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On complete monotonicity for several classes of functions related to ratios of gamma functions

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Abstract

Let $\Gamma(x)$ denote the classical Euler gamma function. The logarithmic derivative $\psi(x) = [\ln \Gamma(x)]' = \frac{\Gamma'(x)}{\Gamma(x)}$, $\psi'(x)$, and $\psi''(x)$ are, respectively, called the digamma, trigamma, and tetragamma functions. In the paper, the authors survey some results related to the function $[\psi'(x)]^2 + \psi''(x)$, its q -analogs, its variants, its divided difference forms, several ratios of gamma functions, and so on. These results include the origins, positivity, inequalities, generalizations, completely monotonic degrees, (logarithmically) complete monotonicity, necessary and sufficient conditions, equivalences to inequalities for sums, applications, and the like. Finally, the authors list several remarks and pose several open problems.

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1 Preliminaries

In order to proceed fluently and smoothly, we prepare some definitions, concepts, and notation in this section.

1.1 Logarithmically completely monotonic functions

Recall from [55, Chapter XIII], [122, Chap. 1], and [127, Chapter IV] that an infinitely differentiable function $f(x)$ is said to be completely monotonic on an interval $I \subseteq \mathbb{R}$ if $0 \leq (-1)^{n-1} f^{(n-1)}(x) < \infty$ for $x \in I$ and $n \in \mathbb{N}$. A characterization of completely monotonic functions is given by the Bernstein–Widder theorem [127, p. 160, Theorem 12a], which states that a function $f(x)$ on $[0, \infty)$ is completely monotonic if and only if there exists a bounded and non-decreasing function $\alpha(t)$ such that the integral

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t) \quad (1.1)$$

converges for $x \in [0, \infty)$. Equivalently speaking, a function $f(x)$ is completely monotonic on $[0, \infty)$ if and only if it is a Laplace transform of a bounded and non-decreasing measure $\alpha(t)$. This is one of many reasons why mathematicians have been studying the class of completely monotonic functions for so many years.

Recall from [122, Chap. 2] that if $f : (0, \infty) \rightarrow [0, \infty)$ can be written as

$$f(x) = \frac{a}{x} + b + \int_0^\infty \frac{1}{s+x} d\mu(s), \quad a, b \geq 0,$$

where μ is a measure on $(0, \infty)$ such that $\int_0^\infty \frac{1}{1+s} d\mu(s) < \infty$, then we call f a Stieltjes transform.

Recall from [11, 85, 98] that a positive and infinitely differentiable function $f(x)$ is said to be logarithmically completely monotonic on $I \subseteq \mathbb{R}$ if its logarithm $\ln f(x)$ satisfies $0 \leq (-1)^k [\ln f(x)]^{(k)} < \infty$ for $x \in I$ and $k \in \mathbb{N}$.

Definition 1.2 in [36] states that a function $f : I \rightarrow [0, \infty)$ is called a Bernstein function on I if $f'(t)$ is completely monotonic on I . This class of functions can be characterized by [122, Theorem 3.2] which states that $f : (0, \infty) \rightarrow [0, \infty)$ is a Bernstein function if and only if it admits the representation

$$f(x) = \alpha + \beta x + \int_0^\infty (1 - e^{-xt}) d\mu(t), \tag{1.2}$$

where $\alpha, \beta \geq 0$ and μ is a positive measure satisfying $\int_0^\infty \min\{1, t\} d\mu(t) < \infty$.

Among the above classes of functions, there exist the following relations:

1. the set of all logarithmically completely monotonic functions on I is a strict subset of all completely monotonic functions on I ; see [85, Theorem 1];
2. the set of all Stieltjes transforms is a strict subset of all logarithmically completely monotonic functions on $(0, \infty)$; see [15, Theorem 1.2];
3. the reciprocal of a Bernstein function is logarithmically completely monotonic; see [21, pp. 161–162, Theorem 3] and [122, p. 64, Proposition 5.25].

For more information on these relations, please refer to [15, 27, 85, 98, 100, 101, 108, 122] and the closely related references therein. This is one of many reasons why mathematicians have been studying the class of logarithmically completely monotonic functions for such a long time.

Let $f(x)$ be completely monotonic on $(0, \infty)$ and $f(\infty) = \lim_{x \rightarrow \infty} f(x)$. Recall from [32, 36, 47–51, 75, 94, 97, 102, 103, 116, 119, 120] that, if for some $r \in \mathbb{R}$ the function $x^r[f(x) - f(\infty)]$ is completely monotonic on $(0, \infty)$, but for any positive number $\varepsilon > 0$ the function $x^{r+\varepsilon}[f(x) - f(\infty)]$ is not, then the number r is called the completely monotonic degree of $f(x)$ with respect to $x \in (0, \infty)$; if $x^r[f(x) - f(\infty)]$ is completely monotonic on $(0, \infty)$ for all $r \in \mathbb{R}$, then the completely monotonic degree of $f(x)$ with respect to $x \in (0, \infty)$ is said to be ∞ . For convenience, the notation $\text{deg}_{\text{cm}}^x[f(x)]$ was designed in [32] to denote the completely monotonic degree r of $f(x)$ with respect to $x \in (0, \infty)$. This notion can help to measure completely monotonic functions more accurately and precisely. Theorem 1.1 in [49], Theorem 1.3 in [50], and Proposition 1.2 in [102] can be modified as that $\text{deg}_{\text{cm}}^x[f(x)] = \alpha > 0$ if and only if

$$f(x) - f(\infty) = \int_0^\infty e^{-xt} I_\alpha(m)(t) dt, \tag{1.3}$$

where

$$I_\alpha(m)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} dm(s)$$

is called the Riemann–Liouville fractional integral of order α for a bounded and non-decreasing Borel measure m on $[0, \infty)$. For information on the Riemann–Liouville fractional integrals, please refer to [114, 126] and the closely related references therein.

1.2 The gamma and polygamma functions and their q -analogs

The classical Euler gamma function $\Gamma(z)$ can be defined by an improper integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0,$$

or by the limit

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad z \neq 0, -1, -2, \dots$$

For more information, please refer to [10, Chap. 1] and [66, 71, 90, 92]. The logarithmic derivative of $\Gamma(z)$, denoted by $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, is called the digamma function. The derivatives $\psi'(z)$ and $\psi''(z)$ are called the trigamma and tetragamma functions, respectively. As a whole, the functions $\psi^{(i)}(z)$ for $i \geq 0$ are called polygamma functions.

The q -analog $\Gamma_q(z)$ of the gamma function $\Gamma(z)$ can be defined for $\Re(z) > 0$ by

$$\Gamma_q(z) = \begin{cases} (1-q)^{1-z} \prod_{i=0}^\infty \frac{1-q^{i+1}}{1-q^{i+z}}, & 0 < q < 1; \\ (q-1)^{1-z} q^{\binom{z}{2}} \prod_{i=0}^\infty \frac{1-q^{-(i+1)}}{1-q^{-(i+z)}}, & q > 1. \end{cases}$$

The q -digamma function $\psi_q(z)$, the q -analog of the digamma function $\psi(z)$, can be defined by

$$\psi_q(z) = \frac{\Gamma'_q(z)}{\Gamma_q(z)} = \begin{cases} -\ln(1-q) + \ln q \sum_{k=0}^\infty \frac{q^{k+z}}{1-q^{k+z}}, & 0 < q < 1; \\ -\ln(q-1) + \ln q \left(z - \frac{1}{2} - \sum_{k=0}^\infty \frac{1}{q^{k+z-1}} \right), & q > 1. \end{cases}$$

These functions satisfy the relations

$$\Gamma_q(z) = q^{\binom{z-1}{2}} \Gamma_{1/q}(z), \quad \lim_{q \rightarrow 1^\pm} \Gamma_q(z) = \Gamma(z), \quad \lim_{q \rightarrow 1^\pm} \psi_q(z) = \psi(z).$$

The proofs of the above two limits can be found in [9, Appendix A], [10, pp. 493–496], and [46, Appendix B]. The functions $\psi_q^{(k)}(z)$, the q -analogs of the polygamma functions $\psi^{(k)}(z)$, for $k \geq 0$, are called the q -polygamma functions.

Equation (1.11) in [42] and its corrected version [43] read

$$\psi_q(x) = -\ln(1-q) - \int_0^\infty \frac{e^{-xt}}{1-e^{-t}} d\gamma_q(t), \quad 0 < q < 1, x > 0, \tag{1.4}$$

where

$$\gamma_q(t) = \begin{cases} -\ln q \sum_{k=1}^{\infty} \delta(t + k \ln q), & 0 < q < 1, \\ t, & q = 1, \end{cases}$$

and $\delta(t)$ represents the Dirac delta function, that is, $d\gamma_q(t)$ is a discrete measure with positive masses $|\ln q|$ at the positive points $k|\ln q|$ for $k \in \mathbb{N}$. Accordingly, we obtain

$$\int_0^{\infty} e^{-xt} d\gamma_q(t) = -\frac{q^x \ln q}{1 - q^x} \quad \text{and} \quad \int_0^{\infty} t e^{-xt} d\gamma_q(t) = \frac{q^x (\ln q)^2}{(1 - q^x)^2}$$

for $0 < q < 1$ and $x > 0$. Differentiating (1.4) with respect to x yields

$$\psi_q^{(k)}(x) = (-1)^{k+1} \int_0^{\infty} \frac{t^k e^{-xt}}{1 - e^{-t}} d\gamma_q(t), \quad 0 < q < 1, k \in \mathbb{N}.$$

In [26, p. 1245, Theorem 4.4, (4.15)], [67, Lemma 2.3 and Remark 2.1], [64, Theorem 7.2, (7.5)], and [106, p. 152, Theorem 4.22, (4.20)], the identity

$$\psi_q^{(k-1)}(x) - \psi_q^{(k-1)}(x + 1) = (\ln q) \frac{d^{k-1}}{dx^{k-1}} \left(\frac{q^x}{1 - q^x} \right), \quad 0 < q < 1$$

was presented for $x \in (0, \infty)$ and $k \in \mathbb{N}$. One can also find this knowledge in [25, 41, 42, 56, 64, 67, 73, 105] and the closely related references therein.

1.3 Divided differences

The divided difference $f[x_1, x_2, \dots, x_n]$ of a function $f(x)$ on n points x_1, x_2, \dots, x_n is defined by $f[x_1] \equiv f(x_1)$ and

$$f[x_1, x_2, \dots, x_n] = \frac{f[x_1, \dots, x_{n-1}] - f[x_2, \dots, x_n]}{x_1 - x_n}$$

for $n \geq 2$. In particular, for $n = 2$,

$$f[x_1, x_2] = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{1}{x_1 - x_2} \int_{x_1}^{x_2} f'(t) dt, \tag{1.5}$$

if f is differentiable. See [1, p. 877, 2.5.14] and [59, p. 23]. On taking $f(x) = \psi^{(i)}(x)$ in (1.5), the divided differences of the polygamma functions are

$$\psi^{(i)}[x_1, x_2] = \frac{\psi^{(i)}(x_1) - \psi^{(i)}(x_2)}{x_1 - x_2} = \frac{1}{x_1 - x_2} \int_{x_1}^{x_2} \psi^{(i+1)}(u) du$$

for $x_1, x_2 \in (0, \infty)$. By the way, we note that

$$A(a, b; f) = \frac{1}{b - a} \int_a^b f(t) dt$$

is called the arithmetic mean of the function f on the interval $[a, b]$; see [18, p. 368] and [117].

2 Three origins

The topic we are about to discuss in this paper have three main origins, as discussed now.

2.1 The first origin

Let $s, t \in \mathbb{R}$ and define

$$z_{s,t}(x) = W_{s,t}(x) - x = \begin{cases} \left[\frac{\Gamma(x+s)}{\Gamma(x+t)} \right]^{1/(s-t)} - x, & s \neq t, \\ e^{\psi(x+s)} - x, & s = t, \end{cases} \tag{2.1}$$

for $x \in (-\min\{s, t\}, \infty)$. The monotonicity and convexity of $z_{s,t}(x)$ have been studied for a long time. An easy computation yields

$$\frac{z''_{s,t}(x)}{z_{s,t}(x) + x} = \left[\frac{\psi(x+t) - \psi(x+s)}{t-s} \right]^2 + \frac{\psi'(x+t) - \psi'(x+s)}{t-s} \triangleq \Delta_{s,t}(x) \tag{2.2}$$

for $s \neq t$ and $t - s \neq \pm 1$. In [23, pp. 243–245], by using the representation

$$\psi(x+1) = -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{x+k} \right)$$

and the elementary algebraic inequality

$$\frac{1}{ab} + \frac{1}{cd} > \frac{1}{ac} + \frac{1}{bd}$$

for $a \leq b < c \leq d$, the positivity for $0 < |t - s| < 1$ and negativeness for $|t - s| > 1$ of the function $\Delta_{s,t}(x)$ were proved. Consequently, it was proved in [23, Theorem 1] that the function $z_{s,t}(x)$ is

1. convex and decreasing for $|t - s| < 1$, or
2. concave and increasing for $|t - s| > 1$.

Hereafter, several alternative proofs for [23, Theorem 1] were supplied in [19, 30, 31, 88, 99]. The differences among [19, 23, 30, 31, 88, 99] are the manners of and the approaches to coping with the positivity or negativity of the function $\Delta_{s,t}(x)$. For more information, please refer to the expository articles [64, 69, 106, 107] and the references therein.

We note that, since

$$\ln W_{s,t}(x) = \frac{\ln \Gamma(x+s) - \ln \Gamma(x+t)}{s-t} = \frac{1}{s-t} \int_t^s \psi(x+u) \, du,$$

the function $\frac{1}{W_{s,t}(x)}$ is logarithmically completely monotonic on $(-\min\{s, t\}, \infty)$.

We guess that the function $W_{s,t}(x)$ is a Bernstein function on $(-\min\{s, t\}, \infty)$.

