\left(t\frac{a_1+a_2}{2}+(1-t)a_2\right)\right|dt\right] \\
&\leq \frac{a_2-a_1}{4}\left\{\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left[\int_0^1|1-\alpha-t|^q\left(\big|f'(a_2)\big|^q+\left(\frac{1+t}{2}\right)\right.\right.\right. \\
&\quad\left.\left.\times\,\eta\big(\big|f'(a_1)\big|^q,\big|f'(a_2)\big|^q\big)\right)dt\right]^{\frac{1}{q}}+\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left[\int_0^1|\beta-t|^q\right. \\
&\quad\left.\left.\times\left(\big|f'(a_2)\big|^q+\left(\frac{t}{2}\right)\eta\big(\big|f'(a_1)\big|^q,\big|f'(a_2)\big|\big)\right)dt\right]^{\frac{1}{q}}\right\} \\
&\leq \frac{a_2-a_1}{4}\left\{\left[\int_0^1|1-\alpha-t|^q\left(\big|f'(a_2)\big|^q+\left(\frac{1+t}{2}\right)\eta\big(\big|f'(a_1)\big|^q,\big|f'(a_2)\big|^q\big)\right)dt\right]^{\frac{1}{q}}\right. \\
&\quad\left.+\left[\int_0^1|\beta-t|^q\left(\big|f'(a_2)\big|^q+\left(\frac{t}{2}\right)\eta\big(\big|f'(a_1)\big|^q,\big|f'(a_2)\big|\big)\right)dt\right]^{\frac{1}{q}}\right\}.
\end{aligned} \tag{20}$$

By Lemma 1.2 we have

$$\begin{aligned}
&\int_0^1|1-\alpha-t|^q\left(\big|f'(a_2)\big|^q+\left(\frac{1+t}{2}\right)\eta\big(\big|f'(a_1)\big|^q,\big|f'(a_2)\big|^q\big)\right)dt \\
&=\left(\big|f'(a_2)\big|^q+\frac{1}{2}\eta\big(\big|f'(a_1)\big|^q,\big|f'(a_2)\big|^q\big)\right)\int_0^1|1-\alpha-t|^q\,dt \\
&\quad+\frac{1}{2}\eta\big(\big|f'(a_1)\big|^q,\big|f'(a_2)\big|^q\big)\int_0^1 t|1-\alpha-t|^q\,dt \\
&=\left(\big|f'(a_2)\big|^q+\frac{1}{2}\eta\big(\big|f'(a_1)\big|^q,\big|f'(a_2)\big|^q\big)\right)\left(\frac{(1-\alpha)^{q+1}+\alpha^{q+1}}{q+1}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\left(\frac{(1-\alpha)^{q+2}+(q+2-\alpha)\alpha^{q+1}}{(q+1)(q+2)}\right)\\
&=\frac{1}{2(q+1)(q+2)}\left[2(q+2)(1-\alpha)^{q+1}+2(q+2)\alpha^{q+1}\right]\left|f'(a_2)\right|^q\\
&\quad+\left[2(q+2)(1-\alpha)^{q+1}+(q+2)\alpha^{q+1}+(1-\alpha)^{q+2}+(q+2-\alpha)\alpha^{q+1}\right]\\
&\quad\times\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\\
&=\frac{1}{2(q+1)(q+2)}\left[2(q+2)(1-\alpha)^{q+1}+2(q+2)\alpha^{q+1}\right]\left|f'(a_2)\right|^q\\
&\quad+\left[(q+3-\alpha)(1-\alpha)^{q+1}+(2q+4-\alpha)\alpha^{q+1}\right]\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1|\beta-t|^q\left(\left|f'(a_2)\right|^q+\left(\frac{t}{2}\right)\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\right)dt\\
&=\left|f'(a_2)\right|^q\int_0^1|\beta-t|^q\,dt+\frac{1}{2}\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\int_0^1 t|\beta-t|^q\,dt\\
&=\left|f'(a_2)\right|^q\left(\frac{\beta^{q+1}+(1-\beta)^{q+1}}{q+1}\right)+\frac{1}{2}\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\\
&\quad\times\left(\frac{\beta^{q+2}+(q+1+\beta)(1-\beta)^{q+1}}{(q+1)(q+2)}\right)\\
&=\frac{1}{2(q+1)(q+2)}\left\{\left[2(q+2)(1-\beta)^{q+1}+2(q+2)\beta^{q+1}\right]\left|f'(a_2)\right|^q\right.\\
&\quad\left.+\left[\beta^{q+2}+(q+1+\beta)(1-\beta)^{q+1}\right]\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\right\}.
\end{aligned}
$$

Substituting the last two equalities into inequality (20) yields inequality (19) for $q > 1$.

