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Approximate solution of generalized inhomogeneous radical quadratic functional equations in 2-Banach spaces

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Abstract

In this paper, using Brzdęk and Ciepliński's fixed point theorems in a 2-Banach space, we investigate approximate solution for the generalized inhomogeneous radical quadratic functional equation of the form

$$f\left(\sqrt{ax^2+by^2}\right)=af(x)+bf(y)+D(x,y),$$

where f is a mapping on the set of real numbers, $a,b\in\mathbf{R}_+$ and $D(x,y)$ is a given function. Some stability and hyperstability properties are presented.

Keywords: Fixed point theorem; Hyperstability; Radical quadratic functional equation; 2-Banach space

1 Introduction

In this paper, $\mathbf{N}$ and $\mathbf{R}$ denote the sets of all positive integers, and real numbers, respectively. We put $\mathbf{N}_0:=\mathbf{N}\cup 0$, $\mathbf{R}_0:=\mathbf{R}\setminus 0$ and $\mathbf{R}_+:=[0,\infty)$. Also, Y^X denotes the set of all functions from a nonempty set X to a nonempty set Y.

The study of stability problems for functional equations originates from a question of Ulam [28] concerning the stability of group homomorphisms. Popularly speaking, the question was "Under what conditions a mathematical object satisfying a certain property approximately must be close to an object satisfying the property exactly?" In the following year, Hyers [20] first partially answered Ulam's question, and proved the Ulam stability of Cauchy function in Banach spaces. Aoki [5] and Rassias [27] generalized the Hyers' results by allowing the Cauchy difference to become unbounded. During the last decades, the Ulam–Hyers–Rassias stability of functional equations has been extensively investigated and generalized by many mathematicians (see [2, 3, 6–9, 11–14, 16, 24, 26, 29] and the references therein).

Recently, a lot of papers (see, for instance, [1, 4, 15, 17–19, 21–23]) on the stability of radical function equations have been published. The functional equation

$$f\left(\sqrt{x^2+y^2}\right)=f(x)+f(y) \tag{1}$$

is called a radical quadratic functional equation. Kim et al. [23] investigated the generalized Hyers–Ulam–Rassias stability problem of Eq. (1) in quasi-β-Banach spaces using the direct method. Khodaei et al. [22] introduced and solved the generalized radical quadratic functional equation

$$f\left(\sqrt{ax^2+by^2}\right) = af(x) + bf(y). \tag{2}$$

They established some stability results in 2-normed spaces by using the direct method, and proved new theorems about the generalized Ulam stability by using subadditive and subquadratic functions in p-2-normed spaces. Cho et al. [15] proved the generalized Hyers–Ulam stability results for Eq. (2) in quasi-β-Banach spaces by using subadditive and subquadratic functions. Using Brzdęk's fixed point theorem, Aiemsomboom et al. [1] and Kang [21] investigated the stability of Eqs. (1) and (2), respectively, where f is a self-mapping on $\mathbf{R}$.

Let $(Y, \|\cdot,\cdot\|)$ be a 2-Banach space, $D : \mathbf{R}^2 \to Y$ a given function, and let $a, b \in \mathbf{R}_+$ be fixed. The purpose of this paper is to prove stability and hyperstability results for the generalized inhomogeneous quadratic radical functional equation

$$f\left(\sqrt{ax^2+by^2}\right) = af(x) + bf(y) + D(x,y), \quad x, y \in \mathbf{R}_0 \tag{3}$$

in a 2-Banach space using Brzdęk and Ciepliński's fixed point results in [11].

2 Preliminaries

Let us recall some basic definitions and facts concerning 2-Banach spaces (see, for instance, [11, 17, 18, 25]).

Definition 1 Let X be a linear space over $\mathbf{R}$ with $\dim X \geq 2$ and let $\|\cdot,\cdot\| : X \times X \to \mathbf{R}_+$ be a function satisfying the following properties:

(1) $\|x,y\| = 0$ if and only if x and y are linearly dependent;

(2) $\|x,y\| = \|y,x\|$ for $x, y \in X$;

(3) $\|rx,y\| = |r|\|x,y\|$ for $r \in \mathbf{R}$ and $x, y \in X$;

(4) $\|x+y,z\| \leq \|x,z\| + \|y,z\|$ for $x, y, z \in X$.

Then the pair $(X, \|\cdot,\cdot\|)$ is called a 2-normed space.

If $x \in X$ and $\|x,y\| = 0$ for all $y \in X$, then $x = 0$. Moreover, the functions $x \to \|x,y\|$ are continuous functions of X into $\mathbf{R}_+$ for each fixed $y \in X$.

Definition 2 Let $\{x_n\}$ be a sequence in a 2-normed space X.

(1) A sequence $\{x_n\}$ in a 2-normed space is called a Cauchy sequence if there are linear independent $y, z \in X$ such that

$$\lim_{n,m\to\infty} \|x_n - x_m, y\| = 0 = \lim_{n,m\to\infty} \|x_n - x_m, z\|;$$

(2) A sequence $\{x_n\}$ is said to be convergent if there exists an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for all $y \in X$. Then, the point x is called the limit of the sequence $\{x_n\}$, which is denoted by $\lim_{n\to\infty} x_n = x$;

(3) If every Cauchy sequence in X converges, then the 2-normed space X is called a 2-Banach space.

It is easily seen that $(\mathbf{R}^2, \|\cdot,\cdot\|)$ is a 2-Banach space, where the Euclidean 2-norm $\|\cdot,\cdot\|$ is defined by

$$\big\|(x_1,x_2),(y_1,y_2)\big\| := |x_1y_2 - x_2y_1|, \quad (x_1,x_2),(y_1,y_2) \in \mathbf{R}^2.$$

The next example following from [11, Proposition 2.3].