2.2 The second origin

In [3, p. 208, (4.39)], to show that the double inequality

$$(n-1)! \exp \left[\frac{\alpha}{x} - n\psi(x) \right] < |\psi^{(n)}(x)| < (n-1)! \exp \left[\frac{\beta}{x} - n\psi(x) \right] \tag{2.3}$$

for $x > 0$ and $n \in \mathbb{N}$ is valid if and only if $\alpha \leq -n$ and $\beta \geq 0$, a single-sided inequality,

$$\begin{aligned} \Delta(x) &\triangleq [\psi'(x)]^2 + \psi''(x) \\ &> \frac{p(x)}{900x^4(x+1)^{10}} = \frac{1}{2x^4} - \frac{1}{x^3} + \frac{34}{15x^2} - \frac{14}{3x} + \frac{14}{3(x+1)} \\ &\quad + \frac{12}{5(x+1)^2} + \frac{17}{15(x+1)^3} + \frac{9}{20(x+1)^4} + \frac{1}{10(x+1)^5} \\ &\quad - \frac{7}{180(x+1)^6} - \frac{1}{30(x+1)^7} - \frac{1}{90(x+1)^8} + \frac{1}{900(x+1)^{10}}, \quad x > 0, \end{aligned} \tag{2.4}$$

was established, where

$$\begin{aligned} p(x) &= 75x^{10} + 900x^9 + 4840x^8 + 15,370x^7 + 31,865x^6 + 45,050x^5 \\ &\quad + 44,101x^4 + 29,700x^3 + 13,290x^2 + 3600x + 450. \end{aligned}$$

By the way, by the same technique as in [3, p. 208, (4.39)], an upper bound

$$\begin{aligned} \Delta(x) &< \frac{36 + 180x + 408x^2 + 504x^3 + 352x^4 + 132x^5 + 21x^6}{36x^4(1+x)^6} \\ &= \frac{1}{x^4} - \frac{1}{x^3} + \frac{7}{3x^2} - \frac{5}{x} + \frac{5}{x+1} + \frac{8}{3(x+1)^2} \\ &\quad + \frac{4}{3(x+1)^3} + \frac{7}{12(x+1)^4} + \frac{1}{6(x+1)^5} + \frac{1}{36(x+1)^6}, \quad x > 0, \end{aligned} \tag{2.5}$$

was claimed in [72].

The double inequality (2.3) can be restated as follows: the double inequality

$$\exp\left[\frac{\alpha}{x} - \psi(x)\right] < \sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} < \exp\left[\frac{\beta}{x} - \psi(x)\right]$$

for $x > 0$ and $n \in \mathbb{N}$ is valid if and only if $\alpha \leq -1$ and $\beta \geq 0$.

2.3 The third origin

In [13, Lemma 1.1], a weaker inequality,

$$\Delta(x) > 0, \quad x > 0, \tag{2.6}$$

than (2.4) was recovered. The sketch of the proof for (2.6) is as follows. By virtue of (3.2), one can find

$$\Delta(x) - \Delta(x+1) = \frac{2}{x^2} \left[\psi'(x) - \frac{1}{x} - \frac{1}{2x^2} \right] > 0, \quad x > 0.$$

This implies the inequality (2.6). In fact, since (5.1), this also implies complete monotonicity of $\Delta(x)$ on $(0, \infty)$.

2.4 A connection between three origins

It is easy to observe that $\lim_{s,t \rightarrow 0} \Delta_{s,t}(x) = \Delta(x)$. Conversely, it is also obvious that the function $\Delta_{s,t}(x)$ is a divided difference form of the function $\Delta(x)$.

3 Generalizations of the second and third origins

We now begin to collect some generalizations related to the second and third origins.

3.1 Some basic tools

One of the difficulties to generalize the above results is how to meaningfully compute any high order derivatives of the term $[\psi'(x)]^2$. If differentiating this term directly and consecutively, the derivatives would become more and more complicated and one cannot read a very useful message from these derivatives.

To overcome the above-mentioned difficulty and to generalize the above three original results, we will employ mathematical induction and the following basic, well-known, and very effectual tools.

- Let $f(x)$ be defined on an infinite interval I whose right endpoint is ∞ . If

$$\lim_{x \rightarrow \infty} f(x) = \epsilon \quad \text{and} \quad f(x) - f(x + \epsilon) > 0 \tag{3.1}$$

for some given $\epsilon > 0$, then $f(x) > \epsilon$ on I . See [13, Lemma 1.1], [30, Lemma 1], [28, p. 223, Lemma 2.1], [29, p. 107, Lemma 4], [62, p. 526, Lemma 2.1], [88, p. 1981, Lemma 2.5], [100, p. 2155, Lemma 1], [108, Lemma 2.1], [38, 89, 107] and closely related references therein. This has been summarized in [76, Remark 1, Theorem 3] which states that a function $f(x)$ on an infinite interval I whose right endpoint is ∞ is completely monotonic on I if and only if

- there exist positive numbers ϵ such that the differences

$$(-1)^i [f(x) - f(x + \epsilon)]^{(i)} = (-1)^i [f^{(i)}(x) - f^{(i)}(x + \epsilon)]$$

are nonnegative for all integers $i \geq 0$ on I ;

- the limits

$$\lim_{x \rightarrow \infty} [(-1)^i f^{(i)}(x)] = (-1)^i \lim_{x \rightarrow \infty} f^{(i)}(x) = \epsilon_i \geq 0$$

exist for all integers $i \geq 0$.

Since the gamma function $\Gamma(x)$ and polygamma functions $\psi^{(i)}(x)$ for $i \geq 0$ are of the recurrent period 1, that is, the recurrence relations $\Gamma(x + 1) = x\Gamma(x)$ and (3.2) hold, when dealing with some problems related to the gamma and polygamma functions $\Gamma(x)$ and $\psi^{(i)}(x)$, one can select $\epsilon = 1$ for possibly simplifying computation.

- For $x > 0$, $n \in \mathbb{N}$, and $r > 0$, we have

$$\psi^{(n-1)}(x + 1) = \psi^{(n-1)}(x) + \frac{(-1)^{n-1}(n-1)!}{x^n}, \tag{3.2}$$

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt} dt, \tag{3.3}$$

and

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} dt. \tag{3.4}$$

See [1, pp. 255–260] or other monographs and handbooks for special functions.

- 3. A product of finitely many completely monotonic functions on an interval I is also completely monotonic on I . See [122, Corollary 1.6].

3.2 Complete monotonicity

In [40, Theorem 1], the function

$$f_1(x) = \Delta(x) - \frac{p(x)}{900x^4(x+1)^{10}},$$

which was constructed from the inequality (2.4), was proved to be completely monotonic on $(0, \infty)$.

Sketch of the proof Making use of (3.2) yields

$$f_1(x) - f_1(x+1) = \frac{2}{x^2} \left[\psi'(x) - \frac{Q_{21}(x)}{1800x^2(1+x)^{10}(2+x)^{10}} \right] \triangleq \frac{2}{x^2} H_1(x),$$

where $Q_{21}(x)$ is a polynomial of degree 21 with positive coefficients. Employing (3.3) and (3.4), one can prove that $H_1(x)$ is a completely monotonic function on $(0, \infty)$. Hence, the function $f_1(x) - f_1(x+1)$ is completely monotonic on $(0, \infty)$. As a result, it follows that

$$(-1)^k [f_1(x) - f_1(x+1)]^{(k)} = (-1)^k f_1^{(k)}(x) - (-1)^k f_1^{(k)}(x+1) \geq 0$$

and, by (3.1), that

$$\begin{aligned} (-1)^k f_1^{(k)}(x) &\geq (-1)^k f_1^{(k)}(x+1) \geq (-1)^k f_1^{(k)}(x+2) \geq \dots \\ &\geq (-1)^k f_1^{(k)}(x+m) \geq \dots \geq \lim_{m \rightarrow \infty} [(-1)^k f_1^{(k)}(x+m)] = 0 \end{aligned}$$

for $k \geq 0$ on $(0, \infty)$. The required result is thus proved. □

3.3 A double inequality and complete monotonicity

The double inequality constituted by (2.4) and (2.5) was simplified in [136, Theorem 1] as

$$\begin{aligned} &\frac{1}{x^4} - \frac{2}{x^3} + \frac{37}{12x^2} - \frac{25}{6x} + \frac{25}{6(x+1)} + \frac{13}{12(x+1)^2} \\ &= \frac{x^2 + 12}{12x^4(x+1)^2} < \Delta(x) \\ &< \frac{x+12}{12x^4(x+1)} = \frac{1}{x^4} - \frac{23}{12x^3} + \frac{17}{6x^2} - \frac{15}{4x} + \frac{15}{4(x+1)} + \frac{11}{12(x+1)^2} \end{aligned}$$

on $(0, \infty)$. More strongly, the functions

$$f_2(x) = \Delta(x) - \frac{x^2 + 12}{12x^4(x+1)^2} \quad \text{and} \quad f_3(x) = \frac{x+12}{12x^4(x+1)} - \Delta(x), \tag{3.5}$$

were proved in [136, Theorem 2] to be completely monotonic on $(0, \infty)$.

Sketch of the proof By (3.2), we can obtain

$$f_2(x) - f_2(x + 1) = \frac{2}{x^2} \left[\psi'(x) - \frac{Q_7(x)}{6x^2(1+x)^4(2+x)^2} \right] \triangleq \frac{2}{x^2} H_2(x)$$

and

$$f_3(x) - f_3(x + 1) = \frac{2}{x^2} \left[\frac{Q_6(x)}{24x^2(x+1)^4(x+2)} - \psi'(x) \right] \triangleq \frac{2}{x^2} H_3(x),$$

where $Q_6(x)$ and $Q_7(x)$ are, respectively, polynomials of degree 6 and 7 with positive coefficients. Utilizing (3.3) and (3.4), one can prove that the functions $H_2(x)$ and $H_3(x)$ are completely monotonic on $(0, \infty)$. Finally, by (3.1), one can conclude that the functions $f_2(x)$ and $f_3(x)$ are completely monotonic on $(0, \infty)$. \square

3.4 Necessary and sufficient conditions

For $\lambda \in \mathbb{R}$, let

$$h_\lambda(x) = \Delta(x) - \frac{x^2 + \lambda x + 12}{12x^4(x+1)^2}$$

on $(0, \infty)$. This is a generalization of the function $f_2(x)$ in (3.5). In [72, Theorem 1], it was established that

1. the function $h_\lambda(x)$ is completely monotonic on $(0, \infty)$ if and only if $\lambda \leq 0$;
2. the function $-h_\lambda(x)$ is completely monotonic on $(0, \infty)$ if and only if $\lambda \geq 4$;
3. consequently, the double inequality

$$\frac{x^2 + \mu x + 12}{12x^4(x+1)^2} < \Delta(x) < \frac{x^2 + \nu x + 12}{12x^4(x+1)^2}$$

holds on $(0, \infty)$ if and only if $\mu \leq 0$ and $\nu \geq 4$.

Sketch of the proofs It is easy to see that $h_\lambda(x) = f_2(x) - \lambda g(x)$ and $g(x) = \frac{1}{12x^3(x+1)^2}$. Since $f_2(x)$ and $g(x)$ are completely monotonic functions on $(0, \infty)$, when $\lambda \leq 0$, the function $h_\lambda(x)$ is also completely monotonic on $(0, \infty)$. Utilizing (3.2) for $n = 1$ and $n = 2$ yields

$$h_\lambda(x) = -\frac{\lambda}{12} \left[\frac{1}{x^3} - \frac{2}{x^2} + \frac{3}{x} - \frac{4+3x}{(1+x)^2} \right] - \frac{37}{12x^2} + \frac{25}{6x} - \frac{63+50x}{12(1+x)^2} + \frac{2}{x^2} \psi'(x+1) + [\psi'(x+1)]^2 + \psi''(x+1).$$

Therefore, if $\lambda > 0$ then $\lim_{x \rightarrow 0^+} h_\lambda(x) = -\infty$. This implies that the function $h_\lambda(x)$ is completely monotonic on $(0, \infty)$ if and only if $\lambda \leq 0$.

From (3.2), (3.3), and (3.4), we can arrive at

$$\begin{aligned} h_\lambda(x) - h_\lambda(x+1) &= \frac{2}{x^2} \left[\psi'(x) - \frac{\lambda(4+8x+5x^2)}{24x(1+x)^3(2+x)^2} - \frac{Q_7(x)}{6x^2(1+x)^4(2+x)^2} \right] \\ &= \frac{1}{24x^2} \int_0^\infty \left[\frac{4P(t)}{Q(t)} - \lambda \right] Q(t)e^{-(x+2)t} dt, \end{aligned}$$

where

$$P(t) = \frac{e^{2t}(t^3 - 12t^2 + 54t - 86) - e^t(t^3 - 12t^2 + 16t - 160) - 26t - 74}{e^t - 1},$$

$$Q(t) = 2e^{2t} - e^t(t^2 - 6t + 18) + 8(t + 2) > 0,$$

$$\lim_{t \rightarrow 0} \frac{P(t)}{Q(t)} = 1, \quad \lim_{t \rightarrow \infty} \frac{P(t)}{Q(t)} = 0,$$

and $\frac{P(x)}{Q(x)}$ is strictly decreasing on $(0, \infty)$. By the Bernstein–Widder theorem stated in (1.1), it follows that

1. the difference $h_\lambda(x) - h_\lambda(x + 1)$ is completely monotonic on $(0, \infty)$ if and only if $\lambda \leq 0$;
2. the difference $-[h_\lambda(x) - h_\lambda(x + 1)]$ is completely monotonic on $(0, \infty)$ if and only if $\lambda \geq 4$.

Finally, by (3.1), one can conclude necessary and sufficient conditions for the function $h_\lambda(x)$ to be completely monotonic on $(0, \infty)$. □

How much are completely monotonic degrees of the function $h_\lambda(x)$ for $\lambda \leq 0$ and its negative $-h_\lambda(x)$ for $\lambda \geq 4$ with respect to $x \in (0, \infty)$?

3.5 A generalization of the third origin

Lemma 1.2 in [14] shows that the inequality

$$(-1)^n \psi^{(n+1)}(x) < \frac{n}{\sqrt[n]{(n-1)!}} [(-1)^{n-1} \psi^{(n)}(x)]^{1+1/n} \tag{3.6}$$

holds for $x > 0$ and $n \in \mathbb{N}$. Remark 1.3 in [14] points out that the inequality (2.6) is the special case $n = 1$ of the inequality (3.6). The inequality (3.6) was reformulated in [39, p. 108] as the form

$$\sqrt[n+1]{\frac{|\psi^{(n+1)}(x)|}{n!}} < \sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} \tag{3.7}$$

for $n \in \mathbb{N}$ on $(0, \infty)$.

