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On the convergence of Lupaş (p,q)-Bernstein operators via contraction principle

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Abstract

The present paper deals with the limit behavior of iterates of Lupaş q- and (p,q)-Bernstein operators. We obtain the convergence for Lupaş q- and (p,q)-Bernstein operators by using the contraction principle.

MSC: 41A10; 41A36; 47H10

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1 Introduction and preliminaries

For any $f \in \mathcal{C}[0,1]$, the sequence of operators $B_n : C[0,1] \to \mathcal{C}[0,1]$ defined by

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1], n \in \mathbb{N}, \tag{1.1}$$

is known as Bernstein polynomials [6].

Lupaş [18] defined the first q-analogue of Bernstein operators (rational) for $q > 0$ as follows:

$$L_n(f,q;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) b_{n,k}(x;q), \tag{1.2}$$

where

$$b_{n,k}(x;q) = \frac{\left[{n \atop k}\right]_q q^{\frac{n(n-1)}{2}} x^k (1-x)^{n-k}}{\prod_{j=0}^{n-1} \{(1-x) + q^j x\}}.$$

Later Phillips [32] proposed another q-analog of Bernstein operators.

For $q = 1$, both are reduced to the original ones. However, for $q \neq 1$, there are considerable differences between them. The convergence properties for iterates of q-Bernstein polynomials have been investigated in [5, 29, 30, 33, 37], and [38].

For elementary properties of q-analogs of Bernstein polynomials, we refer to [12, 18, 19, 31].

The q-integer $[k]_q$ for $k \in \mathbb{N}$ and a fixed real number $q > 0$ is defined by

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q} & \text{if } q \neq 1, \\ k & \text{if } q = 1. \end{cases}$$

Set $[0]_q = 0$. The q-factorial coefficients are defined by

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \cdots [1]_q & \text{if } k \in \mathbb{N}, \\ 1 & \text{if } k = 0, \end{cases}$$

and the q-binomial coefficients by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad 0 \leq k \leq n,$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ for $k > n$.

Also,

$$(x-a)_q^n = \begin{cases} 1 & \text{if } n = 0, \\ (x-a)(x-qa)\cdots(x-q^{n-1}a) & \text{if } n \geq 1. \end{cases}$$

For any function f, the divided differences are denoted by $\Delta_q^0 f_i = f_i$ for $i = 0, 1, 2, \ldots, n-1$ and, recursively, by $\Delta_q^{k+1} f_i = \Delta_q^k f_{i+1} - q^k \Delta_q^k f_i$ for $k \geqq 0$, where f_i denotes $f(\frac{[i]}{[m]})$. It is established by induction that

$$\Delta_q^k f_i = \sum_{r=0}^{k} (-1)^r q^{\frac{r(r-1)}{2}} \begin{bmatrix} k \\ r \end{bmatrix} f_{i+k-r}$$

(see [36] and [17]).

Note that (1.2) may be written in the q-difference form

$$L_n(f, q; x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \Delta_q^k f_0 x^k. \tag{1.3}$$

We may deduce that

$$L_n(ax+b, q; x) = ax + b, \quad a, b \in \mathbb{R}.$$

Also, we can see that these operators verify for the test functions $e_j(x) = x^j$, $j = 0, 1, 2$.

In 1993, Rus [34] introduced and developed the theory of (weakly) Picard operators, which is one of the most strong tools of fixed point theory with several applications to operator equations and inclusions. Berinde [8] showed that an almost contraction is more general than most of the contractions in the literature.

We now recall some basic features from fixed point theory (see [34]).

Definition 1.1 The operator S on a metric space (X,d) is called a weakly Picard operator (WPO) if the sequence of iterates $(S^m(x))_m \geqq 1$ converges to a fixed point of S for all $x \in X$.

We denote, as usual, $S^0 = I_X$, $S^{m+1} = S \circ S^m$, $m \in \mathbb{N}$.

Let $F_S = \{\xi \in X : S(\xi) = \xi\}$. If an operator B is a WPO and F_S has exactly one element, then S is called a Picard operator (PO).

First, we give a characterization for WPOs.