Example 1 If $(X, \langle\cdot,\cdot\rangle)$ is a real Hilbert space, then $(X, \|\cdot,\cdot\|)$ is a 2-Banach space, where $\|\cdot,\cdot\|$ is given by

$$\|x,y\| := \sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2}, \quad x,y \in X.$$

3 Fixed point theorems

Recently, Brzdęk and Ciepliński [11] proved a new fixed point theorem in 2-Banach spaces and showed its applications to the Ulam stability of some single-variable equations and the most important functional equation in several variables, namely, the Cauchy equation. And they extended the fixed point result to the n-normed spaces in [10].

Let us introduce the following hypotheses:

(H1) X is a nonempty set, $(Y, \|\cdot,\cdot\|)$ is a 2-Banach space, Y_0 is a subset of Y containing two linearly independent vectors;

(H2) $j \in \mathbf{N}, f_1,\dots,f_j : X \to X$, $g_1,\dots,g_j : Y_0 \to Y_0$, and $L_1,\dots,L_j : X \times Y_0 \to \mathbf{R}_+$ are given maps;

(H3) $\mathcal{T} : Y^X \to Y^X$ is an operator satisfying the inequality

$$\big\|(\mathcal{T}\xi)(x) - (\mathcal{T}\eta)(x), z\big\| \le \sum_{i=1}^{j} L_i(x,y)\big\|\xi\big(f_i(x)\big) - \eta\big(f_i(x)\big), g_i(z)\big\|, \tag{4}$$

where $\xi,\eta \in Y^X, x \in X, z \in Y_0$;

(H4) $\Lambda : \mathbf{R}_+^{X\times Y_0} \to \mathbf{R}_+^{X\times Y_0}$ is an operator defined by

$$(\Lambda\delta)(x,z) := \sum_{i=1}^{j} L_i(x,z)\delta\big(f_i(x), g_i(z)\big), \delta \in \mathbf{R}_+^{X\times Y_0}, \quad x \in X, z \in Y_0. \tag{5}$$

Now, we are in a position to present the above mentioned fixed point result. We use it to assert the existence of a unique fixed point of operator $\mathcal{T} : Y^X \to Y^X$.

Theorem 1 *Let hypotheses* (H1)–(H4) *hold and functions* $\epsilon : X \times Y_0 \to \mathbf{R}_+$ *and* $\varphi : X \to Y$ *fulfill the following two conditions*:

$$\big\|(\mathcal{T}\varphi)(x) - \varphi(x), z\big\| \le \epsilon(x,z), \quad x \in X, z \in Y_0 \tag{6}$$

and

$$\epsilon^*(x,z) := \sum_{l=0}^{\infty}\big(\Lambda^l\epsilon\big)(x,z) < \infty, \quad x \in X, z \in Y_0. \tag{7}$$

Then, there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x), z\| \leq \epsilon^*(x, z), \quad x \in X, z \in Y_0. \tag{8}$$

Moreover,

$$\psi(x) = \lim_{l \to \infty} (\mathcal{T}^l \varphi)(x), \quad x \in X.$$

4 The main results

In this section, we investigate the stability and hyperstability of the generalized inhomogeneous radical quadratic functional equation (3) in 2-Banach spaces by using Theorem 1. In what follows, we assume that $a, b \in \mathbf{N}$ are fixed, $(Y, \|\cdot,\cdot\|)$ is a 2-Banach space, and Y_0 is a subset of Y containing two linearly independent vectors.

Theorem 2 *Let $h_1, h_2 : \mathbf{R}_0 \times Y_0 \to \mathbf{R}_+$ be two functions such that*

$$\begin{aligned} M_0 := \biggl\{ & n \in \mathbf{N} : k_n := \frac{1}{a}\lambda_1(a + bn^2)\lambda_2(a + bn^2) + \frac{b}{a}\lambda_1(n^2)\lambda_2(n^2) < 1 \biggr\} \\ & \neq \emptyset, \end{aligned} \tag{9}$$

where

$$\lambda_i(n) := \inf\{t \in \mathbf{R}_+ : h_i(nx^2, z) \leq t h_i(x^2, z)\},$$

where $x \in \mathbf{R}_0, z \in Y_0, i = 1, 2, n \in \mathbf{N}$. Suppose that $f : \mathbf{R} \to Y$ satisfies the inequality

$$\|f(\sqrt{ax^2 + by^2}) - af(x) - bf(y), z\| \leq h_1(x^2, z)h_2(y^2, z), \tag{10}$$

where $x, y \in \mathbf{R}_0, z \in Y_0$. Then there exists a unique solution $Q : \mathbf{R} \to Y$ of (2) such that

$$\|f(x) - Q(x), z\| \leq \lambda_0(x, z), \quad x \in \mathbf{R}_0, z \in Y_0, \tag{11}$$

where

$$\lambda_0(x, z) := \inf_{m \in M_0} \left\{ \frac{\lambda_2(m^2)h_1(x^2, z)h_2(x^2, z)}{a(1 - k_m)}, x \in \mathbf{R}_0, z \in Y_0 \right\}.$$

Proof Putting $y = mx$ in (10), we obtain that

$$\|f(\sqrt{(a + bm^2)x^2}) - af(x) - bf(mx), z\| \leq h_1(x^2, z)h_2(m^2x^2, z), \tag{12}$$

where $m \in \mathbf{N}, x \in \mathbf{R}_0, z \in Y_0$, and so

$$\left\| \frac{1}{a}f(\sqrt{(a + bm^2)x^2}) - \frac{b}{a}f(mx) - f(x), z \right\| \leq \frac{1}{a}h_1(x^2, z)h_2(m^2x^2, z), \tag{13}$$

where $m \in \mathbf{N}, x \in \mathbf{R}_0, z \in Y_0$.

