Zhang *Journal of Inequalities and Applications* (2019) 2019:15
https://doi.org/10.1186/s13660-019-1972-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

A note on the convergence rates in precise asymptotics

Yong Zhang[1*]

*Correspondence:
zyong2661@jlu.edu.cn
[1]College of Mathematics, Jilin University, Changchun, P.R. China

Abstract

Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX = 0$, $EX^2 = \sigma^2$. Set $S_n = \sum_{k=1}^{n} X_k$ and let $\mathcal{N}$ be the standard normal random variable. Let $g(x)$ be a positive and twice differentiable function on $[n_0, \infty)$ such that $g(x) \nearrow \infty$, $g'(x) \searrow 0$ as $x \to \infty$. In this short note, under some suitable conditions on both X and $g(x)$, we establish the following convergence rates in precise asymptotics

$$\lim_{\varepsilon \searrow 0}\left[\sum_{n=n_0}^{\infty} g'(n) P\left\{\frac{|S_n|}{\sigma\sqrt{n}} \geq \varepsilon g^s(n)\right\} - \varepsilon^{-1/s} E|\mathcal{N}|^{1/s}\right] = \gamma - \eta,$$

where $\gamma = \lim_{n\to\infty}(\sum_{k=n_0}^{n} g'(k) - g(n))$, $\eta = \sum_{n=n_0}^{\infty} g'(n) P\{S_n = 0\}$. It can describe the relations among the boundary function, weighting function, convergence rate and the limit value in studies of complete convergence. The result extends and generalizes the corresponding results of Gut and Steinebach (Ann. Univ. Sci. Budapest. Sect. Comput. 39:95–110, 2013), Kong (Lith. Math. J. 56(3):318–324, 2016), Kong and Dai (Stat. Probab. Lett. 119(10):295–300, 2016).

Keywords: Convergence rates; Precise asymptotics; Complete convergence

1 Introduction and main results

Throughout this note, let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX = 0$, $EX^2 = \sigma^2$ and set $S_n = \sum_{k=1}^{n} X_k$. Let $\mathcal{N}$ be the standard normal random variable; C denotes a positive constant, possibly varying from place to place, the notions $a_n \sim b_n$ $a_n = O(b_n)$, $a_n \asymp b_n$ stand for $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, $\limsup_{n\to\infty} \frac{a_n}{b_n} < \infty$ and $0 < \liminf_{n\to\infty} \frac{a_n}{b_n} \leq \limsup_{n\to\infty} \frac{a_n}{b_n} < \infty$, respectively. We define $\log x = \ln \max\{e, x\}$ and $\log\log x = \ln\ln\max\{e^e, x\}$.

The concept of complete convergence was first introduced by Hsu and Robbins [4], since then there have been extensions in several directions. One of them is to discuss the precise rate and limit value of $\sum_{n=n_0}^{\infty} \varphi(n) P\{|S_n| \geq \varepsilon g(n)\}$ as $\varepsilon \searrow a$, $a \geq 0$, where the weighting function $\varphi(x)$ and boundary function $g(x)$ are positive functions defined on $[n_0, \infty)$. A first result in this direction was given by Heyde [5], who proved that

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n=1}^{\infty} P\{|S_n| \geq \varepsilon n\} = \sigma^2.$$

The research in this field is called the precise asymptotics. For analogous results in more general case, see [6–9] and the references therein.

Another interesting direction is to consider the convergence rate for the precise asymptotic problems. Klesov [10] obtained the following result:

$$\lim_{\varepsilon\searrow 0}\left[\sum_{n=1}^{\infty}P\bigl\{|S_n|\geq \varepsilon n\bigr\}-\frac{\sigma^2}{\varepsilon^2}\right]=-\frac{1}{2},$$

where $\{X, X_n, n \geq 1\}$ is a sequence of i.i.d. normal random variables with $EX = 0$, $EX^2 = \sigma^2$, $S_n = \sum_{k=1}^{n} X_k$.