Theorem 1.2 ([33]) *The operator S on a metric space (X,d) is a weakly Picard operator $\Leftrightarrow$ there exists a partition $\{X_\lambda : \lambda \in \Lambda\}$ such that for every $\lambda \in \Lambda$ one has $X_\lambda \in I(S)$ and $S|_{X_\lambda} : X_\lambda \to X_\lambda$ is a PO, where $I(S) := \{\emptyset \neq Y \subset X : S(Y) \subset Y\}$ denotes the collection of all nonempty subsets invariant under S.*

Moreover, for a WPO S, we take $S^\infty \in X$ defined as

$$S^\infty(x) = \lim_{m\to\infty} S^m(x), \quad x \in X.$$

Clearly, $S^\infty(x) = F_S$. Also, if S is WPO, then we have $F_S^m = F_S \neq \emptyset$, $m \in \mathbb{N}$.

2 Iterates of Lupaş q-Bernstein operators

In the last decades the iterates of positive linear operators in various classes were intensively investigated. The study of convergence of iterates of Bernstein operators has connections to probability theory, matrix theory, spectral theory, and so on. We emphasize here the importance of the works of some mathematicians such as [1, 7, 9–11, 13, 14, 16, 20, 25], and [35].

We want to extend the study of the iterates of Bernstein operators using (p,q)-calculus. Our aim is to study the convergence for iterates of Lupaşş (p,q)-Bernstein operators using the contraction principle (theory of weakly Picard operators).

Theorem 2.1 ([15]) *For $f \in C[0,1]$ and fixed $n \in \mathbb{N}^*$, we have*

$$\lim_{M\to\infty} L_n^M(f;x) = f(0) + \big(f(1) - f(0)\big)x, \quad x \in C[0,1].$$

Rus [35] proved this result by using the contraction principle.

In [29] the authors defined

$$L_n^{M+1}(f,q;x) = L_n\big(L_n^M(f,q;x)\big), \quad M = 1,2,\dots,$$

and

$$L_n^1(f,q;x) = L_n(f,q;x).$$

In the same paper, the authors proved the convergence of iterates of q-Bernstein operators for $q > 0$ using q-differences, Stirling polynomials, and matrix techniques. Ostrovska [30] used eigenvalues and Radu [33] used the contraction principle to prove the convergence of these iterates. The following result gives the convergence of Lupaş q-Bernstein operators by using also the contraction principle. The following result gives the convergence of Lupaş q-Bernstein operators by using the contraction principle.

Theorem 2.2 *Let $L_n(f,q;x)$ be the Lupaş q-Bernstein operators defined in* (1.2). *Then for all $q>0$,*

$$\lim_{M\to\infty} L_n^M(f,q;x) = f(0) + \big(f(1)-f(0)\big)x \tag{2.1}$$

for all $f\in C[0,1]$ and $x\in[0,1]$.

Proof First, we define $X_{\alpha,\beta} = \{f\in C[0,1] : f(0)=\alpha, f(1)=\beta\}$, $\alpha,\beta\in\mathbb{R}$. Clearly, every $X_{\alpha,\beta}$ is a closed subset of $C[0,1]$, and $\{X_{\alpha,\beta},(\alpha,\beta)\in\mathbb{R}\times\mathbb{R}\}$ is a partition of the space $C[0,1]$.

It follows directly from the definition that Lupaş q-Bernstein polynomials possess the end-point interpolation property.

So we have that $X_{\alpha,\beta}$ is an invariant subset of $f\mapsto L_n(f,q;\cdot)$ for all $(\alpha,\beta)\in\mathbb{R}\times\mathbb{R}$ and $n\in\mathbb{N}$.

We show that the restriction of $f\mapsto L_n(f,q;\cdot)$ to $X_{\alpha,\beta}$ is a contraction for any $\alpha,\beta\in\mathbb{R}$. Put $t_n = \min_{x\in[0,1]}(b_{n,0}(x;q)+b_{n,n}(x;q))$, that is,

$$\begin{aligned} t_n &= \min_{x\in[0,1]}\left(\frac{(1-x)^n}{\prod_{j=0}^{n-1}\{(1-x)+q^jx\}} + \frac{q^{\frac{n(n-1)}{2}}x^n}{\prod_{j=0}^{n-1}\{(1-x)+q^jx\}}\right) \\ &= \min_{x\in[0,1]}\left(\frac{(1-x)^n + q^{\frac{n(n-1)}{2}}x^n}{\prod_{j=0}^{n-1}\{(1-x)+q^jx\}}\right). \end{aligned}$$

Then $0<t_n\le 1$.