For each $m \in \mathbf{N}$, we define the operators $\mathcal{T}_m : Y^{\mathbf{R}_0} \to Y^{\mathbf{R}_0}$ and $\Lambda_m : \mathbf{R}_+^{\mathbf{R}_0 \times Y_0} \to \mathbf{R}_+^{\mathbf{R}_0 \times Y_0}$ by

$$(\mathcal{T}_m\xi)(x) := \frac{1}{a}\xi\bigl(\sqrt{(a+bm^2)x^2}\bigr) - \frac{b}{a}\xi(mx),$$
$$(\Lambda_m\delta)(x,z) := \frac{1}{a}\delta\bigl(\sqrt{(a+bm^2)x^2}, z\bigr) + \frac{b}{a}\delta(mx,z),$$

where $x \in \mathbf{R}_0, \xi \in Y^{\mathbf{R}_0}, \delta \in \mathbf{R}_+^{\mathbf{R}_0 \times Y_0}, z \in Y_0$. Then the operator Λ_m has the form (5) with $X := \mathbf{R}_0$, $j = 2$, $f_1(x) := \sqrt{(a+bm^2)x^2}, f_2(x) := mx$, $g_1(z) = g_2(z) := z, L_1(x,z) := \frac{1}{a}$ and $L_2(x,z) := \frac{b}{a}$ for all $x \in \mathbf{R}_0$ and $z \in Y_0$. Next, put

$$\epsilon_m(x,z) := \frac{1}{a}h_1\bigl(x^2,z\bigr)h_2\bigl(m^2x^2,z\bigr), \quad m \in \mathbf{N}, x \in \mathbf{R}_0, z \in Y_0$$

and observe that

$$\epsilon_m(x,z) = \frac{1}{a}h_1\bigl(x^2,z\bigr)h_2\bigl(m^2x^2,z\bigr) \le \frac{1}{a}\lambda_2\bigl(m^2\bigr)h_1\bigl(x^2,z\bigr)h_2\bigl(x^2,z\bigr),$$

where $m \in \mathbf{N}, x \in \mathbf{R}_0, z \in Y_0$. Then, inequality (13) can be rewritten as

$$\bigl\|(\mathcal{T}_m f)(x) - f(x), z\bigr\| \le \epsilon_m(x,z), \quad m \in \mathbf{N}, x \in \mathbf{R}_0, z \in Y_0,$$

and we have

$$\begin{aligned}
&\bigl\|(\mathcal{T}_m\xi)(x) - (\mathcal{T}_m\eta)(x)), z\bigr\| \\
&\quad = \left\| \frac{1}{a}\xi\bigl(\sqrt{(a+bm^2)x^2}\bigr) - \frac{b}{a}\xi(mx) \right. \\
&\qquad \left. - \frac{1}{a}\eta\bigl(\sqrt{(a+bm^2)x^2}\bigr) + \frac{b}{a}\eta(mx), z \right\| \\
&\quad \le \left\| \frac{1}{a}\xi\bigl(\sqrt{(a+bm^2)x^2}\bigr) - \frac{1}{a}\eta\bigl(\sqrt{(a+bm^2)x^2}\bigr), z \right\| \\
&\qquad + \left\| \frac{b}{a}\xi(mx) - \frac{b}{a}\eta(mx), z \right\| \\
&\quad = \frac{1}{a}\bigl\|\xi\bigl(\sqrt{(a+bm^2)x^2}\bigr) - \eta\bigl(\sqrt{(a+bm^2)x^2}\bigr), z\bigr\| \\
&\qquad + \frac{b}{a}\bigl\|\xi(mx) - \eta(mx), z\bigr\| \\
&\quad = L_1(x,z)\bigl\|\xi\bigl(f_1(x)\bigr) - \eta\bigl(f_1(x)\bigr), z\bigr\| \\
&\qquad + L_2(x,z)\bigl\|\xi\bigl(f_2(x)\bigr) - \eta\bigl(f_2(x)\bigr), z\bigr\|
\end{aligned} \tag{14}$$

for any $x \in \mathbf{R}_0, \xi, \eta \in Y^{\mathbf{R}_0}, z \in Y_0$. Therefore,

$$\bigl\|(\mathcal{T}_m\xi)(x) - (\mathcal{T}_m\eta)(x)), z\bigr\| \le \sum_{i=1}^{2} L_i(x,z)\bigl\|\xi\bigl(f_i(x)\bigr) - \eta\bigl(f_i(x)\bigr), z\bigr\|, \tag{15}$$

so (4) holds for $\mathcal{T} := \mathcal{T}_m$ with $m \in \mathbf{N}$. By the definition of $\lambda_i(n)$, we have

$$h_i\bigl(nx^2,z\bigr) \le \lambda_i(n)h_i\bigl(x^2,z\bigr), \quad x \in \mathbf{R}_0, z \in Y_0, i = 1,2, n \in \mathbf{N}, \tag{16}$$

whence, using induction, we get

$$\big(\Lambda_m^n \epsilon_m\big)(x,z) \le \frac{1}{a}\lambda_2\big(m^2\big)k_m^n h_1\big(x^2,z\big)h_2\big(x^2,z\big), \quad n \in \mathbf{N}_0, x \in \mathbf{R}_0, z \in Y_0. \tag{17}$$