Recently, Gut and Steinebach [1] obtained the following results:

$$\lim_{\varepsilon\searrow 0}\left[\sum_{n=1}^{\infty}n^{\frac{r}{p}-2}P\left\{\frac{|S_n|}{\sigma\sqrt{n}}\geq \varepsilon n^{\frac{1}{p}-\frac{1}{2}}\right\}-\frac{p}{r-p}\varepsilon^{-\frac{2(r-p)}{2-p})}E|\mathcal{N}|^{\frac{2(r-p)}{2-p}}\right]=\gamma_{\frac{r}{p}-2}-\eta_{r,p},$$

where $1 \leq p < 2$, $p < r < 3p/2$, $E|X|^{\frac{2r}{p}} < \infty$, $\gamma_\theta = \lim_{n\to\infty}(\sum_{j=1}^{n} j^\theta - \frac{n^{\theta+1}}{\theta+1})$, $\eta_{r,p} = \sum_{n=1}^{\infty} n^{\frac{r}{p}-2} \times P\{S_n = 0\}$.

Later, Kong [2] proved the convergence rate in precise asymptotics for the law of iterated logarithm as follows:

$$\lim_{\varepsilon\searrow 0}\left[\sum_{n=3}^{\infty}\frac{1}{n\log n}P\left\{\frac{|S_n|}{\sigma\sqrt{n}}\geq \varepsilon\sqrt{\log\log n}\right\}-\varepsilon^{-2}\right]=\gamma-\eta,$$

where $E|X|^q < \infty$, $2 < q \leq 3$, $\gamma = \lim_{n\to\infty}(\sum_{j=3}^{n}\frac{1}{j\log j} - \log\log n)$, and $\eta = \sum_{n=3}^{\infty}\frac{1}{n\log n}P\{S_n = 0\}$.

Also Kong and Dai [3] established the following convergence rate in precise asymptotics for the Davis law of large numbers with $EX^2(\log(1 + |X|))^{1+\delta} < \infty$:

$$\lim_{\varepsilon\searrow 0}\left[\sum_{n=1}^{\infty}\frac{(\log n)^\delta}{n}P\left\{\frac{|S_n|}{\sigma\sqrt{n}}\geq \varepsilon\sqrt{\log n}\right\}-\varepsilon^{-2(\delta+1)}\frac{E|\mathcal{N}|^{2(\delta+1)}}{\delta+1}\right]=\gamma_\delta-\eta_\delta,$$

where $\gamma_\delta = \lim_{n\to\infty}(\sum_{j=1}^{n}\frac{(\log j)^\delta}{j} - \frac{(\log n)^{\delta+1}}{\delta+1})$, $\eta_\delta = \sum_{n=1}^{\infty}\frac{(\log n)^\delta}{n}P\{S_n = 0\}$, and $\delta \geq 0$.

In this note we will extend the scope of the weighting and boundary functions, and give more general convergence rates in precise asymptotics of i.i.d. random variables, which extends and generalizes the above results. The main result of this note is the following.

Theorem 1.1 *Let $g(x)$ be a positive and twice differentiable function defined on $[n_0, \infty)$, which is strictly increasing to ∞. Assume $g'(x)$ is strictly decreasing to 0 and $g'(n) \asymp \frac{1}{n^{\alpha_1}(\log n)^{\alpha_2}(\log\log n)^{\alpha_3}}$ where α_i, $i = 1, 2, 3$ are defined later. Assume that*

(1) $EX^2(\log(1 + |X|)) < \infty$, *for* $\alpha_1 = 1$, $\alpha_2 > 0$, $\alpha_3 \in \mathbb{R}$;

(2) $EX^2(\log(1 + |X|))^{1-\alpha_2} < \infty$, *for* $\alpha_1 = 1$, $\alpha_2 \leq 0$, $\alpha_3 \geq 0$;

(3) $E|X|^{4-2\alpha_1} < \infty$, *for* $1/2 < \alpha_1 < 1$, $\alpha_2 \geq 0$, $\alpha_3 \geq 0$.

Then for any $s > 0$, we have

$$\lim_{\varepsilon\searrow 0}\left[\sum_{n=n_0}^{\infty}g'(n)P\left\{\frac{|S_n|}{\sigma\sqrt{n}}\geq \varepsilon g^s(n)\right\}-\varepsilon^{-1/s}E|\mathcal{N}|^{1/s}\right]=\gamma_g-\eta_g, \tag{1.1}$$

where $\gamma_g = \lim_{n\to\infty}(\sum_{k=n_0}^{n} g'(k) - g(n))$, $\eta_g = \sum_{n=n_0}^{\infty} g'(n)P\{S_n = 0\}$.