For $q = 1$, the proof is the same as that of (15), and the theorem is proved. □

Remark 3.4 If we take $\eta(x,y) = x - y$, then inequality (19) reduces to inequality (3.8) in [9].

Similarly to corollaries of Theorem 3.1, we can obtain the following corollaries of Theorem 3.2.

Corollary 3.4 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° with $f' \in L^1[a_1,a_2]$, where $a_1,a_2 \in I$, $a_1 < a_2$. If $|f'(x)|^q$ for $q \geq 1$ is η-convex on $[a_1,a_2]$ and $0 \leq \alpha \leq 1$, then*

$$
\begin{aligned}
&\left|\frac{\alpha}{2}\left[f(a_1)+f(a_2)\right]+(1-\alpha)f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)\,dx\right|\\
&\leq\frac{a_2-a_1}{4}\left[\frac{1}{2(q+1)(q+2)}\right]^{\frac{1}{q}}\\
&\quad\times\left\{\left[\left(2(q+2)(1-\alpha)^{q+1}+2(q+2)\alpha^{q+1}\right)\left|f'(a_2)\right|^q\right.\right.\\
&\quad\left.+\left((q+3-\alpha)(1-\alpha)^{q+1}+(2q+4-\alpha)\alpha^{q+1}\right)\eta\left(\left|f'(a)\right|^q,\left|f'(a_2)\right|^q\right)\right]^{\frac{1}{q}}\\
&\quad+\left[\left(2(q+2)(1-\alpha)^{q+1}+2(q+2)\alpha^{q+1}\right)\left|f'(a_2)\right|^q\right.\\
&\quad\left.\left.+\left(\alpha^{q+2}+(q+1+\alpha)(1-\alpha)^{q+1}\right)\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\right]^{\frac{1}{q}}\right\}.
\end{aligned}
\tag{21}
$$

Remark 3.5 If we take $\eta(x,y) = x - y$, then inequality (21) reduces to inequality (3.11) in [9].

Corollary 3.5 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° with $f' \in L^1[a_1,a_2]$, where $a_1, a_2 \in I$, $a_1 < a_2$. If $|f'(x)|^q$ for $q \geq 1$ is η-convex on $[a_1,a_2]$ and $0 \leq \alpha, \beta \leq 1$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a_1)+f(a_2)}{2}+f\left(\frac{a_1+a_2}{2}\right)\right]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\,dx\right| \\
&\quad\leq \frac{a_2-a_1}{8}\left[\frac{1}{4(q+1)(q+2)}\right]^{\frac{1}{q}} \\
&\qquad\times\left\{\left[\left((4q+8)\left|f'(a_2)\right|^q+(3q+6)\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\right)\right]^{\frac{1}{q}}\right. \\
&\qquad\left.+\left[\left((4q+8)\left|f'(a_2)\right|^q+(q+2)\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\right)\right]^{\frac{1}{q}}\right\}, \\
&\left|\frac{1}{6}\left[f(a_1)+f(a_2)+4f\left(\frac{a_1+a_2}{2}\right)\right]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\,dx\right| \\
&\quad\leq \frac{a_2-a_1}{12}\left[\frac{1}{18(q+1)(q+2)}\right]^{\frac{1}{q}}\left\{\left[\left((3q+6)(2)^{q+2}+6(q+2)\right)\left|f'(a_2)\right|^q\right.\right. \\
&\qquad\left.+\left((3q+8)(2)^{q+1}+(6q+11)\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\right)\right]^{\frac{1}{q}}+\left[\left((3q+6)(2)^{q+2}\right.\right. \\
&\qquad\left.\left.+6(q+2)\right)\left|f'(a_2)\right|^q+\left(1+(3q+4)(2)^{q+1}\right)\eta\left(\left|f'(a_1)\right|^q,\left|f'(a_2)\right|^q\right)\right]^{\frac{1}{q}}\right\}.
\end{aligned} \tag{22}$$

Remark 3.6 If we take $\eta(x,y) = x - y$, then inequality (22) reduces to inequality (3.12) in [9] respectively.