Indeed, for $n = 0$, (17) is obviously true. Next, we will assume that (17) holds for $n = j$, where $j \in \mathbf{N}$. Then, we have

$$\begin{aligned}
&\big(\Lambda_m^{j+1}\epsilon_m\big)(x,z)\\
&\quad = \big(\Lambda_m\big(\Lambda_m^j \epsilon_m(x,z)\big)\big)\\
&\quad = \frac{1}{a}\big(\Lambda_m^j\epsilon_m\big)\big(\sqrt{\big(a+bm^2\big)x^2},z\big) + \frac{b}{a}\big(\Lambda_m^j\epsilon_m\big)(mx,z)\\
&\quad \le \frac{1}{a^2}\lambda_2\big(m^2\big)k_m^j h_1\big(\big(a+bm^2\big)x^2,z\big)h_2\big(\big(a+bm^2\big)x^2,z\big)\\
&\qquad + \frac{b}{a}\frac{1}{a}\lambda_2\big(m^2\big)k_m^j h_1\big(m^2x^2,z\big)h_2\big(m^2x^2,z\big)\\
&\quad \le \frac{1}{a}\lambda_2\big(m^2\big)k_m^j h_1\big(x^2,z\big)h_2\big(x^2,z\big)\bigg[\frac{1}{a}\lambda_1\big(a+bm^2\big)\lambda_2\big(a+bm^2\big)\\
&\qquad + \frac{b}{a}\lambda_1\big(m^2\big)\lambda_2\big(m^2\big)\bigg]\\
&\quad = \frac{1}{a}\lambda_2\big(m^2\big)k_m^{j+1}h_1\big(x^2,z\big)h_2\big(x^2,z\big), \quad x \in \mathbf{R}_0, z \in Y_0, m \in M_0.
\end{aligned}$$

This shows that (17) holds for $n = j + 1$. Now we can conclude that inequality (17) holds for all $n \in \mathbf{N}_0$. Therefore, by (17), we obtain that

$$\begin{aligned}
\epsilon_m^*(x,z) &= \sum_{n=0}^{\infty}\big(\Lambda_m^n\epsilon_m\big)(x,z)\\
&\le \frac{1}{a}\lambda_2\big(m^2\big)h_1\big(x^2,z\big)h_2\big(x^2,z\big)\sum_{n=0}^{\infty}k_m^n\\
&= \frac{\lambda_2(m^2)h_1(x^2,z)h_2(x^2,z)}{a(1-k_m)}
\end{aligned}$$

for all $x \in \mathbf{R}_0, z \in Y_0$ and $m \in M_0$. Thus, according to Theorem 1, for any $m \in M_0$, there exists a unique fixed point $Q_m' : \mathbf{R}_0 \to Y$ of $\mathcal{T}_m$, which satisfies the estimate

$$\big\|f(x) - Q_m'(x), z\big\| \le \epsilon_m^*(x,z) \le \frac{\lambda_2(m^2)h_1(x^2,z)h_2(x^2,z)}{a(1-k_m)}, \tag{18}$$

where $x \in \mathbf{R}_0, z \in Y_0, m \in M_0$. Moreover,

$$Q_m'(x) := \lim_{n\to\infty}\big(\mathcal{T}_m^n f\big)(x), \quad x \in \mathbf{R}_0, m \in M_0.$$

and for any $m \in M_0$, the function $Q_m : \mathbf{R} \to Y$, given by the formula

$$Q_m(0) = 0, \qquad Q_m(x) := Q_m'(x), \quad x \in \mathbf{R}_0,$$

is a solution of the equation

$$Q(x) = \frac{1}{a}Q\left(\sqrt{(a+bm^2)x^2}\right) - \frac{b}{a}Q(mx), \quad x \in \mathbf{R}, m \in M_0. \tag{19}$$

Now, we show that

$$\begin{aligned} &\left\| \left(\mathcal{T}_m^n f\right)\left(\sqrt{ax^2+by^2}\right) - a\left(\mathcal{T}_m^n f\right)(x) - b\left(\mathcal{T}_m^n f\right)(y), z \right\| \\ &\quad \le k_m^n h_1\left(x^2, z\right) h_2\left(y^2, z\right) \end{aligned} \tag{20}$$

for any $x, y \in \mathbf{R}_0, z \in Y_0, m \in M_0$ and $n \in \mathbf{N}_0$.

Since the case $n = 0$ follows immediately from (10), take $j \in \mathbf{N}_0$ and assume that (20) holds for $n = j$, $x, y \in \mathbf{R}_0$, $m \in M_0$ and $z \in Y$. Then, by (16), we get

$$\begin{aligned} &\left\| \left(\mathcal{T}_m^{j+1} f\right)\left(\sqrt{ax^2+by^2}\right) - a\left(\mathcal{T}_m^{j+1} f\right)(x) - b\left(\mathcal{T}_m^{j+1} f\right)(y), z \right\| \\ &\quad = \left\| \frac{1}{a}\left(\mathcal{T}_m^j f\right)\left(\sqrt{\left(a+bm^2\right)\left(ax^2+by^2\right)}\right) - \frac{b}{a}\left(\mathcal{T}_m^j f\right)\left(m\sqrt{ax^2+by^2}\right) \right. \\ &\qquad - \left(\mathcal{T}_m^j f\right)\left(\sqrt{\left(a+bm^2\right)x^2}\right) + b\left(\mathcal{T}_m^j f\right)(mx) \\ &\qquad \left. - \frac{b}{a}\left(\mathcal{T}_m^j f\right)\left(\sqrt{\left(a+bm^2\right)y^2}\right) + \frac{b^2}{a}\left(\mathcal{T}_m^j f\right)(my), z \right\| \\ &\quad \le \left\| \frac{1}{a}\left(\mathcal{T}_m^j f\right)\left(\sqrt{\left(a+bm^2\right)\left(ax^2+by^2\right)}\right) - \left(\mathcal{T}_m^j f\right)\left(\sqrt{\left(a+bm^2\right)x^2}\right) \right. \\ &\qquad \left. - \frac{b}{a}\left(\mathcal{T}_m^j f\right)\left(\sqrt{\left(a+bm^2\right)y^2}\right), z \right\| \\ &\qquad + \left\| \frac{b}{a}\left(\mathcal{T}_m^j f\right)\left(m\sqrt{ax^2+by^2}\right) - b\left(\mathcal{T}_m^j f\right)(mx) - \frac{b^2}{a}\left(\mathcal{T}_m^j f\right)(my), z \right\| \\ &\quad \le \frac{1}{a}k_m^j h_1\left(\left(a+bm^2\right)x^2, z\right) h_2\left(\left(a+bm^2\right)y^2, z\right) \\ &\qquad + \frac{b}{a}k_m^j h_1\left(m^2x^2, z\right) h_2\left(m^2y^2, z\right) \\ &\quad \le k_m^j h_1\left(x^2, z\right) h_2\left(y^2, z\right)\left[\frac{1}{a}\lambda_1\left(a+bm^2\right)\lambda_2\left(a+bm^2\right) + \frac{b}{a}\lambda_1\left(m^2\right)\lambda_2\left(m^2\right)\right] \\ &\quad = k_m^{j+1} h_1\left(x^2, z\right) h_2\left(y^2, z\right), \quad x, y \in \mathbf{R}_0, z \in Y_0, m \in M_0. \end{aligned}$$