Let $f,g\in X_{\alpha,\beta}$. Then

$$\begin{aligned} &\big|L_n(f,q;x)-L_n(g,q;x)\big| \\ &\quad= \left|\sum_{k=1}^{n-1} b_{n,k}(x;q)(f-g)\left(\frac{[k]_q}{[n]_q}\right)\right| \\ &\quad\le \sum_{k=1}^{n-1} b_{n,k}(x;q)\|f-g\|_{[0,1]} = \big(1-b_{n,0}(x;q)-b_{n,n}(x;q)\big)\|f-g\|_{[0,1]} \\ &\quad= \left(1-\frac{(1-x)^n}{\prod_{j=0}^{n-1}\{(1-x)+q^jx\}} - \frac{q^{\frac{n(n-1)}{2}}x^n}{\prod_{j=0}^{n-1}\{(1-x)+q^jx\}}\right)\|f-g\|_{[0,1]} \\ &\quad= \left(1-\frac{((1-x)^n + q^{\frac{n(n-1)}{2}}x^n)}{\prod_{j=0}^{n-1}\{(1-x)+q^jx\}}\right)\|f-g\|_{[0,1]} \\ &\quad\le (1-t_n)\|f-g\|_{[0,1]}, \end{aligned}$$

and, finally,

$$\big\|L_n(f,q;x)-L_n(g,q;x)\big\|_{[0,1]} \le (1-t_n)\|f-g\|_{[0,1]}.$$

The restriction of $f\mapsto L_n(f,q;x)$ to $X_{\alpha,\beta}$ is a contraction.

On the other hand, $K^*_{\alpha,\beta} = \alpha e_0 + (\beta-\alpha)e_1 \in X_{\alpha,\beta}$. Since $L_n(e_0,q;x)=e_0$ and $L_n(e_1,q;x)=e_1$, it follows that $K^*_{\alpha,\beta}$ is a fixed point of $L_n(f,q;\cdot)$. For any $f\in C[0,1]$, we have $f\in X_{f(0),f(1)}$, and by using the contraction principle we obtain the desired result (2.1). □

In terms of WPOs, using (1.3), we can formulate the above theorem as follows.

Theorem 2.3 *The Lupaş q-Bernstein operator* $f \mapsto L_n(f,q;\cdot)$ *is WPO, and*

$$L_n^{\infty}(f,q;x) = f(0) + \big(f(1) - f(0)\big)x. \tag{2.2}$$

Proof The operator $S : X \to X$ is WPO if the sequence $(S^M(x))_{M\geq 1}$ converges to a fixed point of S for all $x \in X$.

For a WPO, we consider the operator $S^{\infty} : X \to X$ defined as

$$S^{\infty}(x) = \lim_{M\to\infty} S^M(x). \tag{2.3}$$

Now, using Theorem 2.2 and (2.3), we get the result (2.2). □

3 Lupaş (p,q)-Bernstein operator

Mursaleen et al. [23] defined the (p,q)-analogs of Bernstein operators and studied their approximation properties (see [2, 3, 21, 22, 24–28]). For further reading, we refer to [4, 21, 22, 26, 27].

For any $p > 0$ and $q > 0$, we have

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n-q^n}{p-q} & \text{when } p \neq q \neq 1, \\ np^{n-1} & \text{when } p = q \neq 1, \\ [n]_q & \text{when } p = 1, \\ n & \text{when } p = q = 1, \end{cases}$$

$n = 0, 1, 2, 3, 4, \ldots$. Also,

$$[k]_{p,q}! = [k]_{p,q}[k-1]_{p,q}\ldots[1]_{p,q}, \quad k = 1,2,3,\ldots,$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad k = 1,2,3,\ldots,$$

and

$$(ax+by)_{p,q}^n = \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k}b^k x^{n-k}y^k,$$
$$(x+y)_{p,q}^n = (x+y)(px+qy)\big(p^2x+q^2y\big)\ldots\big(p^{n-1}x+q^{n-1}\big),$$
$$(1-x)_{p,q}^n = (1-x)(p-qx)\big(p^2-q^2x\big)\ldots\big(p^{n-1}-q^{n-1}x\big).$$

Khan et al. [16] introduced the following Lupaş-type (p,q)-analog of Bernstein operators (rational):