Sketch of the proof Theorem 2.1 in [8] states that the function

$$F_{n,\alpha}(x) = [\psi^{(n)}(x)]^2 - \alpha \psi^{(n-1)}(x) \psi^{(n+1)}(x), \quad n \geq 2$$

is completely monotonic on $(0, \infty)$ if and only if $\alpha \leq \frac{n-1}{n}$, while its negative $-F_{n,\beta}(x)$ is completely monotonic on $(0, \infty)$ if and only if $\beta \geq \frac{n}{n+1}$. Consequently, a sharp double inequality

$$\frac{n-1}{n} < \frac{[\psi^{(n)}(x)]^2}{\psi^{(n-1)}(x) \psi^{(n+1)}(x)} < \frac{n}{n+1}, \quad n \geq 2 \tag{3.8}$$

was derived in [8, Corollary 2.3]. In [14, Lemma 1.2], by virtue of the left inequality in (3.8), the function $\frac{[\psi^{(n)}(x)]^{n+1}}{[\psi^{(n+1)}(x)]^n}$ for $n \in \mathbb{N}$ was proved to be increasing on $(0, \infty)$. As a result, the inequality (3.6) was deduced straightforwardly. □

3.6 Necessary and sufficient conditions for divided differences

For $n \in \mathbb{N}$ and $p, q \in \mathbb{R}$, let

$$\begin{aligned} \phi_n(x) &= \begin{cases} -\frac{|\psi^{(n-1)}(x+p)| - |\psi^{(n-1)}(x+q)|}{p-q}, & p \neq q, \\ |\psi^{(n)}(x+q)|, & p = q, \end{cases} \\ &= \begin{cases} \frac{1}{p-q} \int_q^p |\psi^{(n)}(x+t)| dt, & p \neq q, \\ |\psi^{(n)}(x+q)|, & p = q, \end{cases} \end{aligned}$$

and

$$\Phi_{n,\lambda}(x) = [\phi_{n+1}(x)]^2 - \lambda \phi_n(x) \phi_{n+2}(x).$$

In [128], Theorem 1 states that

1. when $p = q$,
 - (a) the function $\Phi_{n,\lambda}(x)$ is completely monotonic on $(-\min\{p, q\}, \infty)$ if and only if $\lambda \leq \frac{n}{n+1}$;
 - (b) the function $-\Phi_{n,\lambda}(x)$ is completely monotonic on $(-\min\{p, q\}, \infty)$ if and only if $\lambda \geq \frac{n+1}{n+2}$;
2. when $0 < |p - q| < 1$,
 - (a) the function $\Phi_{n,\lambda}(x)$ is completely monotonic on $(-\min\{p, q\}, \infty)$ if and only if $\lambda \leq \frac{n}{n+1}$;
 - (b) the function $-\Phi_{n,\lambda}(x)$ is completely monotonic on $(-\min\{p, q\}, \infty)$ if and only if $\lambda \geq \sup_{t \in (0, \infty)} Q_{|p-q|}(t)$, where

$$\begin{aligned} Q_\alpha(t) &= \frac{\int_0^1 (1-v^2)^n P_\alpha(t, v) dv}{\int_0^1 (1+v^2)(1-v^2)^{n-1} P_\alpha(t, v) dv}, \\ P_\alpha(t, v) &= h_\alpha\left(\frac{1+v}{2}t\right) h_\alpha\left(\frac{1-v}{2}t\right), \end{aligned}$$

and

$$h_\alpha(t) = \begin{cases} \frac{1}{\alpha} \frac{1-e^{-\alpha t}}{1-e^{-t}}, & \alpha \neq 0; \\ \frac{t}{1-e^{-t}}, & \alpha = 0; \end{cases} \tag{3.9}$$

3. when $|p - q| > 1$,
 - (a) the function $\Phi_{n,\lambda}(x)$ is completely monotonic on $(-\min\{p, q\}, \infty)$ if and only if $\lambda \leq \inf_{t \in (0, \infty)} Q_{|p-q|}(t)$;
 - (b) the function $-\Phi_{n,\lambda}(x)$ is completely monotonic on $(-\min\{p, q\}, \infty)$ if and only if $\lambda \geq \frac{n}{n+1}$.

Theorem 2 in [128] states that the following four statements are equivalent:

1. the function $\frac{[\psi^{(n+1)}(x)]^2}{\psi^{(n)}(x)\psi^{(n+2)}(x)}$ is strictly decreasing from $(0, \infty)$ onto $(\frac{n}{n+1}, \frac{n+1}{n+2})$;
2. the sequence $\{|\frac{\psi^{(n+1)}(x)}{\psi^{(n)}(x)}|, x \in (0, \infty)\}_{n \in \mathbb{N}}$ is concave;
3. the function $\frac{|\psi^{(n)}(x)\psi^{(n+1)}(x)|}{\psi^{(n)}(x)\psi^{(n+2)}(x) - [\psi^{(n+1)}(x)]^2} - x$ is strictly decreasing from $(0, \infty)$ onto $(-\frac{1}{2}, 0)$;
4. the function $|\frac{\psi^{(n)}(x)}{\psi^{(n+1)}(x)}|$ is convex on $(0, \infty)$.

We note that the function $h_\alpha(t)$ in (3.9) can be regarded as a special case of the function $q_{\alpha,\beta}(t)$ defined by (11.6).

3.7 A guess on complete monotonicity

Motivated by Theorem 2.1 in [8] mentioned above, we can consider the function

$$F_{n,k;c}(x) = [\psi^{(n)}(x)]^2 - c\psi^{(n-k)}(x)\psi^{(n+k)}(x)$$

for $n > k \geq 1$ on $(0, \infty)$. Similarly, stimulated by Theorems 1 and 2 in [128], we can also discuss the function

$$\Phi_{n,k;\lambda}(x) = [\phi_{n+1}(x)]^2 - \lambda\phi_{n-k+1}(x)\phi_{n+k+1}(x).$$

We guess that necessary and sufficient conditions for the functions $\pm F_{n,k;c}(x)$ to be completely monotonic on $(0, \infty)$ are

$$c \leq \frac{[(n-1)!]^2}{(n-k-1)!(n+k-1)!} \quad \text{and} \quad c \geq \frac{(n!)^2}{(n-k)!(n+k)!},$$

respectively. We will confirm this guess in a subsequent paper soon.

3.8 Inequalities between polygamma functions

Theorem 2.2 in [14] states that the double inequality

$$(n-1)! \left[\frac{\psi^{(k)}(x+1/2)}{(-1)^{k-1}(k-1)!} \right]^{n/k} < |\psi^{(n)}(x)| < (n-1)! \left[\frac{\psi^{(k)}(x)}{(-1)^{k-1}(k-1)!} \right]^{n/k} \tag{3.10}$$

holds for $1 \leq k \leq n-1$ and $x > 0$. When $n = 1$, the left inequality in (3.10) becomes

$$\psi''(x) + \left[\psi' \left(x + \frac{1}{2} \right) \right]^2 < 0, \quad x > 0.$$

In [33, p. 1008] and [39, p. 110], the double inequality (3.10) was reformulated as

$$\sqrt[k]{\frac{|\psi^{(k)}(x+1/2)|}{(k-1)!}} < \sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} < \sqrt[k]{\frac{|\psi^{(k)}(x)|}{(k-1)!}} \tag{3.11}$$

for $x > 0$ and $1 \leq k \leq n-1$. The right inequality in (3.11) is equivalent to (3.7). In [33], the left inequality in (3.10) and (3.11) was refined as

$$\sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} > \sqrt[k]{\frac{|\psi^{(k)}(1/\ln(1+1/x))|}{(k-1)!}}$$

for $1 \leq k \leq n-1$ on $(0, \infty)$, since $x < \frac{1}{\ln(1+1/x)} < x + \frac{1}{2}$ on $(0, \infty)$.

3.9 Necessary and sufficient conditions

In [89, Theorem 1 and Corollary 1], the following necessary and sufficient conditions were presented:

1. the function $\psi''(x) + [\psi'(x + \alpha)]^2$ is completely monotonic on $(-\min\{0, \alpha\}, \infty)$ if and only if $\alpha \leq 0$;
2. the function

$$-\{\psi''(x) + [\psi'(x + \alpha)]^2\} \tag{3.12}$$

is completely monotonic if

$$\alpha \geq \sup_{x \in (0, \infty)} \frac{x}{\phi^{-1}([2(x + 1)^2 - 1]e^{2x})}, \tag{3.13}$$

where $\phi(x) = x \coth x$ for $x \in (0, \infty)$ and ϕ^{-1} is the inverse function of ϕ ;

3. for $\alpha \geq \frac{1}{6}$, the function (3.12) is completely monotonic on $(0, \infty)$.

Equivalently speaking, for $\beta \in \mathbb{R}$ and $x > -\min\{0, \beta\}$,

1. the function

$$\psi''(x + \beta) + [\psi'(x)]^2 \tag{3.14}$$

is completely monotonic if and only if $\beta \geq 0$;

2. if

$$\beta \leq - \sup_{x \in (0, \infty)} \frac{x}{\phi^{-1}([2(x + 1)^2 - 1]e^{2x})} < 0,$$

the negative of (3.14) is completely monotonic;

3. for $\beta \leq -\frac{1}{6}$, the negative of (3.14) is completely monotonic on the interval $(-\min\{0, \beta\}, \infty)$.

3.10 Complete monotonicity with a parameter

In [89, Theorem 3], it was found that the function $[\psi'(x)]^2 + \lambda \psi''(x)$ is completely monotonic on $(0, \infty)$ if and only if $\lambda \leq 1$.

3.11 Uniqueness of complete monotonicity

In [38, Theorem 1], it was proved that, among the functions

$$f_{m,n}(x) = [\psi^{(m)}(x)]^2 + \psi^{(n)}(x) \tag{3.15}$$

on $(0, \infty)$ for $m, n \in \mathbb{N}$, the functions $f_{1,2}(x)$ and $f_{m,2n-1}(x)$ are completely monotonic on $(0, \infty)$, but complete monotonicity of $f_{m,2n-1}(x)$ is trivial, and the functions $f_{m,2n}(x)$ for $(m, n) \neq (1, 1)$ are not monotonic and does not keep the same sign on $(0, \infty)$.

3.12 Completely monotonic degree

In [47, p. 2273, Corollary 3], among others, it was deduced that the functions $x^2 \Delta(x)$ and $x^3 \Delta(x)$ are completely monotonic on $(0, \infty)$. These imply that $\deg_{\text{cm}}^x[\Delta(x)] \geq 3$. For more information, please refer to the series of papers [47–49, 51].

With the help of [68], it was discovered in [65] that

$$\deg_{\text{cm}}^x[\Delta(x)] = 4. \tag{3.16}$$

3.13 Three q -analogs of the third origin

In [7, pp. 80–81, Lemma 4.6], the inequality (2.6) was generalized as

$$\Delta_q(x) = [\psi'_q(x)]^2 + \psi''_q(x) > 0 \tag{3.17}$$

for $q > 1$ on $(0, \infty)$. In [24, p. 13], it was established that

$$[\psi'_q(x)]^2 - \frac{q^x \ln q}{1 - q} \psi''_q(x) \geq 0, \quad 0 < q < 1, x > 0.$$

In [73, Theorem 1.1], it was presented that

1. for $q > 1$, the function $\Delta_q(x)$ is completely monotonic with respect to $x \in (0, \infty)$;
2. for $0 < q < 1$, the function

$$[\psi'_q(x) - \ln q]^2 + \psi''_q(x) \tag{3.18}$$

is completely monotonic with respect to $x \in (0, \infty)$.

4 Generalizations of the first origin

The generalizations of the first origin have been developing in recent years.

4.1 Some basic methods

In order to present convexity and monotonicity of the function $z_{s,t}(x)$ defined by (2.1), it is enough to show the positivity or negativity of the function $\Delta_{s,t}(x)$. However, it was observed that the function $\Delta_{s,t}(x)$ or its negative should be completely monotonic. In order to verify this observation, there was a barrier to be overcome, the barrier is to significantly simplify and tidy up any high order derivatives of $\Delta_{s,t}(x)$ with respect to x . How to deal with high order derivatives of $\Delta_{s,t}(x)$ easily and simply? This problem was nicely and smartly solved in [30, 88]: utilizing the simple but effectual tools stated in Sect. 3.1.

4.2 Complete monotonicity of divided differences

In [30, Theorem 1] and [88, Theorem 1.2], it was found that the function $\Delta_{s,t}(x)$, defined by (2.2), for $|t - s| < 1$ and $-\Delta_{s,t}(x)$ for $|t - s| > 1$ are completely monotonic with respect to $x \in (-\min\{s, t\}, \infty)$. As a consequence, meanwhile, complete monotonicity of $\Delta(x)$ can be recovered once again.

4.3 Complete monotonicity of divided differences with a parameter

For $s, t, \lambda \in \mathbb{R}$ and $\alpha = \min\{s, t\}$, the function

$$\Delta_{s,t;\lambda}(x) = \begin{cases} \left[\frac{\psi(x+t) - \psi(x+s)}{t-s} \right]^2 + \lambda \frac{\psi'(x+t) - \psi'(x+s)}{t-s}, & s \neq t; \\ [\psi'(x+s)]^2 + \lambda \psi''(x+s), & s = t. \end{cases} \tag{4.1}$$

In [91, Theorem 1.1], complete monotonicity of the function $\Delta_{s,t}(x)$ was generalized as follows:

1. when $0 < |t - s| < 1$,
 - (a) the function $\Delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha, \infty)$ if and only if $\lambda \leq 1$;
 - (b) the function $-\Delta_{s,t;\lambda}(x)$ is completely monotonic if and only if $\lambda \geq \frac{1}{|t-s|}$;

2. when $|t - s| > 1$,
 - (a) the function $\Delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha, \infty)$ if and only if $\lambda \leq \frac{1}{|t-s|}$;
 - (b) the function $-\Delta_{s,t;\lambda}(x)$ is completely monotonic if and only if $\lambda \geq 1$;
3. when $|t - s| = 1$,
 - (a) the function $\Delta_{s,t;\lambda}(x)$ is completely monotonic if and only if $\lambda < 1$;
 - (b) the function $-\Delta_{s,t;\lambda}(x)$ is completely monotonic if and only if $\lambda > 1$;
 - (c) and $\Delta_{s,t;1}(x) \equiv 0$;
4. when $s = t$, the function $\Delta_{s,s;\lambda}(x)$ is completely monotonic on $(-s, \infty)$ if and only if $\lambda \leq 1$.

Consequently, [89, Theorem 3], which shows that the function $[\psi'(x)]^2 + \lambda\psi''(x)$ is completely monotonic on $(0, \infty)$ if and only if $\lambda \leq 1$, can be derived again.

5 Related generalizations

There are several generalizations related to the above results and conclusions.

5.1 The difference between the trigamma function and rational functions

By the formula (3.3), it is easy to see that the function $(-1)^{n+1}\psi^{(n)}(x)$ for $n \in \mathbb{N}$ is completely monotonic on $(0, \infty)$.