Thus, by induction, we have shown that (20) holds for any $n \in \mathbf{N}_0$, $x, y \in \mathbf{R}_0, z \in Y_0$ and $m \in M_0$. Letting $n \to \infty$ in (20) and using Lemmas 2.1 and 2.2 in [11], we obtain that

$$Q_m\left(\sqrt{ax^2+by^2}\right) = aQ_m(x) + bQ_m(y), \quad x, y \in \mathbf{R}_0, m \in M_0. \tag{21}$$

This way, for each $m \in M_0$, we obtain a function Q_m such that (21) holds for $x, y \in \mathbf{R}$ and

$$\left\|f(x) - Q_m(x), z\right\| \le \frac{\lambda_2(m^2)h_1(x^2, z)h_2(x^2, z)}{a(1-k_m)}, \quad x \in \mathbf{R}, z \in Y_0, m \in M_0. \tag{22}$$

Let $L > 0$ be a constant. Next, we will see that every generalized radical quadratic mapping $Q : \mathbf{R} \to Y$ satisfying the inequality

$$\|f(x) - Q(x), z\| \le Lh_1(x^2, z)h_2(x^2, z), \quad x \in \mathbf{R}_0, z \in Y_0 \tag{23}$$

is equal to Q_m for any $m \in M_0$. To do this, fix $s \in M_0$ and let $Q : \mathbf{R} \to Y$ be a generalized radical quadratic mapping satisfying (23). By (18), we have

$$\begin{aligned}\|Q_s(x) - Q(x), z\| &\le \|Q_s(x) - f(x), z\| + \|f(x) - Q(x), z\| \\ &\le \left(\frac{\lambda_2(s^2)}{a(1-k_s)} + L\right)h_1(x^2, z)h_2(x^2, z) \\ &= L_0 h_1(x^2, z)h_2(x^2, z)\sum_{n=0}^{\infty} k_s^n, \quad x \in \mathbf{R}_0, z \in Y_0,\end{aligned} \tag{24}$$

where $L_0 = aL(1 - k_s) + \lambda_2(s^2)$. Observe also that Q and Q_s are solutions of equation (19) for any $m \in M_0$.

Now, we will see that, for any $j \in \mathbf{N}_0$,

$$\|Q_s(x) - Q(x), z\| \le L_0 h_1(x^2, z)h_2(x^2, z)\sum_{n=j}^{\infty} k_s^n, \quad x \in \mathbf{R}_0, z \in Y_0. \tag{25}$$

The case $j = 0$ follows from the previous inequality. Fix a $j \in \mathbf{N}_0$ and assume that (25) holds. Then, by (16), we get

$$\begin{aligned}&\|Q_s(x) - Q(x), z\| \\ &\quad = \left\|\frac{1}{a}Q_s\left(\sqrt{(a+bs^2)x^2}\right) - \frac{b}{a}Q_s(sx) - \frac{1}{a}Q\left(\sqrt{(a+bs^2)x^2}\right) + \frac{b}{a}Q(sx), z\right\| \\ &\quad \le \left\|\frac{1}{a}Q_s\left(\sqrt{(a+bs^2)x^2}\right) - \frac{1}{a}Q\left(\sqrt{(a+bs^2)x^2}\right), z\right\| \\ &\qquad + \left\|\frac{b}{a}Q_s(sx) - \frac{b}{a}Q(sx), z\right\| \\ &\quad \le \frac{1}{a}L_0 h_1\left((a+bs^2)x^2, z\right)h_2\left((a+bs^2)x^2, z\right)\sum_{n=j}^{\infty} k_s^n \\ &\qquad + \frac{b}{a}L_0 h_1(s^2x^2, z)h_2(s^2x^2, z)\sum_{n=j}^{\infty} k_s^n \\ &\quad \le L_0 h_1(x^2, z)h_2(x^2, z)\left(\frac{1}{a}\lambda_1(a+bs^2)\lambda_2(a+bs^2) + \frac{b}{a}\lambda_1(s^2)\lambda_2(s^2)\right)\sum_{n=j}^{\infty} k_s^n \\ &\quad = L_0 h_1(x^2, z)h_2(x^2, z)\sum_{n=j+1}^{\infty} k_s^n, \quad x \in \mathbf{R}_0, z \in Y_0.\end{aligned} \tag{26}$$

Thus (25) is valid for any $j \in \mathbf{N}_0$. Letting $j \to \infty$ in (25) and using Lemma 2.1 in [11], we get

$$Q(x) = Q_s(x), \quad x \in \mathbf{R}_0, \tag{27}$$

which, together with $Q(0) = Q_s(0) = 0$, gives $Q = Q_s$. This means that $Q_m = Q_s$ for any $m \in M_0$. Therefore, by (18), we have

$$\|f(x) - Q_s(x), z\| \leq \frac{\lambda_2(m^2)h_1(x^2, z)h_2(x^2, z)}{a(1 - k_m)}, \quad x \in \mathbf{R}_0, z \in Y_0, m \in M_0. \tag{28}$$

Hence, we get inequality (11) with $Q := Q_s$. □

In a similar way, one can prove the following.