If we take $q = 1$ in Corollary 3.5, then we get Corollary 3.3.

To prove our next results, we consider the following lemma proved in [10].

Lemma 3.1 *Let $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ be a differentiable mapping on I^0 with $f'' \in L^1[a_1,a_2]$, where $a_1, a_2 \in I$ and $a_1 < a_2$. Then*

$$\begin{aligned}
&\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\,dx - f\left(\frac{a_1+a_2}{2}\right) \\
&\quad= \frac{(a_2-a_1)^2}{16}\left[\int_0^1 t^2 f''\left(t\frac{a_1+a_2}{2}+(1-t)a_1\right)dt\right. \\
&\qquad\left.+\int_0^1 (t-1)^2 f''\left(ta_2+(1-t)\frac{a_1+a_2}{2}\right)dt\right].
\end{aligned} \tag{23}$$

Theorem 3.3 *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° with $f'' \in L^1[a_1,a_2]$, where $a_1, a_2 \in I$ and $a_1 < a_2$. If $|f''|$ is η-convex on $[a_1,a_2]$, then*

$$\begin{aligned}
&\left|f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\,dx\right| \\
&\quad\leq \frac{(a_2-a_1)^2}{16}\left[\frac{1}{3}\left(\left|f''(a_1)\right|+\left|f''\left(\frac{a_1+a_2}{2}\right)\right|\right)\right. \\
&\qquad\left.+\frac{1}{4}\left(\eta\left(\left|f''\left(\frac{a_1+a_2}{2}\right)\right|,\left|f''(a_1)\right|\right)+\frac{1}{3}\eta\left(\left|f''(a_2)\right|,\left|f''\left(\frac{a_1+a_2}{2}\right)\right|\right)\right)\right].
\end{aligned} \tag{24}$$

Proof From Lemma 3.1 we have

$$\begin{aligned}
&\left|f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)\,dx\right| \\
&\leq \frac{(a_2-a_1)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{a_1+a_2}{2}+(1-t)a_1\right)\right|dt\right. \\
&\quad\left.+\int_0^1 (t-1)^2\left|f''\left(ta_2+(1-t)\frac{a_1+a_2}{2}\right)\right|dt\right] \\
&\leq \frac{(a_2-a_1)^2}{16}\left[\int_0^1 t^2\left(\left|f''(a_1)\right|+t\eta\left(\left|f''\left(\frac{a_1+a_2}{2}\right)\right|,\left|f''(a_1)\right|\right)\right)dt\right] \\
&\quad+\frac{(a_2-a_1)^2}{16}\left[\int_0^1 (t-1)^2\left(\left|f''\left(\frac{a_1+a_2}{2}\right)\right|\right.\right. \\
&\quad\left.\left.+t\eta\left(\left|f''(a_2)\right|,\left|f''\left(\frac{a_1+a_2}{2}\right)\right|\right)\right)dt\right] \\
&=\frac{(a_2-a_1)^2}{16}\left[\frac{1}{3}\left|f''(a_1)\right|+\frac{1}{3}\left|f''\left(\frac{a_1+a_2}{2}\right)\right|+\frac{1}{4}\eta\left(\left|f''\left(\frac{a_1+a_2}{2}\right)\right|,\left|f''(a_1)\right|\right)\right. \\
&\quad\left.+\frac{1}{12}\eta\left(\left|f''(a_2)\right|,\left|f''\left(\frac{a_1+a_2}{2}\right)\right|\right)\right] \\
&=\frac{(a_2-a_1)^2}{16}\left[\frac{1}{3}\left(\left|f''(a_1)\right|+\left|f''\left(\frac{a_1+a_2}{2}\right)\right|\right)\right. \\
&\quad\left.+\frac{1}{4}\left(\eta\left(\left|f''\left(\frac{a_1+a_2}{2}\right)\right|,\left|f''(a_1)\right|\right)+\frac{1}{3}\eta\left(\left|f''(a_2)\right|,\left|f''\left(\frac{a_1+a_2}{2}\right)\right|\right)\right)\right].
\end{aligned}$$

This proves inequality (24). □

Remark 3.7 If we take $\eta(x,y)=x-y$, then inequality (24) reduces to inequality (4).