For any $p > 0$ and $q > 0$, we get

$$L_n(f,p,q;x) = \sum_{k=0}^{n} f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right) b_{n,k}(x;p,q), \quad x \in [0,1], \tag{3.1}$$

where

$$b_{n,k}(x;p,q) = \frac{\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k}}{\prod_{j=0}^{n-1} \{p^j(1-x) + q^j x\}},$$

and $b_{n,0}(x;p,q), b_{n,1}(x;p,q), \ldots, b_{n,n}(x;p,q)$ are the Lupaş (p,q)-Bernstein basis functions [12]. We recall the following auxiliary results:

$$L_n(e_0,p,q;x) = 1,$$
$$L_n(e_1,p,q;x) = x,$$
$$L_n(e_2,p,q;x) = \frac{p^{n-1}x}{[n]_{p,q}} + \frac{q^2x^2}{p(1-x)+qx}\frac{[n-1]_{p,q}}{[n]_{p,q}}.$$

4 Iterates of Lupaş (p,q)-Bernstein operator

Now we extend the study of the iterates of Bernstein operators in the framework of (p,q)-calculus. The iterates of the Lupaş (p,q)-Bernstein polynomial are defined as

$$L_n^{M+1}(f,p,q;x) = L_n\bigl(L_n^M(f,p,q;x)\bigr), \quad M = 1,2,\ldots,$$

and

$$L_n^1(f,p,q;x) = L_n(f,p,q;x).$$

We study the convergence of the iterates of Lupaş (p,q)-Bernstein operators.

Theorem 4.1 *Let $L_n(f,p,q;x)$ be the Lupaş (p,q)-Bernstein operators defined in (3.1), where $p > 0$, $q > 0$.*

Then the Lupaş (p,q)-Bernstein operator is a weakly Picard operator, and its sequence $(L_n^M)_{M\ge1}$ of iterates satisfies

$$\lim_{M\to\infty} L_n^M(f,p,q;x) = f(0) + \bigl(f(1)-f(0)\bigr)x. \tag{4.1}$$

Proof The proof follows the same steps as in Theorem 2.2. Let

$$X_{\alpha,\beta} = \bigl\{f \in C[0,1] \mid f(0)=\alpha, f(1)=\beta\bigr\}, \quad \alpha,\beta \in \mathbb{R}.$$

The Lupaş (p,q)-Bernstein polynomial possess the end-point interpolation property

$$L_n(f,p,q;0) = f(0), \qquad L_n(f,p,q;1) = f(1).$$

Then $X_{\alpha,\beta}$ is an invariant subset of $f \mapsto L_n(f,p,q;\cdot)$ for all $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$ and $n \in \mathbb{N}$.

We prove that the restriction of $f \mapsto L_n(f,p,q;\cdot)$ to $X_{\alpha,\beta}$ is a contraction for any $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Let $(\alpha,\beta) \in \mathbb{R}$ and $f,g \in X_{\alpha,\beta}$. From the definition of the (p,q)-Bernstein operator,

$$\bigl|L_n(f,p,q;x) - L_n(g,p,q;x)\bigr| = \left|\sum_{k=1}^{n-1} b_{n,k}(x;p,q)(f-g)\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right)\right|$$

$$\begin{aligned}&\leq \sum_{k=1}^{n-1} b_{n,k}(x;p,q)\|f-g\|_{[0,1]}\\ &= \bigl(1-b_{n,0}(x;p,q)-b_{n,n}(x;p,q)\bigr)\|f-g\|_{[0,1]}.\end{aligned}$$

Let $w_n = \min_{x\in[0,1]}(b_{n,0}(x;p,q)+b_{n,n}(x;p,q))$, that is,

$$w_n = \min_{x\in[0,1]}\left(\frac{p^{\frac{n(n-1)}{2}}(1-x)^n}{\prod_{j=0}^{n-1}\{p^j(1-x)+q^jx\}} + \frac{q^{\frac{n(n-1)}{2}}x^n}{\prod_{j=0}^{n-1}\{p^j(1-x)+q^jx\}}\right).$$

Then $0 < w_n \leq 1$. Therefore

$$\begin{aligned}\bigl|L_n(f,p,q;x)-L_n(g,p,q;x)\bigr| &= \left(1-\frac{(p^{\frac{n(n-1)}{2}}(1-x)^n+q^{\frac{n(n-1)}{2}}x^n)}{\prod_{j=0}^{n-1}\{p^j(1-x)+q^jx\}}\right)\|f-g\|_{[0,1]}\\ &\leq (1-w_n)\|f-g\|_{[0,1]}\end{aligned}$$

for any $f,g\in X_{\alpha,\beta}$, that is, the restriction of $f\mapsto L_n(f,p,q;\cdot)$ to $X_{\alpha,\beta}$ is a contraction.