Theorem 3 *Let $H : \mathbf{R}_0 \times Y_0 \to \mathbf{R}_+$ be a function such that*

$$\mathcal{M} := \left\{ n \in \mathbf{N} : \frac{1}{a}\rho\left(a + bn^2\right) + \frac{b}{a}\rho\left(n^2\right) < 1 \right\} \neq \emptyset, \tag{29}$$

where

$$\rho(n) := \inf\left\{t \in \mathbf{R}_+ : H\left(nx^2, z\right) \leq tH\left(x^2, z\right), x \in \mathbf{R}_0, z \in Y_0, n \in \mathbf{N}\right\}.$$

Suppose that $f : \mathbf{R} \to Y$ satisfies the inequality

$$\left\|f\left(\sqrt{ax^2 + by^2}\right) - af(x) - bf(y), z\right\| \leq H\left(x^2, z\right) + H\left(y^2, z\right), \tag{30}$$

where $x, y \in \mathbf{R}_0, z \in Y_0$. Then there exists a unique solution $Q : \mathbf{R} \to Y$ of (2) such that

$$\|f(x) - Q(x), z\| \leq \rho_0(x, z), \quad x \in \mathbf{R}_0, z \in Y_0, \tag{31}$$

where

$$\rho_0(x, z) := \inf_{m \in \mathcal{M}} \left\{ \frac{(1 + \rho(m^2))H(x^2, z)}{a - \rho(a + bm^2) - b\rho(m^2)}, x \in \mathbf{R}_0, z \in Y_0 \right\}.$$

From the above theorems we can obtain results analogous to Theorems 2 and 3 for the inhomogeneous radical quadratic functional equation.

Corollary 1 *Let $h_1, h_2 : \mathbf{R}_0 \times Y_0 \to \mathbf{R}_+, f : \mathbf{R} \to Y$ and $D : \mathbf{R}^2 \to Y$ such that (9) holds, and*

$$\left\|f\left(\sqrt{ax^2 + by^2}\right) - af(x) - bf(y) - D(x, y), z\right\| \leq h_1\left(x^2, z\right)h_2\left(y^2, z\right), \tag{32}$$

where $x, y \in \mathbf{R}_0, z \in Y_0$. Assume that (3) has a solution $f_0 : \mathbf{R} \to Y$. Then there exists a unique solution $F : \mathbf{R} \to Y$ of (3) such that

$$\|f(x) - F(x), z\| \leq \lambda_0(x, z), \quad x \in \mathbf{R}_0, z \in Y_0, \tag{33}$$

where $\lambda_0(x, z)$ is defined as in Theorem 2.

Proof Write $f_1 := f - f_0$. Then, we have

$$\begin{aligned}
&\left\|f_1\left(\sqrt{ax^2+by^2}\right) - af_1(x) - bf_1(y), z\right\| \\
&\quad = \left\|f\left(\sqrt{ax^2+by^2}\right) - af(x) - bf(y) - D(x,y)\right. \\
&\qquad \left. - \left(f_0\left(\sqrt{ax^2+by^2}\right) - af_0(x) - bf_0(y) - D(x,y)\right), z\right\| \\
&\quad = \left\|f\left(\sqrt{ax^2+by^2}\right) - af(x) - bf(y) - D(x,y), z\right\| \\
&\quad \leq h_1\left(x^2,z\right)h_2\left(y^2,z\right), \quad x,y \in \mathbf{R}_0, z \in Y_0,
\end{aligned}$$

and, according to Theorem 2, there is a unique solution $Q : \mathbf{R} \to Y$ of (2) such that

$$\left\|f_1(x) - Q(x), z\right\| \leq \lambda_0(x,z), \quad x \in \mathbf{R}_0, z \in Y_0,$$

where $\lambda_0(x,z)$ is defined as in Theorem 2. Let $F = f_0 + Q$. Then F is a solution to (3) and (33) holds. The uniqueness of F follows from the uniqueness of Q (see [6, Corollary 4]). □

Corollary 2 *Let $H : \mathbf{R}_0 \times Y_0 \to \mathbf{R}_+$, $f : \mathbf{R} \to Y$ and $D : \mathbf{R}^2 \to Y$ such that (29) holds, and*

$$\left\|f\left(\sqrt{ax^2+by^2}\right) - af(x) - bf(y) - D(x,y), z\right\| \leq H\left(x^2,z\right) + H\left(y^2,z\right), \tag{34}$$

where $x,y \in \mathbf{R}_0, z \in Y_0$. Assume that (3) admits a solution $f_0 : \mathbf{R} \to Y$. Then there exists a unique solution $F : \mathbf{R} \to Y$ of (3) such that

$$\left\|f(x) - F(x), z\right\| \leq \rho_0(x,z), \quad x \in \mathbf{R}_0, z \in Y_0, \tag{35}$$

where $\rho_0(x,z)$ is defined as in Theorem 3.

Corollaries 1 and 2 yield at once the following hyperstability results.