Theorem 3.4 *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° with $f''\in L^1[a_1,a_2]$, where $a_1,a_2\in I$ and $a_1<a_2$. If $|f''|^q$ for $q\geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$ is η-convex on $[a_1,a_2]$, then*

$$\begin{aligned}
&\left|f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)\,dx\right| \\
&\leq \frac{(a_2-a_1)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left[\left(\frac{1}{3}\left|f''(a_1)\right|^q+\frac{1}{4}\eta\left(\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q,\left|f''(a_1)\right|^q\right)\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{1}{3}\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q+\frac{1}{12}\eta\left(\left|f''(a_2)\right|^q,\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}\right]. \qquad (25)
\end{aligned}$$

Proof Suppose that $p\geq 1$. From Lemma 3.1, using the power mean inequality, we have

$$\begin{aligned}
&\left|f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)\,dx\right| \\
&\leq \frac{(a_2-a_1)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{a_1+a_2}{2}+(1-t)a_1\right)\right|dt\right. \\
&\quad\left.+\int_0^1 (t-1)^2\left|f''\left(ta_2+(1-t)\frac{a_1+a_2}{2}\right)\right|dt\right]
\end{aligned}$$

$$\leq \frac{(a_2-a_1)^2}{16}\left(\int_0^1 t^2\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t^2\left|f''\left(t\frac{a_1+a_2}{2}+(1-t)a_1\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(a_2-a_1)^2}{16}\left(\int_0^1 (t-1)^2\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (t-1)^2\left|f''\left(ta_2+(1-t)\frac{a_1+a_2}{2}\right)\right|^q dt\right)^{\frac{1}{q}}.$$

Because $|f''|^q$ is η-convex, we have

$$\int_0^1 t^2\left|f''\left(t\frac{a_1+a_2}{2}+(1-t)a_1\right)\right|^q dt$$
$$\leq \frac{1}{3}\left|f''(a_1)\right|^q+\frac{1}{4}\left(\eta\left(\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q,\left|f''(a_1)\right|^q\right)\right)$$

and

$$\int_0^1 (t-1)^2\left|f''\left(ta_2+(1-t)\frac{a_1+a_2}{2}\right)\right|^q dt$$
$$\leq \frac{1}{3}\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q+\frac{1}{12}\eta\left(\left|f''(a_2)\right|^q,\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q\right).$$

Therefore we have

$$\left|f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\,dx\right|$$
$$\leq \frac{(a_2-a_1)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left\{\left(\frac{1}{3}\left|f''(a_1)\right|^q+\frac{1}{4}\eta\left(\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q,\left|f''(a_1)\right|^q\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{1}{3}\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q+\frac{1}{12}\eta\left(\left|f''(a_2)\right|^q,\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\}.$$ □

Remark 3.8 If we take $\eta(x,y)=x-y$, then inequality (25) reduces to inequality (5).

4 Application to means

For two positive numbers $a_1>0$ and $a_2>0$, define

$$A(a_1,a_2)=\frac{a_1+a_2}{2},\qquad G(a_1,a_2)=\sqrt{a_1a_2},\qquad H(a_1,a_2)=\frac{2a_1a_2}{a_1+a_2},$$
$$L(a_1,a_2)=\begin{cases}\left[\frac{a_2^{s+1}-a_1^{s+1}}{(s+1)(a_2-a_1)}\right]^{\frac{1}{s}}, & a_1\neq a_2,\\ a_1, & a_1=a_2,\end{cases}$$
$$I(a_1,a_2)=\begin{cases}\frac{1}{e}\left(\frac{a_2^{a_2}}{a_1^{a_1}}\right)^{\frac{1}{a_2-a_1}}, & a_1\neq a_2,\\ a_1, & a_1=a_2,\end{cases} \tag{26}$$
$$H_{w,s}(a_1,a_2)=\begin{cases}\left[\frac{a_1^s+w(a_1a_2)^{\frac{s}{2}}+a_2^s}{w+2}\right]^{\frac{1}{s}}, & s\neq 0,\\ \sqrt{a_1a_2}, & s=0,\end{cases}$$

for $0\leq w<\infty$. These means are respectively called the arithmetic, geometric, harmonic, generalized logarithmic, identric, and Heronian means of two positive numbers a_1 and a_2.

Applying Theorems 3.1 and 3.2 to $f(x) = x^s$ for $s \neq 0$ and $x > 0$ results in the following inequalities for means.