Remark 1.2 There are many functions satisfying the assumptions of $g(x)$, such as $g(x) = x^{\beta_1}(\log x)^{\beta_2}(\log\log x)^{\beta_3}$ with some suitable conditions of β_i, $i = 1, 2, 3$. The following corollaries are some typical examples.

Corollary 1.3 *Let* $g(x) = (\log x)^{\delta+1}$, $\delta > -1$, *and* $s > 0$ *in Theorem* 1.1. *If* $EX^2(\log(1 + |X|))^{\max\{1,1+\delta\}} < \infty$, *then*

$$\lim_{\varepsilon\searrow 0}\left[\sum_{n=1}^{\infty}\frac{(\log n)^{\delta}}{n}P\left\{\frac{|S_n|}{\sigma\sqrt{n}} \geq \varepsilon(\log n)^{(\delta+1)s}\right\} - \varepsilon^{-1/s}\frac{E|\mathcal{N}|^{1/s}}{\delta+1}\right] = \gamma_\delta - \eta_\delta,$$

where $\gamma_\delta = \lim_{n\to\infty}(\sum_{j=1}^{n}\frac{(\log j)^{\delta}}{j} - \frac{(\log n)^{\delta+1}}{\delta+1})$, $\eta_\delta = \sum_{n=1}^{\infty}\frac{(\log n)^{\delta}}{n}P\{S_n = 0\}$.

Corollary 1.4 *Let* $g(x) = (\log\log x)^{b+1}$, $b > -1$, *and* $s > 0$ *in Theorem* 1.1. *If* $EX^2(\log(1 + |X|)) < \infty$, *then*

$$\lim_{\varepsilon\searrow 0}\left[\sum_{n=1}^{\infty}\frac{(\log n)^{b}}{n\log n}P\left\{\frac{|S_n|}{\sigma\sqrt{n}} \geq \varepsilon(\log\log n)^{(b+1)s}\right\} - \varepsilon^{-1/s}\frac{E|\mathcal{N}|^{1/s}}{b+1}\right] = \gamma_b - \eta_b,$$

where $\gamma_b = \lim_{n\to\infty}(\sum_{j=1}^{n}\frac{(\log j)^{b}}{j\log j} - \frac{(\log\log n)^{b+1}}{b+1})$, $\eta_b = \sum_{n=1}^{\infty}\frac{(\log n)^{b}}{n\log n}P\{S_n = 0\}$.

Corollary 1.5 *Let* $g(x) = x^{\frac{r}{p}-1}$, $0 < p < 2$, $p < r < 3p/2$, *and* $s > 0$ *in Theorem* 1.1. *If* $E|X|^{\frac{2r}{p}} < \infty$, *then*

$$\lim_{\varepsilon\searrow 0}\left[\sum_{n=1}^{\infty}n^{\frac{r}{p}-2}P\left\{\frac{|S_n|}{\sigma\sqrt{n}} \geq \varepsilon n^{\frac{(r-p)s}{p}}\right\} - \frac{p}{r-p}\varepsilon^{-1/s}E|\mathcal{N}|^{1/s}\right] = \gamma_{\frac{r}{p}-2} - \varrho_{r,p},$$

where $\gamma_\theta = \lim_{n\to\infty}(\sum_{j=1}^{n}j^{\theta} - \frac{n^{\theta+1}}{\theta+1})$, $\varrho_{r,p} = \sum_{n=1}^{\infty}n^{\frac{r}{p}-2}P\{S_n = 0\}$.

Corollary 1.6 *Let* $g(x) = \frac{x^a}{(\log x)^b}$, $0 < a < 1/2$, $b > 0$, *and* $s > 0$ *in Theorem* 1.1. *If* $E|X|^{2(1+a)} < \infty$, *then*

$$\lim_{\varepsilon\searrow 0}\left[\sum_{n=1}^{\infty}\left[a - \frac{b}{\log n}\right]\frac{1}{n^{1-a}(\log n)^b}P\left\{\frac{|S_n|}{\sigma\sqrt{n}} \geq \varepsilon\frac{n^{as}}{(\log n)^{bs}}\right\} - \varepsilon^{-1/s}E|\mathcal{N}|^{1/s}\right] = \gamma_{a,b} - \varrho_{a,b},$$

where $\gamma_{a,b} = \lim_{n\to\infty}(\sum_{j=1}^{n}[a - \frac{b}{\log j}]\frac{1}{j^{1-a}(\log j)^b} - \frac{n^a}{(\log n)^b})$ *and* $\varrho_{a,b} = \sum_{n=1}^{\infty}[a - \frac{b}{\log n}]\frac{1}{n^{1-a}(\log n)^b} \times P\{S_n = 0\}$.