The functions

$$\psi'(x) - \frac{1}{x} \quad \text{and} \quad \psi'(x) - \frac{1}{x} - \frac{1}{2x^2} = \psi'(x) - \frac{2x + 1}{2x^2} \tag{5.1}$$

are completely monotonic on $(0, \infty)$. This can be derived from [2, pp. 374–375, Theorem 1], [20, Theorems 1 and 2], [29, Theorem 1.1], [63, Theorem 1.3], [86, Theorem 1] and the closely related references therein. For more general results, please refer to [39, Sect. 1.4] and the closely related references therein.

5.2 The difference between the trigamma function and a rational function

It was concluded in [40, Remark 2] that the function

$$\begin{aligned} H_1(x) &= \psi'(x) - \frac{Q_{21}(x)}{1800x^2(1+x)^{10}(2+x)^{10}} \\ &= \psi'(x) - \frac{1}{2x} - \frac{3}{4x^2} - \frac{28}{5(x+1)} + \frac{51}{10(x+2)} + \frac{251}{120(x+1)^2} \\ &\quad + \frac{331}{120(x+2)^2} - \frac{7}{6(x+1)^3} + \frac{17}{12(x+2)^3} + \frac{13}{90(x+1)^4} \\ &\quad + \frac{49}{72(x+2)^4} - \frac{13}{180(x+1)^5} + \frac{47}{180(x+2)^5} - \frac{1}{120(x+1)^6} \\ &\quad - \frac{1}{60(x+2)^6} + \frac{1}{180(x+1)^7} - \frac{2}{45(x+2)^7} + \frac{1}{200(x+1)^8} \\ &\quad - \frac{13}{600(x+2)^8} + \frac{1}{900(x+1)^9} - \frac{1}{450(x+2)^9} \\ &\quad - \frac{1}{1800(x+1)^{10}} + \frac{1}{450(x+2)^{10}} \end{aligned}$$

is completely monotonic on $(0, \infty)$, where

$$Q_{21}(x) = 1,382,400 + 21,657,600x + 162,792,960x^2 + 778,137,600x^3$$

$$\begin{aligned}
 &+ 2,645,782,983x^4 + 6,789,381,590x^5 + 13,626,443,025x^6 \\
 &+ 21,889,330,810x^7 + 28,579,049,475x^8 + 30,634,381,522x^9 \\
 &+ 27,125,436,630x^{10} + 19,896,883,200x^{11} + 12,088,287,630x^{12} \\
 &+ 6,063,596,590x^{13} + 2,494,770,300x^{14} + 832,958,400x^{15} \\
 &+ 222,060,150x^{16} + 46,134,540x^{17} + 7,195,500x^{18} \\
 &+ 792,300x^{19} + 54,900x^{20} + 1800x^{21}.
 \end{aligned}$$

This result was proved in [40, Theorem 1].

5.3 Completely monotonic degree is one

In [32, Theorem 1], complete monotonicity of $H_1(x)$ was strengthened as

$$\text{deg}_{\text{cm}}^x[H_1(x)] = 1$$

and the integral representation

$$H_1(x) = \int_0^\infty q(t)e^{-xt} dt$$

of the form (1.1) or (1.3) was concluded in [32, Remark 2], where

$$q(t) = \frac{\theta(t)}{653,184,000e^{2t}(e^t - 1)}$$

and

$$\begin{aligned}
 \theta(t) = &163,296,000(t - 2)e^{3t} - e^{2t}(t^9 - 18t^8 - 648t^7 - 5040t^6 \\
 &+ 45,360t^5 + 1,965,600t^4 - 15,724,800t^3 + 381,024,000t^2 \\
 &- 1,856,131,200t + 3,331,238,400) + e^t(5t^9 - 54t^8 - 3456t^7 \\
 &- 45,360t^6 - 45,360t^5 + 9,072,000t^4 + 58,363,200t^3 \\
 &+ 843,696,000t^2 + 435,456,000t + 6,989,068,800) - 4(t^9 - 9t^8 \\
 &- 702t^7 - 10,080t^6 - 22,680t^5 + 1,776,600t^4 + 18,522,000t^3 \\
 &+ 115,668,000t^2 + 450,424,800t + 832,809,600)
 \end{aligned}$$

with $q(0) = 0$, $q(t) \geq 0$, and $q'(t) > 0$ on $(0, \infty)$.

5.4 A difference between the trigamma function and a rational function

In [136, Remark 1], it was concluded that the functions

$$\begin{aligned}
 H_2(x) &= \psi'(x) - \frac{Q_7(x)}{6x^2(1+x)^4(2+x)^2} \\
 &= \psi'(x) - \frac{24 + 120x + 283x^2 + 399x^3 + 345x^4 + 181x^5 + 51x^6 + 6x^7}{6x^2(1+x)^4(2+x)^2}
 \end{aligned}$$

$$= \psi'(x) - \left[\frac{1}{x^2} + \frac{43}{6(x+1)} - \frac{7}{2(x+1)^2} + \frac{2}{(x+1)^3} - \frac{1}{2(x+1)^4} - \frac{37}{6(x+2)} - \frac{13}{6(x+2)^2} \right]$$

and

$$\begin{aligned} H_3(x) &= \frac{Q_6(x)}{24x^2(x+1)^4(x+2)} - \psi'(x) \\ &= \frac{24x^6 + 156x^5 + 412x^4 + 621x^3 + 523x^2 + 242x + 48}{24x^2(x+1)^4(x+2)} - \psi'(x) \\ &= \frac{1}{x^2} + \frac{13}{24x} + \frac{55}{24(x+1)} - \frac{15}{8(x+1)^2} + \frac{35}{24(x+1)^3} \\ &\quad - \frac{1}{2(x+1)^4} - \frac{11}{6(x+2)} - \psi'(x) \end{aligned}$$

are completely monotonic on $(0, \infty)$. Consequently, as claimed in [136, Remark 2], integrating the inequalities $H_2(x) > 0$ and $H_3(x) > 0$ yields a double inequality,

$$\begin{aligned} &\frac{42t^3 + t^2 - 81t - 48}{48t(t+1)^3} + \frac{13 \ln t + 55 \ln(t+1) - 44 \ln(t+2)}{24} \\ &< \psi(t) < \frac{28t^4 + 87t^3 + 73t^2 + 3t - 12}{6t(t+1)^3(t+2)} + \frac{43 \ln(t+1) - 37 \ln(t+2)}{6}, \quad t > 0. \end{aligned}$$

Can one compute completely monotonic degrees of the functions $H_2(x)$ and $H_3(x)$ on $(0, \infty)$, as one did in [32]?

5.5 Necessary and sufficient conditions

In [72, Remark 3], it was deduced that the function

$$\psi'(x) - \frac{\lambda(4 + 8x + 5x^2)}{24x(1+x)^3(2+x)^2} - \frac{Q_7(x)}{6x^2(1+x)^4(2+x)^2} \tag{5.2}$$

is completely monotonic on $(0, \infty)$ if and only if $\lambda \leq 0$, and its negative is completely monotonic on $(0, \infty)$ if and only if $\lambda \geq 4$.

What are completely monotonic degrees of the function (5.2) for $\lambda \leq 0$ and its negative for $\lambda \geq 4$ on $(0, \infty)$?

5.6 A difference between the trigamma function and a rational function of a parameter

Let $\alpha \in \mathbb{R}$ and $x > -\min\{0, \alpha\}$. It was proved in [89, Theorem 2 and Corollary 1] that

1. the function

$$\psi'(x + \alpha) - \frac{1}{2(x + \alpha)^2} - \frac{(x + \alpha)^2}{x^3} \tag{5.3}$$

is completely monotonic if and only if $\alpha \leq 0$;

2. the negative of the function (5.3) is completely monotonic if the inequality (3.13) is valid;

- the necessary condition for the negative of (5.3) to be completely monotonic on $x \in (0, \infty)$ is

$$\alpha \geq \sup_{x \in (0, \infty)} \left\{ x \left[1 - \sqrt[3]{\frac{2x}{2x^2 \psi'(x) - 1}} \right] \right\}; \tag{5.4}$$

- for $\alpha \geq \frac{1}{6}$, the negative of (5.3) is completely monotonic on $(0, \infty)$; equivalently speaking, for $\beta \leq -\frac{1}{6}$, the function $\frac{1}{2x^2} + \frac{x^2}{(x+\beta)^3} - \psi'(x)$ is completely monotonic on $(-\min\{0, \beta\}, \infty)$.

We guess that the conditions in (3.13) and (5.4) should be necessary and sufficient.

5.7 A difference between a divided difference of and a rational function

As by-product of [62, p. 524, Theorem 1.1] and [88, p. 1978, Theorem 1.1], the following complete monotonicity for the divided difference form of the second function in (5.1) was acquired: the function

$$\delta_{s,t}(x) = \begin{cases} \frac{\psi(x+t) - \psi(x+s)}{t-s} - \frac{2x+s+t+1}{2(x+s)(x+t)}, & s \neq t, \\ \psi'(x+s) - \frac{1}{x+s} - \frac{1}{2(x+s)^2}, & s = t, \end{cases} \tag{5.5}$$

for $|t - s| < 1$ and $-\delta_{s,t}(x)$ for $|t - s| > 1$ are completely monotonic on $x \in (-\alpha, \infty)$, where s and t are real numbers and $\alpha = \min\{s, t\}$.

5.8 A difference of a divided difference of and a rational function with a parameter

In [91, Theorem 1.3], along with generalizing $\Delta_{s,t}(x)$ to $\Delta_{s,t;\lambda}(x)$ in (4.1), the authors generalized the function $\delta_{s,t}(x)$ as

$$\theta_{s,t;\lambda}(x) = \begin{cases} \frac{\psi(x+t) - \psi(x+s)}{t-s} - \frac{1+\lambda(2x+s+t)}{2(x+s)(x+t)}, & s \neq t, \\ \psi'(x+s) - \frac{\lambda}{x+s} - \frac{1}{2(x+s)^2}, & s = t, \end{cases} \tag{5.6}$$

on $(-\min\{s, t\}, \infty)$ for $s, t, \lambda \in \mathbb{R}$, and presented the following complete monotonicity:

- when $0 < |t - s| < 1$,
 - the function $\theta_{s,t;\lambda}(x)$ is completely monotonic if and only if $\lambda \leq 1$;
 - the function $-\theta_{s,t;\lambda}(x)$ is completely monotonic if and only if $\lambda \geq \frac{1}{|t-s|}$;
- when $|t - s| > 1$,
 - the function $\theta_{s,t;\lambda}(x)$ is completely monotonic if and only if $\lambda \leq \frac{1}{|t-s|}$;
 - the function $-\theta_{s,t;\lambda}(x)$ is completely monotonic if and only if $\lambda \geq 1$;
- when $|t - s| = 1$,
 - the function $\theta_{s,t;\lambda}(x)$ is completely monotonic if and only if $\lambda < 1$;
 - the function $-\theta_{s,t;\lambda}(x)$ is completely monotonic if and only if $\lambda > 1$;
 - and $\theta_{s,t;1}(x) \equiv 0$;
- when $s = t$, the function $\theta_{s,s;\lambda}(x)$ is completely monotonic if and only if $\lambda \leq 1$.

5.9 Necessary and sufficient conditions of complete monotonicity

In [83, 108], the authors alternatively generalized the function $\delta_{s,t}(x)$ in (5.5) as

$$\delta_{s,t;\lambda}(x) = \begin{cases} \frac{\psi(x+t)-\psi(x+s)}{t-s} - \frac{2x+s+t+2\lambda}{2(x+s)(x+t)}, & s \neq t, \\ \psi'(x+s) - \frac{1}{x+s} - \frac{\lambda}{(x+s)^2}, & s = t, \end{cases} \tag{5.7}$$

on $(-\alpha, \infty)$ for $s, t, \lambda \in \mathbb{R}$ and $\alpha = \min\{s, t\}$, and discovered the following necessary and sufficient conditions of complete monotonicity:

1. when $|t - s| < 1$,
 - (a) the function $\delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha, \infty)$ if and only if $\lambda \leq \frac{1}{2}$;
 - (b) the function $-\delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha, \infty)$ if and only if $\lambda \geq 1 - \frac{|t-s|}{2}$;
2. when $|t - s| > 1$,
 - (a) the function $\delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha, \infty)$ if and only if $\lambda \leq 1 - \frac{|t-s|}{2}$;
 - (b) the function $-\delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha, \infty)$ if and only if $\lambda \geq \frac{1}{2}$;
3. when $|t - s| = 1$,
 - (a) the function $\delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha, \infty)$ if and only if $\lambda < \frac{1}{2}$;
 - (b) the function $-\delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha, \infty)$ if and only if $\lambda > \frac{1}{2}$;
 - (c) the function $\delta_{s,t;\lambda}(x)$ is identically zero if and only if $\lambda = \frac{1}{2}$.

5.10 Complete monotonicity of q -analogs

In [67, 105, 135], the second function in (5.1) was generalized to the q -analogs

$$f_q(x) = \psi'_q(x) - \frac{(1-q)q^x}{1-q^x} - \frac{1}{2} \left[\frac{(1-q)q^x}{1-q^x} \right]^2$$

and

$$F_q(x) = \psi'_q(x) - \frac{1-q}{1-q^x} - \frac{1}{2} \left(\frac{1-q}{1-q^x} \right)^2 + \frac{1}{2} (1-q)(3-q)$$

on $(0, \infty)$ for $0 < q < 1$ and these two q -analogs were proved to be completely monotonic on $(0, \infty)$.

For $0 < q < 1$, we guess that the functions

$$I_q(x) = \psi'_q(x) - \frac{1-q}{1-q^x} - \frac{1}{2} \left(\frac{1-q}{1-q^x} \right)^2 - \frac{1}{6} \left(\frac{1-q}{1-q^x} \right)^3 + \frac{1}{30} \left(\frac{1-q}{1-q^x} \right)^5 + \frac{1-q}{30} (49 - 21q - q^2 + 4q^3 - q^4)$$

and

$$J_q(x) = \psi'_q(x) - \frac{1-q}{1-q^x} - \frac{1}{2} \left(\frac{1-q}{1-q^x} \right)^2 - \frac{1}{6} \left(\frac{1-q}{1-q^x} \right)^2 + \frac{1}{30} \left[\frac{(1-q)q^x}{1-q^x} \right]^5 + \frac{1-q}{6} (10 - 5q + q^2)$$

are completely monotonic with respect to $x \in (0, \infty)$.

6 Applications

Some of the above results and conclusions have been applied to find inequalities, monotonicity, logarithmically completely monotonicity of some functions involving the gamma function $\Gamma(x)$, the ratio $\frac{\Gamma(x+a)}{\Gamma(x+b)}$, polygamma functions $\psi^{(k)}(x)$ for $k \geq 0$, and so on.