Theorem 4.1 *Let $a_1 > 0$, $a_2 > 0$, $a_1 \neq a_2$, $q \geq 1$, and either $s > 1$ and $(s-1)q \geq 1$ or $s < 0$. Then*

$$\begin{aligned} &\left|A\left(\alpha a_1^s, \beta a_2^s\right) + \frac{2-\alpha-\beta}{2} A^s(a_1,a_2) - L^s(a_1,a_2)\right| \\ &\quad \leq \frac{a_2-a_1}{8}\left(\frac{1}{6}\right)^{\frac{1}{q}} \left\{\left(1-2\alpha+2\alpha^2\right)^{1-\frac{1}{q}}\left[\left(6-12\alpha+12\alpha^2\right)\left|sa_2^{s-1}\right|^q\right.\right. \\ &\quad\quad \left.+ \left(4-9\alpha+12\alpha^2-2\alpha^3\right)\eta\left(\left|sa_1^{s-1}\right|^q, \left|sa_2^{s-1}\right|^q\right)\right]^{\frac{1}{q}} \\ &\quad\quad + \left(1-2\beta+2\beta^2\right)^{1-\frac{1}{q}}\left[\left(6-12\beta+12\beta^2\right)\left|sa_2^{s-1}\right|^q\right. \\ &\quad\quad \left.\left.+ \left(2-3\beta+2\beta^3\right)\eta\left(\left|sa_1^{s-1}\right|^q, \left|sa_2^{s-1}\right|^q\right)\right]^{\frac{1}{q}}\right\}. \end{aligned} \tag{27}$$

Theorem 4.2 *Let $a_1 > 0$, $a_2 > 0$, $a_1 \neq a_2$, $q \geq 1$, and either $s > 1$ and $(s-1)q \geq 1$ or $s < 0$. Then*

$$\begin{aligned} &\left|A\left(\alpha a_1^s, \beta a_2^s\right) + \frac{2-\alpha-\beta}{2} A^s(a_1,a_2) - L^s(a_1,a_2)\right| \\ &\quad \leq \frac{a_2-a_1}{4}\left[\frac{1}{2(q+1)(q+2)}\right]^{\frac{1}{q}} \\ &\quad\quad \times \left\{\left[\left(\left[2(q+2)(1-\alpha)^{q+1} + 2(q+2)\alpha^{q+1}\right]\right)\left|sa_2^{s-1}\right|^q\right.\right. \\ &\quad\quad \left.+ \left[(q+3-\alpha)(1-\alpha)^{q+1} + (2q+4-\alpha)\alpha^{q+1}\right]\eta\left(\left|sa_1^{s-1}\right|^q, \left|sa_2^{s-1}\right|^q\right)\right]^{\frac{1}{q}} \\ &\quad\quad + \left[\left(2(q+2)(1-\beta)^{q+1} + 2(q+2)\beta^{q+1}\right)\left|sa_2^{s-1}\right|^q\right. \\ &\quad\quad \left.\left.+ \left(\beta^{q+2} + (q+1+\beta)(1-\beta)^{q+1}\right)\eta\left(\left|sa_1^{s-1}\right|^q, \left|sa_2^{s-1}\right|^q\right)\right]^{\frac{1}{q}}\right\}. \end{aligned} \tag{28}$$

Taking $f(x) = \ln x$ for $x > 0$ in Theorems 3.1 and 3.2 results in the following inequalities for means.

Theorem 4.3 *For $a_1 > 0$, $a_2 > 0$, $a_1 \neq a_2$ and $q \geq 1$, we have*

$$\begin{aligned} &\left|\frac{\ln G^2(a_1^\alpha, a_2^\beta)}{2} + \frac{2-\alpha-\beta}{2}\ln A(a_1,a_2) - \ln I(a_1,a_2)\right| \\ &\quad \leq \frac{a_2-a_1}{8}\left(\frac{1}{6}\right)^{\frac{1}{q}} \left\{\left(1-2\alpha+2\alpha^2\right)^{1-\frac{1}{q}}\left[\left(6-12\alpha+12\alpha^2\right)\left(\frac{1}{a_2}\right)^q\right.\right. \\ &\quad\quad \left.+ \left(4-9\alpha+12\alpha^2-2\alpha^3\right)\eta\left(\left(\frac{1}{a_1}\right)^q, \left(\frac{1}{a_2}\right)^q\right)\right]^{\frac{1}{q}} \\ &\quad\quad + \left(1-2\beta+2\beta^2\right)^{1-\frac{1}{q}}\left[\left(6-12\beta+12\beta^2\right)\left(\frac{1}{a_2}\right)^q\right. \\ &\quad\quad \left.\left.+ \left(2-3\beta+2\beta^3\right)\eta\left(\left(\frac{1}{a_1}\right)^q, \left(\frac{1}{a_2}\right)^q\right)\right]^{\frac{1}{q}}\right\}. \end{aligned} \tag{29}$$