Corollary 3 *Let $h_1, h_2 : \mathbf{R}_0 \times Y_0 \to \mathbf{R}_+$ be functions such that*

$$\begin{aligned}
&\sup_{n\in\mathbf{N}}\left\{\lambda_1\left(a+bn^2\right)\lambda_2\left(a+bn^2\right) + b\lambda_1\left(n^2\right)\lambda_2\left(n^2\right)\right\} < a, \\
&\inf_{n\in\mathbf{N}}\left\{\lambda_2\left(n^2\right)\right\} = 0,
\end{aligned} \tag{36}$$

where $\lambda_i(\cdot)$ $(i = 1,2)$ are defined as in Theorem 2. Assume that Eq. (3) has a solution f_0. Then any function $f : \mathbf{R} \to Y$, which satisfies $f(0) = f_0(0)$ and inequality (32), is a solution of (3).

Corollary 4 *Let $H : \mathbf{R}_0 \times Y_0 \to \mathbf{R}_+$ be function such that*

$$\sup_{n\in\mathbf{N}}\left\{\rho\left(a+bn^2\right) + b\rho\left(n^2\right)\right\} < a, \qquad \inf_{n\in\mathbf{N}}\left\{\rho\left(n^2\right)\right\} = -1, \tag{37}$$

where $\rho(\cdot)$ is defined as in Theorem 3. Assume that (3) has a solution f_0. Then any function $f : \mathbf{R} \to Y$, which satisfies $f(0) = f_0(0)$ and inequality (34), is a solution of (3).

According to Corollaries 3 and 4, we derive the following particular cases.

Corollary 5 *Let $h_1, h_2 : \mathbf{R}_0 \times Y_0 \to \mathbf{R}_+$ be functions such that*

$$\lim_{n\to\infty} \left(\lambda_1\left(a+bn^2\right)\lambda_2\left(a+bn^2\right) + b\lambda_1\left(m^2\right)\lambda_2\left(n^2\right)\right) = 0, \qquad \lim_{n\to\infty} \lambda_2\left(n^2\right) = 0, \tag{38}$$

where $\lambda_i(\cdot)$ $(i = 1, 2)$ are defined as in Theorem 2. Assume that (3) has a solution f_0. Then any function $f : \mathbf{R} \to Y$, which satisfies $f(0) = f_0(0)$ and inequality (32), is a solution of (3).

Corollary 6 *Let $H : \mathbf{R}_0 \times Y_0 \to \mathbf{R}_+$ be function such that*

$$\lim_{n\to\infty} \left(\rho\left(a+bn^2\right) + b\rho\left(n^2\right)\right) = 0, \qquad \lim_{n\to\infty} \rho\left(n^2\right) = -1,$$

where $\rho(\cdot)$ is defined as in Theorem 3. Assume that (3) has a solution f_0. Then any function $f : \mathbf{R} \to Y$, which satisfies $f(0) = f_0(0)$ and inequality (34), is a solution of (3).

Next, we derive some hyperstability results for particular forms of h_1, h_2, H and (3).

Corollary 7 *Let $\theta \in \mathbf{R}_+$ and let $p, q \in \mathbf{R}$ be such that $p + q < 0$. Assume that Eq. (3) has a solution f_0. If $f : \mathbf{R} \to Y$ satisfies $f(0) = f_0(0)$ and the inequality*

$$\left\| f\left(\sqrt{ax^2+by^2}\right) - af(x) - bf(y) - D(x,y), z \right\| \le \theta |x|^p |y|^q, \quad x, y \in \mathbf{R}_0, z \in Y, \tag{39}$$

then f is a solution of (3).

Proof Define $h_1, h_2 : \mathbf{R}_0 \times Y_0 \to \mathbf{R}_+$ by $h_1(x^2, z) = \theta_1 |x|^p$ and $h_2(y^2, z) = \theta_2 |y|^p$, where $\theta_1, \theta_2 \in \mathbf{R}_+$ with $\theta = \theta_1\theta_2$. Then, we have

$$\begin{aligned}
\lambda_1(n) &= \inf\left\{t \in \mathbf{R}_+ : h_1\left(nx^2, z\right) \le t h_1\left(x^2, z\right), x \in \mathbf{R}_0, z \in Y_0\right\} \\
&= \inf\left\{t \in \mathbf{R}_+ : \theta_1 \left|n^{1/2} x^2\right|^p \le t\theta_1 |x|^p, x \in \mathbf{R}_0, z \in Y_0\right\} \\
&= n^{p/2}, \quad n \in \mathbf{N}.
\end{aligned}$$

Similarly, we get $\lambda_2(n) = n^{q/2}$ for any $n \in \mathbf{N}$. Thus,

$$\begin{aligned}
&\lim_{n\to\infty} \left(\lambda_1\left(a+bn^2\right)\lambda_2\left(a+bn^2\right) + b\lambda_1\left(n^2\right)\lambda_2\left(n^2\right)\right) \\
&\quad = \lim_{n\to\infty} \left(\left(a+bn^2\right)^{(p+q)/2} + bn^{p+q}\right) \\
&\quad = 0.
\end{aligned}$$

As $p, q \in \mathbf{R}$ with $p + q < 0$, either $p < 0$ or $q < 0$. Hence, by inequality (39), one can see that it is sufficient to consider only the case $q < 0$, and thus

$$\lim_{n\to\infty} \lambda_2\left(n^2\right) = \lim_{n\to\infty} n^q = 0.$$

So, we can apply Corollary 5. □

Corollary 8 *Let $\theta \in \mathbf{R}_+$ and let $p, q \in \mathbf{R}$ be such that $p + q < 0$. If $f : \mathbf{R} \to Y$ satisfies $f(0) = 0$ and the inequality*

$$\left\| f\left(\sqrt{ax^2 + by^2}\right) - af(x) - bf(y), z \right\| \le \theta |x|^p |y|^q, \quad x, y \in \mathbf{R}_0, z \in Y,$$

then f is a solution of (2).

Similarly, we can prove the following.