6.1 Monotonicity and convexity

In [34, Theorem 2], by virtue of the inequality (2.6), it was proved that the function

$$e^{\psi(x+1)} - x \tag{6.1}$$

is strictly decreasing and strictly convex on $(-1, \infty)$.

In [123], Theorems 1 and 2 state that

1. the function

$$\exp\left[\psi\left(x + \frac{1}{2}\right) - \frac{1}{24x^2 + 7/40}\right] - x$$

is decreasing from $(-\frac{1}{2}, \infty)$ onto $(0, \frac{1}{2})$ and convex on $(-\frac{1}{2}, \infty)$;

2. the function

$$\exp\left[\psi\left(x + \frac{1}{2}\right)\right] - x \exp\left(\frac{1}{24x^2 + 7/40}\right)$$

is decreasing from $(0, \infty)$ onto $(0, \frac{1}{4e^\gamma})$ and convex on $(\frac{1}{2}, \infty)$, where $\gamma = 0.57721566\dots$ is the Euler–Mascheroni constant.

6.2 Alternative proofs of the first origin

By virtue of the equality (2.2), we see readily that complete monotonicity of the function $\Delta_{s,t}(x)$ implies the convexity, and then the monotonicity, of the function $z_{s,t}(x)$ defined by (2.1). Therefore, alternative proofs of [23, Theorem 1] were provided in [19, 30, 31, 88, 99] and the closely related references therein.

6.3 A generalization of the first origin

As an application of [91, Theorem 1.1] and as a generalization of [23, Theorem 1, Corollaries 2 and 3], [34, Theorem 2], and [89, p. 75, Corollary 2], Theorem 1.5 in [91] states that the function

$$z_{s,t;\lambda}(x) = \begin{cases} \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/[\lambda(t-s)]}, & s \neq t, \\ e^{\psi(x+s)/\lambda}, & s = t, \end{cases}$$

for $s, t \in \mathbb{R}$, $\alpha = \min\{s, t\}$, and $\lambda \neq 0$ has the following properties:

1. when $0 < |t - s| < 1$,
 - (a) the function $z_{s,t;\lambda}(x)$ is convex on $(-\alpha, \infty)$ if and only if $\lambda \leq 1$,
 - (b) the function $z_{s,t;\lambda}(x)$ is concave on $(-\alpha, \infty)$ if and only if $\lambda \geq \frac{1}{|t-s|}$;
2. when $|t - s| > 1$,
 - (a) the function $z_{s,t;\lambda}(x)$ is convex on $(-\alpha, \infty)$ if and only if $\lambda \leq \frac{1}{|t-s|}$,
 - (b) the function $z_{s,t;\lambda}(x)$ is concave on $(-\alpha, \infty)$ if and only if $\lambda \geq 1$;

3. when $s = t$, the function $z_{s,s;\lambda}(x)$ is convex on $(-s, \infty)$ if and only if $\lambda \leq 1$;
4. when $|t - s| = 1$,
 - (a) the function $z_{s,t;\lambda}(x)$ is convex on $(-\alpha, \infty)$ if and only if $\lambda < 1$,
 - (b) the function $z_{s,t;\lambda}(x)$ is concave on $(-\alpha, \infty)$ if and only if $\lambda > 1$,
 - (c) $z_{s,t;1}(x) = x + \alpha$ on $(-\alpha, \infty)$.

6.4 Monotonicity and inequalities for q -analogs

Using complete monotonicity of the function $\Delta_q(x)$ on $(0, \infty)$, the following monotonicity cases and inequalities were established in [73, Corollary 1.1]: the functions

$$\begin{aligned} \phi_q(x) &= \begin{cases} \psi_q(x) + \ln[\exp \frac{(\ln q)q^x}{q^x-1} - 1], & q > 1, \\ \psi(x) + \ln(e^{1/x} - 1), & q = 1, \end{cases} \\ \varphi_q(x) &= \begin{cases} \psi_q(x) + \ln(\exp \frac{\ln q}{q^x-1} - 1), & q > 1, \\ \psi(x) + \ln(e^{1/x} - 1), & q = 1, \end{cases} \\ \Phi_q(x) &= \begin{cases} \psi_q(x) - (\ln q)x + \ln[\exp \frac{(\ln q)q^x}{q^x-1} - 1], & 0 < q < 1, \\ \psi(x) + \ln(e^{1/x} - 1), & q = 1, \end{cases} \\ \Theta_q(x) &= \begin{cases} \psi_q(x) - (\ln q)x + \ln(\exp \frac{\ln q}{q^x-1} - 1), & 0 < q < 1, \\ \psi(x) + \ln(e^{1/x} - 1), & q = 1, \end{cases} \end{aligned}$$

are strictly increasing on $(0, \infty)$; consequently,

1. the double inequality

$$a - \ln(e^{1/x} - 1) < \psi(x) < b - \ln(e^{1/x} - 1)$$

holds on $(0, \infty)$ if and only if $a \leq -\gamma = -0.577\dots$ and $b \geq 0$;

2. for $0 < q < 1$, the double inequality

$$\begin{aligned} &\psi_q(1) + (\ln q)x - \ln \left[\exp \frac{(\ln q)q^x}{q^x - 1} - 1 \right] \\ &< \psi_q(x) < \ln \frac{\ln q}{q - 1} + (\ln q)x - \ln \left[\exp \frac{(\ln q)q^x}{q^x - 1} - 1 \right] \end{aligned}$$

holds on $(0, \infty)$, where the constants $\psi_q(1)$ and $\ln(\frac{\ln q}{q-1})$ are the best possible;

3. for $q > 1$, the double inequality

$$\begin{aligned} &\psi_q(1) - \ln q - \ln \left(\exp \frac{\ln q}{q^x - 1} - 1 \right) \\ &< \psi_q(x) < \ln \frac{\ln q}{q - 1} - \frac{\ln q}{2} - \ln \left(\exp \frac{\ln q}{q^x - 1} - 1 \right) \end{aligned}$$

holds on $(0, \infty)$, where the scalars $\psi_q(1) - \ln q$ and $\ln \frac{\ln q}{q-1} - \frac{\ln q}{2}$ are the best possible.

For properties of the function $\psi(x) + \ln(e^{1/x} - 1)$ and related functions, please refer to [4, 13, 14, 28, 53] and the closely related references therein.

6.5 Double inequalities for the gamma function and its q -analog

The inequality (2.6) was applied in [12, Theorem 2.1] to provide a double inequality,

$$\Gamma(x^*) \exp\{\alpha[e^{\psi(x)}(\psi(x) - 1) + 1]\} \leq \Gamma(x) \leq \Gamma(x^*) \exp\{\beta[e^{\psi(x)}(\psi(x) - 1) + 1]\} \quad (6.2)$$

for $x > x^*$, where $\alpha = 1$, $\beta = \frac{6e^\gamma}{\pi^2} = 1.0827\dots$, and $x^* = 1.4616\dots$ is the only positive zero of $\psi(x)$ such that $\Gamma(x^*) = 0.8856\dots$

In [7, Theorem 4.3], the inequality (6.2) was proved to be valid for $x \in (a, b)$ and with the best possible constants

$$\alpha = \begin{cases} Q(b), & \text{if } b < \infty, \\ 1, & \text{if } b = \infty, \end{cases} \quad \text{and} \quad \beta = Q(a),$$

where $0 < a < b \leq \infty$ and

$$Q(x) = \begin{cases} \frac{\ln \Gamma(x) - \ln \Gamma(x^*)}{[\psi(x) - 1]e^{\psi(x)} + 1}, & x \neq x^*; \\ \frac{1}{\psi'(x^*)}, & x = x^*. \end{cases}$$

In [7, Theorem 4.8], by establishing and utilizing the inequality (3.17), the inequality (6.2) was generalized to the q -analog

$$\Gamma_q(x_0) e^{\alpha[e^{\psi_q(x)}(\psi_q(x) - 1) + 1]} \leq \Gamma_q(x) \leq \Gamma_q(x_0) e^{\beta[e^{\psi_q(x)}(\psi_q(x) - 1) + 1]}$$

for $x \in (a, b)$ and with the best possible constants

$$\alpha = \begin{cases} Q_q(b), & \text{if } b < \infty, \\ 0, & \text{if } b = \infty, \end{cases} \quad \text{and} \quad \beta = Q_q(a),$$

where $q > 1$, $0 < a < b \leq \infty$, x_0 is the only zero of $\psi_q(x)$ on $(0, \infty)$, and

$$Q_q(x) = \begin{cases} \frac{\ln \Gamma_q(x) - \ln \Gamma_q(x_0)}{[\psi_q(x) - 1]e^{\psi_q(x)} + 1}, & x \neq x_0; \\ \frac{1}{\psi'_q(x_0)}, & x = x_0. \end{cases}$$

6.6 A monotonicity result

In [30, Theorem 2], complete monotonicity of the function $\Delta_{s,t}(x)$ defined in (2.2) was applied to generalize the inequality (6.2) to a monotonicity result below. For real numbers s and t , $\alpha = \min\{s, t\}$, and $c \in (-\alpha, \infty)$, let

$$g_{s,t}(x) = \begin{cases} \frac{1}{t-s} \int_c^x \ln \left[\frac{\Gamma(u+t)}{\Gamma(u+s)} \frac{\Gamma(c+s)}{\Gamma(c+t)} \right] du, & s \neq t, \\ \int_c^x [\psi(u+s) - \psi(c+s)] du, & s = t \end{cases}$$

on $x \in (-\alpha, \infty)$. Then the function

$$f_{s,t}(x) = \begin{cases} \frac{g_{s,t}(x)}{[g'_{s,t}(x) - 1] \exp[g'_{s,t}(x)] + 1}, & x \neq c, \\ \frac{1}{g''_{s,t}(c)}, & x = c, \end{cases}$$

on $x \in (-\alpha, \infty)$ is decreasing for $|t - s| < 1$ and increasing for $|t - s| > 1$.

6.7 Monotonicity and convexity

In [89, p. 75, Corollary 2], complete monotonicity of $[\psi'(x)]^2 + \lambda\psi''(x)$ was utilized to discuss the monotonicity and convexity of the function

$$f_{p,q}(x) = e^{p\psi(x+1)} - qx,$$

which is a generalization of the function in (6.1), with respect to $x \in (-1, \infty)$ for $p \neq 0$ and $q \in \mathbb{R}$: the function $f_{p,q}(x)$

1. is strictly decreasing if and only if $p = 1$ and $q \geq 1$ or $p < 0$ and $q \geq 0$;
2. is strictly increasing if and only if $p = 1$ and $q \leq 0$ or $p > 1$ and $q \leq 0$;
3. is strictly convex if and only if $p \geq 1$ or $p < 0$;
4. has a unique minimum if and only if $p \geq 1$ and $0 < q < 1$ or $p < 0$ and $q < 0$.

These results extend and generalize [23, pp. 241–242, Corollary 2].

6.8 Logarithmically complete monotonicity related to the ratio of gamma functions

In [88, p. 1980, Theorem 1.4], complete monotonicity of the function $\delta_{s,t}(x)$ defined by (5.5) was applied to derive the following logarithmically complete monotonicity of a function involving the ratio of two gamma functions: for $s, t \in \mathbb{R}$ and $\alpha = \min\{s, t\}$,

1. when $|t - s| > 1$, the function

$$H_{s,t}(x) = \begin{cases} \frac{(x+t)^{1/2(t-s)-1/2}}{(x+s)^{1/2(t-s)+1/2}} \left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)}, & s \neq t, \\ \frac{1}{x+t} \exp\left[\psi(x+t) + \frac{1}{2(x+t)}\right], & s = t, \end{cases}$$

is logarithmically completely monotonic on $(-\alpha, \infty)$;

2. when $|t - s| < 1$, the function $\frac{1}{H_{s,t}(x)}$ is logarithmically completely monotonic on $(-\alpha, \infty)$;
3. when $|t - s| = 1$, the function $H_{s,t}(x)$ identically equals 1 on $(-\alpha, \infty)$;
4. when $|t - s| < 1$, the inequality

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} < \frac{(x+s)^{1/2(t-s)+1/2}}{(x+t)^{1/2(t-s)-1/2}} \tag{6.3}$$

holds on $(-\alpha, \infty)$;

5. when $|t - s| > 1$, the inequality (6.3) is reversed on $(-\alpha, \infty)$.

6.9 Logarithmically complete monotonicity involving the ratio of gamma functions

By applying complete monotonicity of the function $\theta_{s,t;\lambda}(x)$ defined in (5.6), the following results were discovered in [91, Theorem 1.2]. For $s, t, \lambda \in \mathbb{R}$ and $\alpha = \min\{s, t\}$, let

$$\mathcal{H}_{s,t;\lambda}(x) = \begin{cases} \frac{(x+t)^{[1/(t-s)-\lambda]/2}}{(x+s)^{[1/(t-s)+\lambda]/2}} \left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)}, & s \neq t, \\ \frac{1}{(x+s)^\lambda} \exp\left[\psi(x+s) + \frac{1}{2(x+s)}\right], & s = t, \end{cases} \tag{6.4}$$

on $(-\alpha, \infty)$. Then

1. when $0 < |t - s| < 1$,
 - (a) the function $\mathcal{H}_{s,t;\lambda}(x)$ is logarithmically completely monotonic if and only if $\lambda \geq \frac{1}{|t-s|}$;
 - (b) the function $\frac{1}{\mathcal{H}_{s,t;\lambda}(x)}$ is logarithmically completely monotonic if and only if $\lambda \leq 1$;
2. when $|t - s| > 1$,
 - (a) the function $\mathcal{H}_{s,t;\lambda}(x)$ is logarithmically completely monotonic if and only if $\lambda \geq 1$;
 - (b) the function $\frac{1}{\mathcal{H}_{s,t;\lambda}(x)}$ is logarithmically completely monotonic if and only if $\lambda \leq \frac{1}{|t-s|}$;
3. when $s = t$, the function $\frac{1}{\mathcal{H}_{s,s;\lambda}(x)}$ is logarithmically completely monotonic if and only if $\lambda \leq 1$;
4. when $|t - s| = 1$,
 - (a) the function $\mathcal{H}_{s,t;\lambda}(x)$ is logarithmically completely monotonic if and only if $\lambda > 1$;
 - (b) the function $\frac{1}{\mathcal{H}_{s,s;\lambda}(x)}$ is logarithmically completely monotonic if and only if $\lambda < 1$;
 - (c) and $\mathcal{H}_{s,t;1}(x) \equiv 1$.