Theorem 4.4 *For $a_1 > 0$, $a_2 > 0$, $a_1 \neq a_2$ and $q \geq 1$, we have*

$$\begin{aligned}
&\left|\frac{\ln G^2(a_1^\alpha, a_2^\beta)}{2} + \frac{2-\alpha-\beta}{2}\ln A(a_1,a_2) - \ln I(a_1,a_2)\right| \\
&\quad\leq \frac{a_2-a_1}{4}\left[\frac{1}{2(q+1)(q+2)}\right]^{\frac{1}{q}} \\
&\qquad\times\left\{\left[\left(\left[2(q+2)(1-\alpha)^{q+1} + 2(q+2)\alpha^{q+1}\right]\right)\left(\frac{1}{a_2}\right)^q\right.\right. \\
&\qquad\left. + \left[(q+3-\alpha)(1-\alpha)^{q+1} + (2q+4-\alpha)\alpha^{q+1}\right]\eta\left(\left(\frac{1}{a_1}\right)^q, \left(\frac{1}{a_2}\right)^q\right)\right]^{\frac{1}{q}} \\
&\qquad + \left[\left(2(q+2)(1-\beta)^{q+1} + 2(q+2)\beta^{q+1}\right)\left(\frac{1}{a_2}\right)^q\right. \\
&\qquad\left.\left. + \left(\beta^{q+2} + (q+1+\beta)(1-\beta)^{q+1}\right)\eta\left(\left(\frac{1}{a_1}\right)^q, \left(\frac{1}{a_2}\right)^q\right)\right]^{\frac{1}{q}}\right\}. \qquad (30)
\end{aligned}$$

Finally, we can establish an inequality for the Heronian mean as follows.

Theorem 4.5 *For $a_2 > a_1 > 0$, $a_1 \neq a_2$, $w \geq 0$, and $s \geq 4$ or $0 \neq s < 1$, we have*

$$\begin{aligned}
&\left|\frac{H_{w,s}^s(a_1,a_2)}{H(a_1^s, a_2^s)} + H_{w,(\frac{s}{2}+1)}^{\frac{s}{2}+1}\left(\frac{a_2}{a_1} + \frac{a_1}{a_2}, 1\right) - H_{w,s}^s\left(\frac{L(a_1^2,a_2^2)}{G^2(a_1,a_2)}, 1\right)\right| \\
&\quad\leq \frac{(a_2-a_1)A(a_1,a_2)}{8G^2(a_1,a_2)}\left[\frac{2|s|}{w+2}\left(G^{2(s-1)}\left(a_2, \frac{1}{a_1}\right) + \frac{w}{2}G^{s-\frac{1}{2}}\left(a_2, \frac{1}{a_1}\right)\right)\right. \\
&\qquad + \eta\left(\frac{|s|}{w+2}\left(G^{2(s-1)}\left(a_1, \frac{1}{a_2}\right) + \frac{w}{2}G^{s-\frac{1}{2}}\left(a_1, \frac{1}{a_2}\right)\right),\right. \\
&\qquad\left.\left.\frac{|s|}{w+2}\left(G^{2(s-1)}\left(a_2, \frac{1}{a_1}\right) + \frac{w}{2}G^{s-\frac{1}{2}}\left(a_2, \frac{1}{a_1}\right)\right)\right)\right]. \qquad (31)
\end{aligned}$$

Proof Let $f(x) = \frac{x^s + wx^{\frac{s}{2}+1}}{w+2}$ for $x > 0$ and $s \notin (1,4)$. Then

$$f'(x) = \frac{s}{w+2}\left(x^{s-1} + \frac{w}{2}x^{\frac{s}{2}-1}\right). \qquad (32)$$