On the other hand, $K^*_{\alpha,\beta} = \alpha e_0 + (\beta-\alpha)e_1 \in X_{\alpha,\beta}$, where $e_0(x)=1$ and $e_1(x)=x$ for all $x\in[0,1]$. Since $L_n(e_0,p,q;x)=e_0$, and $L_n(e_1,p,q;x)=e_1$, it follows that $K^*_{\alpha,\beta}$ is a fixed point of $L_n(f,p,q;\cdot)$.

By Theorem 1.2 the Lupaş (p,q)-Bernstein operator is a WPO, and using the contraction principle, we obtain the claim (4.1). □

5 Conclusion

In this paper, we have studied the convergence for Lupaş q-Bernstein operators by using the contraction principle. We further extended the study of the iterates of Bernstein operators using (p,q)-calculus.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The first author has 50 percent contribution, the third author also has 50 percent contribution, whereas the second author checked and drafted the manuscript in its present form after making necessary corrections. All authors read and approved the final manuscript.

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References

1. Abel, U., Ivan, M.: Over-iterates of Bernstein operators: a short and elementary proof. Am. Math. Mon. **116**, 535–538 (2009)
2. Acar, T.: (p,q)-Generalization of Szász–Mirakyan operators. Math. Methods Appl. Sci. **39**(10), 2685–2695 (2016)
3. Acar, T., Mohiuddine, S.A., Mursaleen, M.: Approximation by (p,q)-Baskakov–Durrmeyer–Stancu operators. Complex Anal. Oper. Theory (2018). https://doi.org/10.1007/s11785-016-0633-5