6.10 Logarithmically complete monotonicity concerning the ratio of gamma functions

By applying complete monotonicity of the function $\delta_{s,t;\lambda}(x)$ defined in (5.7), the following logarithmically complete monotonicity were obtained in [83, Theorem 1.2] and [108, Theorem 1.3]. For $s, t, \lambda \in \mathbb{R}$ and $\alpha = \min\{s, t\}$, define

$$H_{s,t;\lambda}(x) = \begin{cases} \frac{(x+t)^{\lambda/(t-s)-1/2}}{(x+s)^{\lambda/(t-s)+1/2}} \left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)}, & s \neq t, \\ \frac{1}{x+t} \exp\left[\psi(x+t) + \frac{\lambda}{x+t}\right], & s = t, \end{cases} \tag{6.5}$$

on $(-\alpha, \infty)$. Then

1. when $|t - s| < 1$,
 - (a) the function $H_{s,t;\lambda}(x)$ is logarithmically completely monotonic if and only if $\lambda \geq 1 - \frac{|t-s|}{2}$;
 - (b) the function $\frac{1}{H_{s,t;\lambda}(x)}$ is logarithmically completely monotonic if and only if $\lambda \leq \frac{1}{2}$;
2. when $|t - s| > 1$,
 - (a) the function $H_{s,t;\lambda}(x)$ is logarithmically completely monotonic if and only if $\lambda \geq \frac{1}{2}$;
 - (b) the function $\frac{1}{H_{s,t;\lambda}(x)}$ is logarithmically completely monotonic if and only if $\lambda \leq 1 - \frac{|t-s|}{2}$;
3. when $|t - s| = 1$,
 - (a) the function $H_{s,t;\lambda}(x)$ is logarithmically completely monotonic if and only if $\lambda > \frac{1}{2}$;
 - (b) the function $\frac{1}{H_{s,t;\lambda}(x)}$ is logarithmically completely monotonic if and only if $\lambda < \frac{1}{2}$;
 - (c) the function $H_{s,t;\lambda}(x)$ identically equals 1 on $(-\alpha, \infty)$ if and only if $\lambda = \frac{1}{2}$.

7 Equivalent inequalities of complete monotonicity

It is very surprising that some (logarithmically) complete monotonicity cases discussed above are equivalent to some inequalities for sums.

7.1 First equivalence

Theorem 1.2 in [62, p. 526] shows that, for $k \geq 0$ and $\theta > 0$,

1. if $a > 0$ and $b > 0$, then

$$\sum_{i=0}^k \frac{1}{(a + \theta)^{i+1}(b + \theta)^{k-i+1}} + \sum_{i=0}^k \frac{1}{a^{i+1}b^{k-i+1}} > 2 \sum_{i=0}^k \frac{1}{(a + \theta)^{i+1}b^{k-i+1}} \tag{7.1}$$

holds for $b - a > -\theta$ and reverses for $b - a < -\theta$;

2. if $a < -\theta$ and $b < -\theta$, then inequalities

$$\begin{aligned} &\sum_{i=0}^{2k} \frac{1}{(a + \theta)^{i+1}(b + \theta)^{2k-i+1}} + \sum_{i=0}^{2k} \frac{1}{a^{i+1}b^{2k-i+1}} \\ &> 2 \sum_{i=0}^{2k} \frac{1}{(a + \theta)^{i+1}b^{2k-i+1}} \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} &\sum_{i=0}^{2k+1} \frac{1}{(a + \theta)^{i+1}(b + \theta)^{2k-i+2}} + \sum_{i=0}^{2k+1} \frac{1}{a^{i+1}b^{2k-i+2}} \\ &< 2 \sum_{i=0}^{2k+1} \frac{1}{(a + \theta)^{i+1}b^{2k-i+2}} \end{aligned} \tag{7.3}$$

hold for $b - a > -\theta$ and reverse for $b - a < -\theta$;

- 3. if $-\theta < a < 0$ and $-\theta < b < 0$, then inequality (7.2) holds and inequality (7.3) is valid for $a + b + \theta > 0$ and is reversed for $a + b + \theta < 0$;
- 4. if $a < -\theta$ and $b > 0$, then inequality (7.2) holds and inequality (7.3) is valid for $a + b + \theta > 0$ and is reversed for $a + b + \theta < 0$;
- 5. if $a > 0$ and $b < -\theta$, then inequality (7.2) is reversed and inequality (7.3) holds for $a + b + \theta < 0$ and reverses for $a + b + \theta > 0$;
- 6. if $b = a - \theta$, then inequalities (7.1), (7.2) and (7.3) become equalities.

Theorem 1.3 in [62, p. 526] states that the inequality (7.1) for $a, b > 0$ is equivalent to [62, p. 524, Theorem 1.1] and [88, p. 1978, Theorem 1.1] mentioned in Sect. 5.7 in this paper.

7.2 Second equivalence

The (logarithmically) complete monotonicity of the functions $\delta_{s,t;\lambda}(x)$ and $H_{s,t;\lambda}(x)$ are equivalent to the following inequality (7.4) for the case $a, b > 0$. See [108, Remarks 1.8 and 1.9].

Theorem 1.7 in [108], which is a generalization of [62, Theorem 1.2] mentioned above, states that, for $k \in \mathbb{N}$, $\theta > 0$, and $a, b, \lambda \in \mathbb{R}$,

1. the inequality

$$\begin{aligned} \sum_{i=0}^k \frac{1}{(a + \theta)^{i+1}b^{k-i+1}} &< \frac{2\theta(1 - \lambda) - b + a}{2(\theta - b + a)} \sum_{i=0}^k \frac{1}{a^{i+1}b^{k-i+1}} \\ &+ \frac{2\theta\lambda - b + a}{2(\theta - b + a)} \sum_{i=0}^k \frac{1}{(a + \theta)^{i+1}(b + \theta)^{k-i+1}} \end{aligned} \tag{7.4}$$

holds

- (a) if $a, b > 0$ and
 - (1) either $0 < |b - a| < \theta$ and $\lambda \leq \frac{1}{2}$,
 - (2) or $b - a < -\theta$ and $\lambda \leq 1 + \frac{b-a}{2\theta}$,
 - (3) or $b - a > \theta$ and $\lambda \geq \frac{1}{2}$;
- (b) if $a, b < -\theta$,
 - (1) k is even and

$$\left\{ \begin{array}{l} \text{either } 0 < |b - a| < \theta \text{ and } \lambda \geq \frac{1}{2}, \\ \text{or } a - b < -\theta \text{ and } \lambda \geq \frac{b-a}{2\theta}, \\ \text{or } a - b > \theta \text{ and } \lambda \leq \frac{1}{2}; \end{array} \right.$$

- (2) k is odd and

$$\left\{ \begin{array}{l} \text{either } 0 < |b - a| < \theta \text{ and } \lambda \leq \frac{|b-a|}{2\theta}, \\ \text{or } a - b < -\theta \text{ and } \lambda \leq \frac{1}{2}, \\ \text{or } a - b > \theta \text{ and } \lambda \geq \frac{a-b}{2\theta}; \end{array} \right.$$

- (c) if $-\theta < a < 0, -\theta < b < 0$ and
 - (1) k is even and $0 < \lambda \leq 1$,
 - (2) k is odd,

$$\left\{ \begin{array}{l} \text{either } -2\theta < a + b < -\theta \text{ and } \lambda \geq \frac{1}{2}, \\ \text{or } a + b + \theta = 0 \text{ and } \lambda > \frac{1}{2}; \end{array} \right.$$

- (d) if either $a < -\theta$ and $b > 0$ or $a > 0$ and $b < -\theta$,
 - (1) k is even and $0 \leq \lambda \leq 1$,
 - (2) k is odd and

$$\left\{ \begin{array}{l} \text{either } a + b + \theta < 0 \text{ and } \lambda \leq \frac{1}{2}, \\ \text{or } a + b + \theta = 0 \text{ and } \lambda < \frac{1}{2}; \end{array} \right.$$

2. the inequality (7.1) is reversed

- (a) if $a > 0, b > 0$ and
 - (1) either $0 < |b - a| < \theta$ and $\lambda \geq 1 - \frac{|b-a|}{2\theta}$,
 - (2) or $b - a < -\theta$ and $\lambda \geq \frac{1}{2}$,
 - (3) or $b - a > \theta$ and $\lambda \leq 1 - \frac{b-a}{2\theta}$;
- (b) if $a < -\theta$ and $b < -\theta$,
 - (1) k is even and

$$\left\{ \begin{array}{l} \text{either } 0 < |b - a| < \theta \text{ and } \lambda \leq \frac{|b-a|}{2\theta}, \\ \text{or } a - b < -\theta \text{ and } \lambda \leq \frac{1}{2}, \\ \text{or } a - b > \theta \text{ and } \lambda \geq \frac{a-b}{2\theta}; \end{array} \right.$$

(2) k is odd and

$$\begin{cases} \text{either } 0 < |b - a| < \theta \text{ and } \lambda \geq \frac{1}{2}, \\ \text{or } a - b < -\theta \text{ and } \lambda \geq \frac{b-a}{2\theta}, \\ \text{or } a - b > \theta \text{ and } \lambda \leq \frac{1}{2}; \end{cases}$$

(c) if $-\theta < a < 0, -\theta < b < 0, k$ is odd and

(1) either $0 > a + b > -\theta$ and $\lambda \leq \frac{1}{2}$,

(2) or $a + b + \theta = 0$ and $\lambda < \frac{1}{2}$;

(d) if either $a < -\theta$ and $b > 0$ or $a > 0$ and $b < -\theta, k$ is odd and

(1) either $a + b + \theta > 0$ and $\lambda \geq \frac{1}{2}$,

(2) or $a + b + \theta = 0$ and $\lambda > \frac{1}{2}$.

3. the sign $<$ in (7.1) can be replaced by the equal sign $=$ if $a + b + \theta = 0, \lambda = \frac{1}{2}$ and

(a) either $-1 < a < 0$ and $-1 < b < 0$,

(b) or $a < -1$ and $b > 0$,

(c) or $a > 0$ and $b < -1$.

8 Related inequalities

Making use of (logarithmically) complete monotonicity of some functions discussed above, we can derive some inequalities for the ratio of two gamma functions and for the divided differences of polygamma functions.

8.1 First group of inequalities

From the logarithmically complete monotonicity of the function $\mathcal{H}_{s,t;\lambda}(x)$ defined in (6.4), the following inequalities were derived in [91, Theorem 1.4].

For $a, b > 0$, the inequality

$$\left[\frac{\Gamma(b)}{\Gamma(a)} \right]^{1/(b-a)} < \sqrt{ab} \left(\frac{a}{b} \right)^{1/2(b-a)}$$

holds for $0 < |b - a| < 1$ and reverses for $|b - a| > 1$. For $0 < |b - a| < 1$, the double inequality

$$\begin{aligned} & \frac{(k-1)!}{2} \left[\left(\frac{1}{b-a} + \beta \right) \frac{1}{a^k} + \left(\beta - \frac{1}{b-a} \right) \frac{1}{b^k} \right] \\ & < \frac{(-1)^{k-1} [\psi^{(k-1)}(b) - \psi^{(k-1)}(a)]}{b-a} \\ & < \frac{(k-1)!}{2} \left[\left(\frac{1}{b-a} + \gamma \right) \frac{1}{a^k} + \left(\gamma - \frac{1}{b-a} \right) \frac{1}{b^k} \right] \end{aligned} \tag{8.1}$$

holds if and only if $\beta \leq 1$ and $\gamma \geq \frac{1}{|b-a|}$. For $|b - a| > 1$, inequalities in (8.1) are valid if and only if $\beta \leq \frac{1}{|b-a|}$ and $\gamma \geq 1$.

8.2 Second group of inequalities

From the logarithmically complete monotonicity of the function $H_{s,t;\lambda}(x)$ defined in (6.5), the following double inequalities were procured in [83, Theorem 1.3] and [108, Theorem 1.5].

For $a, b > 0$,

- when $0 < |b - a| < 1$, the double inequality

$$\beta_2 \frac{(k-1)!}{b-a} \left(\frac{1}{a^k} - \frac{1}{b^k} \right) < \frac{(-1)^{k-1} [\psi^{(k-1)}(b) - \psi^{(k-1)}(a)]}{b-a} - \frac{(k-1)!}{2} \left(\frac{1}{a^k} + \frac{1}{b^k} \right) < \gamma_2 \frac{(k-1)!}{b-a} \left(\frac{1}{a^k} - \frac{1}{b^k} \right) \tag{8.2}$$

holds if and only if $\beta_2 \leq \frac{1}{2}$ and $\gamma_2 \geq 1 - \frac{|b-a|}{2}$;

- when $|b - a| > 1$, the double inequality (8.2) is valid if and only if $\beta_2 \leq 1 - \frac{|b-a|}{2}$ and $\gamma_2 \geq \frac{1}{2}$.
- when $0 < |b - a| < 1$, the double inequality

$$\sqrt{ab} \left(\frac{a}{b} \right)^{\beta_3/(b-a)} < \left[\frac{\Gamma(b)}{\Gamma(a)} \right]^{1/(b-a)} < \sqrt{ab} \left(\frac{a}{b} \right)^{\gamma_3/(b-a)} \tag{8.3}$$

holds if and only if $\beta_3 \geq 1 - \frac{|b-a|}{2}$ and $\gamma_3 \leq \frac{1}{2}$;

- when $|b - a| > 1$, the double inequality (8.3) is reversed if and only if $\beta_3 \leq 1 - \frac{|b-a|}{2}$ and $\gamma_3 \geq \frac{1}{2}$;
- the double inequality

$$\beta_1 \frac{k!}{x^{k+1}} < (-1)^{k+1} \psi^{(k)}(x) - \frac{(k-1)!}{x^k} < \gamma_1 \frac{k!}{x^{k+1}}$$

holds on $(0, \infty)$ if and only if $\beta_1 \leq \frac{1}{2}$ and $\gamma_1 \geq 1$.

The inequality (8.3) can be rearranged in the following beautiful form:

$$\frac{1}{\gamma_3} \frac{\ln b - \ln a}{b-a} < \ln \sqrt{ab} - \frac{\ln \Gamma(b) - \ln \Gamma(a)}{b-a} < \frac{1}{\beta_3} \frac{\ln b - \ln a}{b-a}.$$

9 Some new properties for the ratio of two gamma functions

In [64, 69, 101, 106, 107, 112] and the closely related references therein, some kinds of functions involving the ratio of two gamma functions were surveyed. In what follows, we summarize several new kinds of functions involving the ratio of two gamma functions, including some relating to the Catalan numbers in combinatorics and the Bernoulli trials in probability. For information as regards the Catalan numbers, please refer to the survey articles [93, 96] and the closely related references therein.