Corollary 9 *Let $\theta \in \mathbf{R}_+$ and consider $p \in \mathbf{R}$ with $p < 0$. Assume that* (3) *has a solution f_0. If $f : \mathbf{R} \to Y$ satisfies $f(0) = f_0(0)$ and the inequality*

$$\left\| f\left(\sqrt{ax^2 + by^2}\right) - af(x) - bf(y) - D(x, y), z \right\| \le \theta \left(|x|^p + |y|^p\right),$$

where $x, y \in \mathbf{R}_0, z \in Y$, then f is a solution of (3).

Corollary 10 *Let $\theta \in \mathbf{R}_+$ and consider $p \in \mathbf{R}$ with $p < 0$. If $f : \mathbf{R} \to Y$ satisfies $f(0) = 0$ and the inequality*

$$\left\| f\left(\sqrt{ax^2 + by^2}\right) - af(x) - bf(y), z \right\| \le \theta \left(|x|^p + |y|^p\right), \quad x, y \in \mathbf{R}_0, z \in Y,$$

then f is a solution of (2).

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References
1. Aiemsomboon, L., Sintunavarat, W.: On a new type of stability of a radical quadratic functional equation using Brzdęk fixed point theorem. Acta Math. Hung. **151**(1), 35–46 (2017)
2. Aiemsomboon, L., Sintunavarat, W.: A note on the generalised hyperstability of the general linear equation. Bull. Aust. Math. Soc. **96**(2), 263–273 (2017)
3. Almahalebi, M.: On the hyperstability of Drygas functional equation on semigroups. Aequ. Math. **90**, 849–857 (2016)
4. Almahalebi, M., Charifi, A., Park, C., Kabbaj, S.: Hyperstability results for a generalized radical cubic functional equation related to additive mapping in non-Archimedean Banach spaces. J. Fixed Point Theory Appl. **20**(1), 40 (2018)
5. Aoki, T.: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**, 64–66 (1950)
6. Bahyrycz, A., Brzdęk, J., Jabłońska, E., Malejki, R.: Ulam's stability of a generalization of the Fréchet functional equation. J. Math. Anal. Appl. **442**, 537–553 (2016)
7. Brzdęk, J.: Stability of additivity and fixed point methods. Fixed Point Theory Appl. **2013**(1), 1 (2013)
8. Brzdęk, J.: Remarks on solutions to the functional equations of the radical type. Adv. Theory Nonlinear Anal. Appl. **1**, 125–135 (2017)
9. Brzdęk, J., Cădariu, L.: Stability for a family of equations generalizing the equation of p-Wright affine functions. Appl. Math. Comput. **276**, 158–171 (2016)

10. Brzdek, J., Cieplinski, K.: A fixed point theorem in n-Banach spaces and Ulam stability. J. Math. Anal. Appl. **470**, 632–646 (2019)
11. Brzdęk, J., Ciepliski, K.: On a fixed point theorem in 2-Banach spaces and some of its applications. Acta Math. Sci. **38B**(2), 377–390 (2018)
12. Brzdęk, J., Fechner, W., Moslehian, M.S., Sikorska, J.: Recent developments of the conditional stability of the homomorphism equation. Banach J. Math. Anal. **9**(3), 278–326 (2015)
13. Brzdek, J., Popa, D., Raşa, I., Xu, B.: Ulam Stability of Operators. Math. Anal. Appl. Academic Press, Oxford (2018)
14. Brzdek, J., Schwaiger, J.: Remarks on solutions to a generalization of the radical functional equations. Aequ. Math. **92**, 975–991 (2018)
15. Cho, Y.J., Gordji, M.E., Kim, S.S.: On the stability of radical functional equations in quasi-β-normed spaces. Bull. Korean Math. Soc. **51**(5), 1511–1525 (2014)
16. Dung, N.V., Hang, V.T.L.: The generalized hyperstability of general linear equations in quasi-Banach spaces. J. Math. Anal. Appl. **462**, 131–147 (2018)
17. El-Fassi, I.: Approximate solution of radical quartic functional equation related to additive mapping in 2-Banach spaces. J. Math. Anal. Appl. **455**, 2001–2013 (2017)
18. El-Fassi, I.: A new type of approximation for the radical quintic functional equation in non-Archimedean $(2,\beta)$-Banach spaces. J. Math. Anal. Appl. **457**, 322–335 (2018)
19. Gordji, M.E., Khodaei, H., Kim, G.H.: Nearly radical quadratic functional equations in p-2-normed spaces. Abstr. Appl. Anal. **2012**(2012), 185 (2012)
20. Hyers, D.H.: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222–224 (1941)
21. Kang, D.: Brzdęk fixed point approach for generalized quadratic radical functional equations. J. Fixed Point Theory Appl. **20**(1), 50 (2018)
22. Khodaei, H., Gordji, M.E., Kim, S.S., Cho, Y.J.: Approximation of radical functional equations related to quadratic and quartic mappings. J. Math. Anal. Appl. **395**(1), 284–297 (2012)
23. Kim, S.S., Cho, Y.J., Gordji, M.E.: On the generalized Hyers–Ulam–Rassias stability problem of radical functional equations. J. Inequal. Appl. **2012**(1), 186 (2012)
24. Moszner, Z.: Stability has many names. Aequ. Math. **90**, 983–999 (2016)
25. Park, W.G.: Approximate additive mappings in 2-Banach spaces and related topics. J. Math. Anal. Appl. **376**, 193–202 (2011)
26. Piszczek, M.: Hyerstability of the general linear functional equation. Bull. Korean Math. Soc. **52**(6), 1827–1838 (2015)
27. Rassias, T.M.: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297–300 (1978)
28. Ulam, S.M.: Problems in Modern Mathematics. Wiley, New York (1960)
29. Xu, T.Z.: Stability of multi-Jensen mappings in non-Archimedean normed spaces. J. Math. Phys. **53**(2), 852–869 (2012)