4. Acar, T., Mursaleen, M., Mohiuddine, S.A.: Stancu type (p,q)-Szász–Mirakyan–Baskakov operators. Commun. Fac. Sci. Univ. Ank. Sér. A1 **67**(1), 116–128 (2018)
5. Acu, A.M., Radu, V.A.: About the iterates of some operators depending on a parameter and preserving the affine functions. Iran. J. Sci. Technol., Trans. A, Sci. https://doi.org/10.1007/s40995-017-0461-0
6. Bernstein, S.N.: Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités. Commun. Soc. Math. Kharkow **2**(13), 1–2 (1912)
7. Bica, A., Galea, L.F.: On Picard iterates properties of the Bernstein operators and an application to fuzzy numbers. Commun. Math. Anal. **5**(1), 8–19 (2008)
8. Brinde, V.: Approximating fixed points of weak contractions using the Picard iteration. Nonlinear Anal. Forum **9**(1), 43–53 (2004)
9. Cătinaş, T., Otrocol, D.: Iterates of Bernstein type operators on a square with one curved side via contraction principle. Int. J. Fixed Point Theory Comput. Appl. **14**(1), 97–106 (2013)
10. Devdhara, A.R., Mishra, V.N.: Stancu variant of (p,q)-Szász–Mirakyan operators. J. Inequal. Spec. Funct. **8**(5), 1–7 (2017)
11. Gonska, H.H., Raşa, I.: The limiting semigroup of the Bernstein iterates: degree of convergence. Acta Math. Hung. **111**, 119–130 (2006)
12. Han, L.-W., Chu, Y., Qiu, Z.-Y.: Generalized Bèzier curves and surfaces based on Lupaş q-analogue of Bernstein operator. J. Comput. Appl. Math. **261**, 352–363 (2014)
13. Kadak, U.: On weighted statistical convergence based on (p,q)-integers and related approximation theorems for functions of two variables. J. Math. Anal. Appl. **443**, 752–764 (2016)
14. Kadak, U.: Weighted statistical convergence based on generalized difference operator involving (p,q)-gamma function and its applications to approximation theorems. J. Math. Anal. Appl. **448**, 1633–1650 (2017)
15. Kelisky, R.P., Rivlin, T.J.: Iterates of Bernstein polynomials. Pac. J. Math. **21**, 511–520 (1967)
16. Khan, K., Lobiyal, D.K.: Bèzier curve based on Lupaş (p,q)-analogue of Bernstein functions in CAGD. J. Comput. Appl. Math. **317**, 458–477 (2017)
17. Lee, S.L., Phillips, G.M.: Polynomial interpolation at points of a geometric mesh on a triangle. Proc. R. Soc. Edinb. **108A**, 75–87 (1988)
18. Lupaş, A.: A q-analogue of the Bernstein operator. Univ. Cluj-Napoca Semin. Numer. Stat. Calc. **9**, 85–92 (1987)
19. Mahmudov, N.I., Sabancgil, P.: Some approximation properties of Lupaş q-analogue of Bernstein operators. arXiv:1012.4245v1 [math.FA]. Accessed 20 Dec 2010
20. Mishra, V.N., Khatri, K., Mishra Deepmala, L.N.: Inverse result in simultaneous approximation by Baskakov–Durrmeyer–Stancu operators. J. Inequal. Appl. **2013**, 586 (2013)
21. Mishra, V.N., Mursaleen, M., Pandey, S., Alotaibi, A.: Approximation properties of Chlodowsky variant of (p,q)-Bernstein–Stancu–Schurer operators. J. Inequal. Appl. **2017**, 176 (2017)
22. Mishra, V.N., Pandey, S.: On (p,q)-Baskakov–Durrmeyer–Stancu operators. Adv. Appl. Clifford Algebras **27**(2), 1633–1646 (2017)
23. Mursaleen, M., Ansari, K.J., Khan, A.: On (p,q)-analogue of Bernstein operators. Appl. Math. Comput. **266**, 874–882 (2015). Erratum to "On (p,q)-analogue of Bernstein operators", Appl. Math. Comput. **278**, 70–71 (2016)
24. Mursaleen, M., Ansari, K.J., Khan, A.: Some approximation results by (p,q)-analogue of Bernstein–Stancu operators. Appl. Math. Comput. **264**, 392–402 (2015)
25. Mursaleen, M., Khan, A.: Generalized q-Bernstein–Schurer operators and some approximation theorems. J. Funct. Spaces **2013**, Article ID 719834 (2013)
26. Mursaleen, M., Khan, F., Khan, A.: Approximation by (p,q)-Lorentz polynomials on a compact disk. Complex Anal. Oper. Theory **10**(8), 1725–1740 (2016)
27. Mursaleen, M., Nasiruzzaman, Md., Khan, A., Ansari, K.J.: Some approximation results on Bleimann–Butzer–Hahn operators defined by (p,q)-integers. Filomat **30**(3), 639–648 (2016)
28. Mursaleen, M., Nasiruzzaman, Md., Nurgali, A.: Some approximation results on Bernstein–Schurer operators defined by (p,q)-integers. J. Inequal. Appl. **2015**, 249 (2015)
29. Oruç, H., Tuncer, N.: On the convergence and iterates of q-Bernstein polynomials. J. Approx. Theory **117**, 301–313 (2002)
30. Ostrovska, S.: q-Bernstein polynomials and their iterates. J. Approx. Theory **123**, 232–255 (2003)
31. Ostrovska, S.: On the Lupaş q-analogue of the Bernstein operator. Rocky Mt. J. Math. **36**(5), 1615–1629 (2006)
32. Phillips, G.M.: Bernstein polynomials based on the q-integers. Ann. Numer. Math. **4**, 511–518 (1997)
33. Radu, V.A.: Note on the iterates of q- and (p,q)-Bernstein operators, Sci. Stud. Res. Ser. Math. Inform., **26**(2), 83–94 (2016)
34. Rus, I.A.: Weakly Picard mappings. Comment. Math. Univ. Carol. **34**(4), 769–773 (1993)
35. Rus, I.A.: Iterates of Bernstein operators, via contraction principle. J. Math. Anal. Appl. **292**, 259–261 (2004)
36. Schoenberg, I.J.: On polynomial interpolation at the points of a geometric progression. Proc. R. Soc. Edinb. **90A**, 195–207 (1981)
37. Srivastava, H.M., Mursaleen, M., Alotaibi, A.M., Nasiruzzaman, Md., Al-Abied, A.A.H.: Some approximation results involving the q-Szász–Mirakjan–Kantorovich type operators via Dunkl's generalization. Math. Methods Appl. Sci. **40**(15), 5437–5452 (2017)
38. Xiang, X., He, Q., Yang, W.: Convergence rate for iterates of q-Bernstein polynomials. Anal. Theory Appl. **23**(3), 243–254 (2007)