9.1 An exponential integral representation and logarithmically complete monotonicity

Let

$$F_{a,b}(x) = \frac{\Gamma(x+a)(x+b)^{x+b-a}}{\Gamma(x+b)(x+a)^x}, \quad a, b \in \mathbb{R}, a \neq b, x > -\min\{a, b\}.$$

In [54, Theorem 2] and [115, Theorem 3], it was established that, when $a, b > 0$,

1. the function $F_{a,b}(x)$ has the exponential representation

$$F_{a,b}(x) = \exp \left[b - a + \int_0^\infty \frac{1}{t} \left(a + \frac{1}{t} - \frac{1}{1 - e^{-t}} \right) (e^{-bt} - e^{-at}) e^{-xt} dt \right]$$

on $[0, \infty)$;

2. the function $[F_{a,b}(x)]^{\pm 1}$ is logarithmically completely monotonic on $[0, \infty)$ if and only if

$$(a, b) \in D_{\pm}(a, b) = \left\{ (a, b) : a \geq b, a \geq 1 \right\} \cup \left\{ (a, b) : a \leq b, a \leq \frac{1}{2} \right\}.$$

9.2 An exponential expansion

Theorem 3 in [110] reads

$$\frac{\Gamma(a)}{\Gamma(b)} = \frac{a^{a-1/2}}{b^{b-1/2}} \exp \left[\sum_{j=1}^\infty \frac{B_{2j}}{2j(2j-1)} \left(\frac{1}{a^{2j-1}} - \frac{1}{b^{2j-1}} \right) \right], \quad a, b > 0.$$

9.3 A double inequality

Theorem 11 in [110] states that

$$\begin{aligned} & \sqrt{\frac{b}{a}} [I(a, b)]^{a-b} \exp \left[\sum_{j=1}^{2m} \frac{B_{2j}}{2j(2j-1)} \left(\frac{1}{a^{2j-1}} - \frac{1}{b^{2j-1}} \right) \right] \\ & < \frac{\Gamma(a)}{\Gamma(b)} < \sqrt{\frac{b}{a}} [I(a, b)]^{a-b} \exp \left[\sum_{j=1}^{2m-1} \frac{B_{2j}}{2j(2j-1)} \left(\frac{1}{a^{2j-1}} - \frac{1}{b^{2j-1}} \right) \right], \quad a, b > 0, \end{aligned}$$

where B_i for $i \in \mathbb{N}$ are the Bernoulli numbers defined [80, 81, 84] by

$$\frac{x}{e^x - 1} = \sum_{i=0}^\infty B_i \frac{x^i}{i!} = 1 - \frac{x}{2} + \sum_{j=1}^\infty B_{2j} \frac{x^{2j}}{(2j)!}, \quad |x| < 2\pi$$

and $I(\alpha, \beta)$ is the exponential mean defined for $\alpha, \beta > 0$ by

$$I(\alpha, \beta) = \begin{cases} \frac{1}{e} \left(\frac{\beta^\beta}{\alpha^\alpha} \right)^{1/(\beta-\alpha)}, & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

9.4 Logarithmically complete monotonicity and applications in probability

Let $m \in \mathbb{N}$, $\mathbf{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \leq i \leq m$, and $\mathbf{p} = (p_1, p_2, \dots, p_m)$ with $p_i \in (0, 1)$ for $1 \leq i \leq m$ and $\sum_{i=1}^m p_i = 1$. In [57, 113], it was proved that the function

$$Q_{\mathbf{a}, \mathbf{p}; m}(x) = \frac{\Gamma(1 + x \sum_{i=1}^m a_i)}{\prod_{i=1}^m \Gamma(1 + x a_i)} \prod_{i=1}^m p_i^{x a_i} \tag{9.1}$$

is logarithmically completely monotonic on $(0, \infty)$. This is a generalization of a problem originating from the Bernoulli trials in probability (see [6, 52]). This logarithmically complete monotonicity can be used to derive some inequalities for multinomial coefficients and multivariate beta functions (see [6, 113]).

In [79], the q -analog of the function (9.1) was considered and its logarithmically complete monotonicity was investigated.

At the deep night on 6 May 2018, Dr. Frédéric Ouimet (Université de Montréal, Canada) acknowledged that, in his arXiv preprint [57], which was formally published as [58] later, he alternatively and essentially proved [113, Theorem 2.2] and applied it to deriving asymptotic formulas for quantities of interest in the context of a statistical density estimation based on the Bernstein polynomials on the n -dimensional simplex.

9.5 Completely monotonic degrees of functions involving ratios of gamma functions

Recall from [125] that a function f is said to be strongly completely monotonic on $(0, \infty)$ if it has derivatives of all orders and $(-1)^n x^{n+1} f^{(n)}(x)$ is nonnegative and decreasing on $(0, \infty)$ for all $n \geq 0$.

Let

$$L_{s,t}(x) = x - \frac{\Gamma(x+t)}{\Gamma(x+s)} x^{s-t+1} = x \left[1 - \frac{\Gamma(x+t)/x^t}{\Gamma(x+s)/x^s} \right]$$

and suppose that either

1. $s \geq 1 > t$ and $0 < s - t \leq \frac{1}{2}$, or
2. $t \geq 1$ and $0 < s - t < 1$, or
3. $\frac{\sqrt{3}}{6} + \frac{1}{2} \leq t < s < 1$.

Theorem 1 in [47] reads

1. the function

$$\Phi_{s,t}(x) = \frac{L''_{s,t}(x)}{L_{s,t}(x) - x} = -\frac{\Gamma(x+s)}{\Gamma(x+t)} x^{t-s-1} L''_{s,t}(x)$$

is strongly completely monotonic on $(0, \infty)$;

2. the function $L_{s,t}(x)$ is strictly increasing and concave on $(0, \infty)$ and the function $-L''(x)$ is completely monotonic on $(0, \infty)$.

Proposition 1.1 in [50, p. 34], which was proved in [65, Sect. 4.2], states that a function $f(x)$ is strongly completely monotonic if and only if the function $xf(x)$ is completely monotonic. In other words, the set of functions of completely monotonic degree not less than 1 with respect to $x \in (0, \infty)$ coincides with the set of strongly completely monotonic functions on $(0, \infty)$. This implies that $\text{deg}_{\text{cm}}^x[\Phi(x)] \geq 1$.

Corollary 3 in [47] can be restated as follows: the functions

$$\Delta(x), \quad \frac{2}{3} \left[\psi'(x) + \frac{1}{2x} \right]^2 + \psi''(x) - \frac{1}{2x^2}, \quad \text{and} \quad -\psi''(x) - \frac{2}{x} \psi'(x) + \frac{1}{x^2}$$

are of completely monotonic degree not less than 2. In [48, Theorem 1.3], the first two functions above were proved to be of completely monotonic degree at least 3, but the third function is of completely monotonic degree less than 3.

In [48, 49], the functions $L_{s,t}(x)$ and $\Phi_{s,t}(x)$ were further investigated and more conclusions were discovered therein.

From the identity (3.16), we surely conclude that main results and their proofs in [47–49] can be improved. We also believe that there should be a simple method or a nice approach

to find some properties of the functions $L_{s,t}(x)$ and $\Phi_{s,t}(x)$. For this purpose, the integral representation

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} = \frac{1}{\Gamma(b-a)} \int_0^\infty (1-e^{-t})^{b-a-1} e^{-(x+a)t} dt$$

for $\Re(b-a) > 0$ and $\Re(x+a) > 0$, which can be found in [10, p. 165] and [124, p. 67], would play a key role.

9.6 Necessary and sufficient conditions

In [129], Theorems 3.1 and 3.2 show that

1. the function

$$v(s, t, r; x) = \begin{cases} e^{-\psi(x+r)} \left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)}, & t \neq s, \\ e^{\psi(x+s) - \psi(x+r)}, & t = s \end{cases}$$

is logarithmically completely monotonic with respect to $x \in (-\rho, \infty)$ if and only if $r \leq \min\{s, t\}$ and its reciprocal $\frac{1}{v(s,t,r;x)}$ is logarithmically completely monotonic with respect to $x \in (-\rho, \infty)$ if and only if $r \geq \frac{s+t}{2}$, where $s, t, r \in \mathbb{R}$ and $\rho = \min\{s, t, r\}$; see also [131, Corollary 3];

2. the function

$$U(a, b, c; x) = \begin{cases} \frac{1}{x+c} \left[\frac{\Gamma(x+a)}{\Gamma(x+b)} \right]^{1/(a-b)}, & a \neq b, \\ \frac{1}{x+c} e^{\psi(x+a)}, & a = b, \end{cases}$$

is logarithmically completely monotonic with respect to $x \in (-\rho, \infty)$ if and only if $c \leq \frac{a+b-\max\{|a-b|, 1\}}{2}$ and its reciprocal $\frac{1}{U(a,b,c;x)}$ is logarithmically completely monotonic with respect to $x \in (-\rho, \infty)$ if and only if $c \geq \frac{a+b-\min\{|a-b|, 1\}}{2}$, where $a, b, c \in \mathbb{R}$ and $\rho = \min\{a, b, c\}$.

9.7 Five classes of functions involving ratios of gamma functions

In [133], the logarithmically complete monotonicity of the functions

$$\frac{W_{r,s}(x)}{W_{u,v}(x)}, \quad \frac{W_{\bar{r},\bar{s}}(x)}{\prod_{i=1}^n W_{r_i,s_i}(x)}, \quad \frac{\prod_{i=1}^n \Gamma^{\lambda_i}(x+s_i)}{\Gamma(x+s)},$$

$$\prod_{i=1}^n \frac{\Gamma(x+u_i)}{\Gamma(x+r_i)}, \quad \prod_{i=1}^n \frac{W_{r_i,s}(x)}{W_{u_i,s}(x)}$$

was investigated, where $W_{u,v}(x)$ is defined in (2.1). Concretely speaking,

1. Theorem 3.1 in [133] states that, for $u, v, r, s \in \mathbb{R}$ and $\rho = \min\{u, v, r, s\}$, the function $\frac{W_{r,s}(x)}{W_{u,v}(x)}$ is logarithmically completely monotonic on $(-\rho, \infty)$ if and only if $u + v \leq r + s$ and $\min\{u, v\} \leq \min\{r, s\}$; see also [131, Proposition 1] and [132, Corollary 3];
2. Theorem 3.7 in [133] states that, if $r, s, r_i, s_i \in \mathbb{R}$ and $\lambda_i \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$, then
 - (a) the function $\frac{W_{\bar{r},\bar{s}}(x)}{\prod_{i=1}^n W_{r_i,s_i}(x)}$ is logarithmically completely monotonic on $(-\rho_1, \infty)$, where $\rho_1 = \min\{r_i, s_i : 1 \leq i \leq n\}$ and $(\bar{r}, \bar{s}) = (\sum_{i=1}^n \lambda_i r_i, \sum_{i=1}^n \lambda_i s_i)$;

- (b) the function $\frac{W_{r,s}(x)}{\prod_{i=1}^n W_{r,s_i}^{\lambda_i}(x)}$ is logarithmically completely monotonic on $(-\rho_2, \infty)$ if and only if $s \geq \bar{s}$;
- (c) the function $\frac{\prod_{i=1}^n W_{r,s_i}^{\lambda_i}(x)}{W_{r,s}(x)}$ is logarithmically completely monotonic on $(-\rho_2, \infty)$ if and only if $s \leq \min\{s_i : 1 \leq i \leq n\}$, where $\rho_2 = \min\{r, s, s_i : 1 \leq i \leq n\}$;
- 3. Theorems 4.2 and 4.3 in [133] show that, if $\lambda_i > 0$ such that $\sum_{i=0}^n \lambda_i = 1$, then
 - (a) the function $\frac{\prod_{i=1}^n \Gamma^{\lambda_i}(x+s_i)}{\Gamma(x+s)}$ is greater than 1 and logarithmically completely monotonic on $(-\min\{s, s_i : 1 \leq i \leq n\}, \infty)$ if and only if $s = \bar{s}$;
 - (b) the function $\frac{\Gamma(x+s)}{\prod_{i=1}^n \Gamma^{\lambda_i}(x+s_i)}$ is logarithmically completely monotonic on $(-\min\{s, s_i : 1 \leq i \leq n\}, \infty)$ if and only if $s \leq \min\{s_i : 1 \leq i \leq n\}$;
- 4. Theorems 4.7, 4.9, and 4.12 in [133] show that, if $u_1 \leq u_2 \leq \dots \leq u_n$ and $r_1 \leq r_2 \leq \dots \leq r_n$ such that $\sum_{i=1}^k u_i \leq \sum_{i=1}^k r_i$ for $k = 2, \dots, n-1$, then
 - (a) the function $\prod_{i=1}^n \frac{\Gamma(x+u_i)}{\Gamma(x+r_i)}$ is logarithmically completely monotonic on $(-\min\{u_1, r_1\}, \infty)$ if and only if $u_1 \leq r_1$ and $\sum_{i=1}^n u_i \leq \sum_{i=1}^n r_i$;
 - (b) the function $\ln \prod_{i=1}^n \frac{\Gamma(x+u_i)}{\Gamma(x+r_i)}$ is completely monotonic on the interval $(-\min\{u_1, r_1\}, \infty)$ if and only if $u_1 \leq r_1$ and $\sum_{i=1}^n u_i = \sum_{i=1}^n r_i$;
 - (c) the function $\ln \prod_{i=1}^n \frac{W_{r_i,s}(x)}{W_{u_i,s}(x)}$ is completely monotonic on the interval $(-\min\{u_1, r_1, s\}, \infty)$ if and only if $\sum_{i=1}^n u_i \leq \sum_{i=1}^n r_i$.