By Corollary 3.3 it follows that

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(\frac{a_2}{a_1}) + f(\frac{a_1}{a_2})}{2} + f\left(\frac{\frac{a_2}{a_1} + \frac{a_1}{a_2}}{2}\right)\right] - \frac{1}{\frac{a_2}{a_1} - \frac{a_1}{a_2}}\int_{\frac{a_1}{a_2}}^{\frac{a_2}{a_1}} f(x)\,dx\right| \\
&\quad= \left|\frac{1}{2}\left\{\frac{1}{2}\left[\frac{a_2^s + w(a_1a_2)^{\frac{s}{2}} + a_1^s}{a_1^s(w+2)} + \frac{a_1^s + w(a_1a_2)^{\frac{s}{2}} + a_2^s}{a_2^s(w+2)}\right]\right.\right. \\
&\qquad\left. + \frac{(\frac{a_2}{a_1} + \frac{a_1}{a_2})^s + w(\frac{a_2}{a_1} + \frac{a_1}{a_2})^{\frac{s}{2}} + 1}{w+2}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{w+2}\left[\frac{(\frac{a_2}{a_1})^{s+1}-(\frac{a_1}{a_2})^{s+1}}{(s+1)(\frac{a_2}{a_1}-\frac{a_1}{a_2})} + w\frac{(\frac{a_2}{a_1})^{\frac{s}{2}+1}-(\frac{a_1}{a_2})^{\frac{s}{2}+1}}{(\frac{s}{2}+1)(\frac{a_2}{a_1}-\frac{a_1}{a_2})}+1\right]\Bigg| \\
&= \left|\frac{H^s_{w,s}(a_1,a_2)}{H(a_1^s,a_2^s)} + H^{\frac{s}{2}+1}_{w,(\frac{s}{2}+1)}\left(\frac{a_2}{a_1}+\frac{a_1}{a_2},1\right) - H^s_{w,s}\left(\frac{L(a_1^2,a_2^2)}{G^2(a_1,a_2)},1\right)\right|. \qquad (33)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\frac{\frac{a_2}{a_1}-\frac{a_1}{a_2}}{16}\left[2\left|f'\left(\frac{a_2}{a_1}\right)\right| + \eta\left(\left|f'\left(\frac{a_1}{a_2}\right)\right|,\left|f'\left(\frac{a_2}{a_1}\right)\right|\right)\right] \\
&= \frac{a_2^2-a_1^2}{16a_1a_2}\left[2\left|\frac{s}{w+2}\left(\left(\frac{a_2}{a_1}\right)^{s-1}+\frac{w}{2}\left(\frac{a_2}{a_1}\right)^{\frac{s}{2}-1}\right)\right|\right. \\
&\quad \left. + \eta\left(\left|\frac{s}{w+2}\left(\left(\frac{a_1}{a_2}\right)^{s-1}+\frac{w}{2}\left(\frac{a_1}{a_2}\right)^{\frac{s}{2}-1}\right)\right|,\left|\frac{s}{w+2}\left(\left(\frac{a_2}{a_1}\right)^{s-1}+\frac{w}{2}\left(\frac{a_2}{a_1}\right)^{\frac{s}{2}-1}\right)\right|\right)\right] \\
&= \frac{(a_2-a_1)A(a_1,a_2)}{8G^2(a_1,a_2)}\left[\frac{2|s|}{w+2}\left(G^{2(s-1)}\left(a_2,\frac{1}{a_1}\right)+\frac{w}{2}G^{s-\frac{1}{2}}\left(a_2,\frac{1}{a_1}\right)\right)\right. \\
&\quad + \eta\left(\frac{|s|}{w+2}\left(G^{2(s-1)}\left(a_1,\frac{1}{a_2}\right)+\frac{w}{2}G^{s-\frac{1}{2}}\left(a_1,\frac{1}{a_2}\right)\right),\right. \\
&\quad \left.\left.\frac{|s|}{w+2}\left(G^{2(s-1)}\left(a_2,\frac{1}{a_1}\right)+\frac{w}{2}G^{s-\frac{1}{2}}\left(a_2,\frac{1}{a_1}\right)\right)\right)\right]. \qquad (34)
\end{aligned}
$$

Obviously (33) and (34) yield (31). □

Acknowledgements
This work was presented at the 6th International Conference on Education (ICE), which was organized by Division of Science and Technology, University of Education, Lahore, Pakistan.

Funding
This work was supported by the Dong-A University research fund.

Availability of data and materials
All data required for this research are available in this paper.

Competing interests
The authors have no competing interests.

Authors' contributions
All authors contributed equally in this paper. All authors read and approved the final manuscript.

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Publisher's Note

Received: 1 October 2018 Accepted: 8 February 2019 Published online: 22 February 2019

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