9.8 Complete monotonicity of a multivariate function involving ratios of the gamma functions

It is well known [16, Theorem 4.2.2] that a function $f(x_1, x_2, \dots, x_k)$ is called completely monotonic in a cone $X = X_1 \times X_2 \times \dots \times X_k$ if it is C^∞ and

$$(-1)^{n_1+n_2+\dots+n_k} \frac{\partial^{n_1+n_2+\dots+n_k} f(x_1, x_2, \dots, x_k)}{\partial x_1^{n_1} \partial x_2^{n_2} \dots \partial x_k^{n_k}} \geq 0$$

for all $n_1, n_2, \dots, n_k \geq 0$. See also [55, Chapter XIII, Sect. 6] and the closely related references therein. For $x, y > -\min\{p, q\}$, let

$$G(p, q; x, y) = \begin{cases} \left[\frac{\Gamma(x+p)\Gamma(y+q)}{\Gamma(y+p)\Gamma(x+q)} \right]^{1/(p-q)(x-y)}, & (p-q)(x-y) \neq 0; \\ \exp\left[\frac{\psi(x+p) - \psi(y+p)}{x-y} \right], & p = q, x \neq y; \\ \exp\left[\frac{\psi(x+p) - \psi(x+q)}{p-q} \right], & p \neq q, x = y; \\ e^{\psi'(x+p)}, & p = q, x = y. \end{cases}$$

Then

1. Theorem 1 in [132] states that the function $\ln G(p, q; x, y)$ is completely monotonic on $\Omega = \{(p, q, x, y) : x, y > -\min\{p, q\}, p, q, x, y \in \mathbb{R}\}$;
2. Theorem 2 in [132] states that, for fixed $p, q, r, s \in \mathbb{R}$ and $\rho = \min\{p, q, r, s\}$, the function $(x, y) \mapsto \ln \frac{G(p,q;x,y)}{G(r,s;x,y)}$ is completely monotonic on $(-\rho, \infty)^2$ if and only if $p + q \leq r + s$ and $\min\{p, q\} \leq \min\{r, s\}$.

10 Difference between trigamma and exponential functions

In [33, Lemma 2], the inequality

$$\psi'(x) < e^{1/x} - 1, \quad x > 0, \tag{10.1}$$

was obtained and applied. Hereafter, there have been some developments on this topic.

In [82, Theorem 3.1], [116, Theorem 1.1], and [118], among other things, by different and elementary approaches, the inequality (10.1) was generalized to complete monotonicity, which shows that the difference

$$h(x) = e^{1/x} - \psi'(x)$$

is completely monotonic on $(0, \infty)$.

For $\alpha, \beta > 0$, let

$$h_{\alpha,\beta}(x) = \alpha e^{\beta/x} - \psi'(x)$$

on $(0, \infty)$. Theorem 4.1 in [82] reads

1. the function $h_{1,\beta}(x)$ is completely monotonic on $(0, \infty)$ if and only if $\beta \geq 1$;
2. if $\beta \geq 1$ and $\alpha\beta \geq 1$, the function $h_{\alpha,\beta}(x)$ is completely monotonic on $(0, \infty)$;
3. a necessary condition for the function $h_{\alpha,\beta}(x)$ to be completely monotonic on $(0, \infty)$ is $\alpha\beta \geq 1$;
4. if $0 < \beta < 1$, the condition

$$\alpha\beta \geq \max_{u \in (0, \infty)} F_\beta(u) > 1$$

is necessary and sufficient for $h_{\alpha,\beta}(x)$ to be completely monotonic on $(0, \infty)$, where

$$\lim_{u \rightarrow 0^+} F_\beta(u) = 1 \quad \text{and} \quad \lim_{u \rightarrow \infty} F_\beta(u) = 0$$

for all $\beta > 0$.

For $\alpha, \beta > 0$, let

$$H_{\alpha,\beta}(x) = h_{\alpha,\beta}(x) - \alpha$$

on $(0, \infty)$. Theorem 1.3 in [75] states that

1. if $(\alpha, \beta) = (1, 1)$, then

$$\text{deg}_{\text{cm}}^x [H_{1,1}(x)] = 4;$$

2. if $\beta > 1$, then

$$\text{deg}_{\text{cm}}^x [H_{1/\beta,\beta}(x)] = 2;$$

3. if $\alpha\beta > 1$ and $\beta \geq 1$, or if $\alpha\beta > 1$, $0 < \beta < 1$, and

$$\alpha\beta^2 \geq \max_{u \in (0, \infty)} \{G_\beta(u)\},$$

then

$$\text{deg}_{\text{cm}}^x [H_{\alpha,\beta}(x)] = 1.$$

In [130, Theorem 1] that the double inequality

$$\frac{1}{x^2} + \theta(x, p) < \psi'(x) < \frac{1}{x^2} + \theta(x, q), \quad x > 0,$$

was proved to be valid if and only if $p \leq 1$ and $q \geq 2$, where

$$\theta(x, m) = \frac{e^{m/(x+1)} - e^{-m/x}}{2m}.$$

This means that

$$\sinh \frac{1}{x+1} < \psi'(x+1) < \frac{1}{2} \sinh \frac{2}{x}, \quad x > 0.$$

In [111], when $x > 3$, the inequality (10.1) was refined as

$$e^{m(x)} - 1 < \psi'(x) < e^{M(x)} - 1,$$

where

$$m(x) = \frac{1}{x} - \frac{1}{24x^4} + \frac{7}{360x^6} \quad \text{and} \quad M(x) = m(x) + \frac{1}{90x^7}.$$

Are the functions

$$\psi'(x+1) - \sinh \frac{1}{x+1} \quad \text{and} \quad \frac{1}{2} \sinh \frac{2}{x} - \psi'(x+1)$$

completely monotonic on $(0, \infty)$?

By the way, we note the function $e^{\pm 1/x}$ has something to do with the existence of partitions of unity in differential geometry and with the Lah numbers in combinatorics. For more information, please refer to [17, 22, 35, 37, 44, 70, 74, 77, 78, 95, 104, 121, 134] and the closely related references therein.

11 Remarks

11.1 First remark

In [45, p. 20], Kazarinoff proved the inequality

$$[\ln \phi(t)]'' - \{[\ln \phi(t)]'\}^2 > 0, \tag{11.1}$$

where

$$\phi(t) = \int_0^{\pi/2} \sin^t x \, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma((t+1)/2)}{\Gamma((t+2)/2)}$$

for $-1 < t < \infty$. We observe that the inequality (11.1) can be rewritten as

$$\left[\frac{\psi'((t+2)/2) - \psi'((t+1)/2)}{(t+2)/2 - (t+1)/2} \right]^2 + 2 \frac{\psi'((t+2)/2) - \psi'((t+1)/2)}{(t+2)/2 - (t+1)/2} < 0 \tag{11.2}$$

for $t > -1$. Letting $u = \frac{t+1}{2}$ in (11.2) yields

$$\left[\frac{\psi(u + 1/2) - \psi(u)}{(u + 1/2) - u} \right]^2 + 2 \frac{\psi'(u + 1/2) - \psi'(u)}{(u + 1/2) - u} < 0 \tag{11.3}$$

for $u > 0$. Consequently, it is easy to see that the inequality (11.2) or (11.3) can be derived from complete monotonicity of the function $\Delta_{s,t;\lambda}(x)$ defined in (4.1).

11.2 Second remark

In the proof of [19, Theorem 1.1], by using the convolution theorem for Laplace transforms, the inequality

$$\psi'(x + b) - \psi'(x + a) + \frac{1}{b - a} [\psi(x + b) - \psi(x + a)]^2 > 0,$$

which is equivalent to

$$\left[\frac{\psi(x + b) - \psi(x + a)}{b - a} \right]^2 + \frac{\psi'(x + b) - \psi'(x + a)}{b - a} > 0,$$

for $0 \leq a < b < a + 1$ was proved. This result can be directly deduced from complete monotonicity of the functions $\Delta_{s,t}(x)$ and $\Delta_{s,t;\lambda}(x)$.

11.3 Third remark

In [5, Lemma 7], the inequality

$$\psi'(x) - \psi'(x + c) > [\psi(x + c) - \psi(x)]^2,$$

which can be rearranged as

$$\left[\frac{\psi(x + c) - \psi(x)}{(x + c) - x} \right]^2 - \frac{1}{c} \frac{\psi'(x + c) - \psi'(x)}{(x + c) - x} < 0,$$

on $(0, \infty)$ for $0 < c < 1$ and its reversed version for $c > 1$ were proved. This inequality is also a special case of complete monotonicity of the function $\Delta_{s,t;\lambda}(x)$ defined in (4.1).

11.4 Fourth remark

In [24, p. 13], the following inequalities were obtained:

$$\frac{1}{c} [\psi(x + c) - \psi(x)]^2 > \psi'(x) - \psi'(x + c) > [\psi(x + c) - \psi(x)]^2, \tag{11.4}$$

$$\begin{aligned} \frac{1 - q}{1 - q^c} [\psi_q(x + c) - \psi_q(x)]^2 &> q^x [\psi'_q(x) - \psi'_q(x + c)] \\ &> [\psi_q(x + c) - \psi_q(x)]^2, \end{aligned} \tag{11.5}$$

where $0 < c < 1, 0 < q < 1$, and $x > 0$. When $c > 1$, the inequalities in (11.4) and (11.5) reverse.

It is clear that both inequalities in (11.4) are special cases of complete monotonicity of the function $\Delta_{s,t;\lambda}(x)$.

The inequalities (11.4) and (11.5) can be reformulated as

$$-\left[\frac{\psi(x+c) - \psi(x)}{(x+c) - x} \right]^2 < \frac{\psi'(x+c) - \psi'(x)}{(x+c) - x} < -c \left[\frac{\psi(x+c) - \psi(x)}{(x+c) - x} \right]^2$$

and

$$\begin{aligned} -\frac{c}{q^x} \frac{1-q}{1-q^c} \left[\frac{\psi_q(x+c) - \psi_q(x)}{(x+c) - x} \right]^2 &< \frac{\psi'_q(x+c) - \psi'_q(x)}{(x+c) - x} \\ &< -\frac{c}{q^x} \left[\frac{\psi_q(x+c) - \psi_q(x)}{(x+c) - x} \right]^2. \end{aligned}$$

Stimulated by (11.5), we naturally ask a question: what are the q -analogs of the function $\Delta_{s,t;\lambda}(x)$? Are they completely monotonic functions?

11.5 Fifth remark

For $\alpha, \beta \in \mathbb{R}$ satisfying $\alpha \neq \beta$ and $(\alpha, \beta) \notin \{(0, 1), (1, 0)\}$, let

$$q_{\alpha,\beta}(t) = \begin{cases} \frac{e^{\beta t} - e^{\alpha t}}{e^t - 1}, & t \neq 0; \\ \beta - \alpha, & t = 0. \end{cases} \tag{11.6}$$

It is clear that $q_{\alpha,0}(t) = -\alpha h_\alpha(-t)$, or say, $h_\alpha(t) = -\frac{q_{\alpha,0}(-t)}{\alpha}$, where $h_\alpha(t)$ is defined by (3.9). This implies that some properties of the function $q_{\alpha,\beta}(t)$ can be applied in [128].

The function $q_{\alpha,\beta}(t)$ plays an important role in some investigations on functions involving the ratio of two gamma functions. For more details, please refer to [64, Sect. 1.7], [106, Sect. 4.1], the survey articles [107, 109], [26, 31, 47, 48, 60, 61, 87, 99], and the closely related references therein.

11.6 Sixth remark

For more information on the history and properties of logarithmically completely monotonic functions, please refer to [15], [27, pp. 21–23], [64, pp. 5–6, Sect. 1.5], [100, pp. 2154–2155, Remark 8], and the closely related references therein. The notion “logarithmically completely monotonic function” has been collected in the monograph [122]: see [122, Definition 5.10], [122, Theorem 5.11], [122, Proposition 5.17], [122, Proposition 5.25], [122, Comments 5.29], and [122, Remark 9.18].

12 Open problems

Finally, we would like to pose more open problems.

12.1 First problem

What are the q -analogs of $\Delta_{s,t}(x)$, $\Delta_{s,t;\lambda}(x)$, $\delta_{s,t}(x)$, $\delta_{s,t;\lambda}(x)$, $\theta_{s,t;\lambda}(x)$, and others in this paper? What about the (logarithmically) complete monotonicity of these q -analogs?

12.2 Second problem

Can one find integral representations of the form in (1.1) for the (logarithmically) completely monotonic functions collected in this paper?

12.3 Third problem

Motivated by [89, p. 75, Corollary 2] and others, we can naturally generalize the function (2.1) to

$$f_{p,q;s,t}(x) = \left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{p/(t-s)} - qx$$

on $(-\alpha, \infty)$, where $0 < |t-s| \neq 1$, $\alpha = \min\{s, t\}$, $p \neq 0$, and $q \in \mathbb{R}$. How about the monotonicity and convexity of the function $f_{p,q;s,t}(x)$?

12.4 Fourth problem

Motivated by the results in [40], we guess that the difference between the right and left hand sides of (2.5) is a completely monotonic function on $(0, \infty)$.

12.5 Fifth problem

For $m, n \in \mathbb{N}$ and $i, j \in \mathbb{N}$, let

$$f_{m,n;i,j}(x) = [\psi^{(m)}(x)]^i + [\psi^{(n)}(x)]^j \tag{12.1}$$

on $(0, \infty)$. It is clear that $f_{1,2;2,1}(x) = f_{1,2}(x)$, a special case of the function (3.15), and that $f_{m,n;i,j}(x) = f_{n,m;j,i}(x)$. Stimulated by [38], we pose to discuss complete monotonicity of the function (12.1).

12.6 Sixth problem

For $n \in \mathbb{N}$, let

$$f_n(x) = \sum_{km=n} [\psi^{(k)}(x)]^m$$

on $(0, \infty)$. It is clear that

$$\begin{aligned} f_1(x) &= \psi'(x), & f_2(x) &= \Delta(x), & f_3(x) &= [\psi'(x)]^3 + \psi'''(x), \\ f_4(x) &= [\psi'(x)]^4 + [\psi''(x)]^2 + \psi^{(4)}(x), & f_5(x) &= [\psi'(x)]^5 + \psi^{(5)}(x), \\ f_6(x) &= [\psi'(x)]^6 + [\psi''(x)]^3 + [\psi^{(3)}(x)]^2 + \psi^{(6)}(x). \end{aligned}$$

For $\ell \in \mathbb{N}$, prove that the functions $f_{2\ell-1}(x)$ is completely monotonic on $(0, \infty)$, but $f_{2\ell}(x)$ is not. This problem was suggested by Dr. Li Yin (Binzhou University, Shangdong, China) on 26 August 2018.

12.7 Seventh problem

We conjectured [72, Remark 6] that the double inequality

$$\frac{1}{x^4} \left[\frac{x^2 + 4x + 12}{12(x+1)^2} \right]^\alpha < [\psi'(x)]^2 + \psi''(x) < \frac{1}{x^4} \left[\frac{x^2 + 4x + 12}{12(x+1)^2} \right]^\beta$$

holds on $(0, \infty)$ if and only if $\alpha \geq \frac{6}{5}$ and $\beta \leq 1$.

12.8 Eighth problem

The function $\frac{1}{\arctan x}$ is logarithmically completely monotonic on $(0, \infty)$, but not a Stieltjes transform. This problem was first proposed in 2010.

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