Remark 1.7 Obviously, Corollary 1.3 with $s = \frac{1}{2(\delta+1)}$ extends Theorem 1.1 in Kong and Dai [3] with the scope of δ from $\delta \geq 0$ to $\delta > -1$; Corollary 1.4 with $s = \frac{1}{2(b+1)}$ extends Theorem 1 from Kong [2] with the scope of b from $b = 0$ to $b > -1$ and the moment condition from $E|X|^q < \infty$ $(2 < q \leq 3)$ to $EX^2(\log(1 + |X|)) < \infty$; Corollary 1.5 with $s = \frac{2-p}{2(r-p)}$ extends Theorem 2.2(a) from Gut and Steinebach [1] with the scope of r, p from $1 \leq p < 2$, $p < r < 3p/2$ to $0 < p < 2$, $p < r < 3p/2$. Therefore our results extend the known results.

2 Proof of Theorem 1.1

The following lemmas are useful for the proof of Theorem 1.1.

Lemma 2.1 *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX = 0$, $EX^2 = \sigma^2$ and set $S_n = \sum_{k=1}^{n} X_k$. Then*

$$\sum_{n=1}^{\infty} \frac{1}{n^{1-\delta/2}} \sup_x \left| P\left\{ \frac{S_n}{\sigma\sqrt{n}} \leq x \right\} - P\{\mathcal{N} \leq x\} \right| < \infty,$$

if and only if $E|X|^{2+\delta} < \infty$ for $0 < \delta < 1$. Also

$$\sum_{n=1}^{\infty} \frac{(\log n)^{\delta}}{n} \sup_x \left| P\left\{ \frac{S_n}{\sigma\sqrt{n}} \leq x \right\} - P\{\mathcal{N} \leq x\} \right| < \infty,$$

if and only if $E|X|^2(\log(1+|X|))^{1+\delta} < \infty$ for $\delta \geq 0$.

Proof The first part can be found in a theorem from Heyde [11] or Theorem 1 from Heyde and Leslie [12] (with $k = 0$), the second part can be found in Proposition 3.2 from Kong and Dai [3]. □

Lemma 2.2 *Under the conditions of Theorem 1.1, we have*

$$\sum_{n=n_0}^{\infty} g'(n) \sup_x \left| P\left\{ \frac{S_n}{\sigma\sqrt{n}} \leq x \right\} - P\{\mathcal{N} \leq x\} \right| < \infty.$$

Proof If $\alpha_1 = 1$, $\alpha_2 > 0$, $\alpha_3 \in \mathbb{R}$, we know that $EX^2 \log(1+|X|) < \infty$, and then, by the second part of Lemma 2.1 (with $\delta = 0$),

$$\begin{aligned}
&\sum_{n=n_0}^{\infty} g'(n) \sup_x \left| P\left\{ \frac{S_n}{\sigma\sqrt{n}} \leq x \right\} - P\{\mathcal{N} \leq x\} \right| \\
&\quad \leq \sum_{n=n_0}^{\infty} \frac{C}{n(\log n)^{\alpha_2}(\log\log n)^{\alpha_3}} \sup_x \left| P\left\{ \frac{S_n}{\sigma\sqrt{n}} \leq x \right\} - P\{\mathcal{N} \leq x\} \right| \\
&\quad \leq \sum_{n=n_0}^{\infty} \frac{C}{n} \sup_x \left| P\left\{ \frac{S_n}{\sigma\sqrt{n}} \leq x \right\} - P\{\mathcal{N} \leq x\} \right| < \infty.
\end{aligned}$$

If $\alpha_1 = 1$, $\alpha_2 \leq 0$, $\alpha_3 \geq 0$, we know that $EX^2(\log(1+|X|))^{1-\alpha_2} < \infty$, and then, by the second part of Lemma 2.1 (with $\delta = -\alpha_2 \geq 0$),

$$\begin{aligned}
&\sum_{n=n_0}^{\infty} g'(n) \sup_x \left| P\left\{ \frac{S_n}{\sigma\sqrt{n}} \leq x \right\} - P\{\mathcal{N} \leq x\} \right| \\
&\quad \leq \sum_{n=n_0}^{\infty} \frac{C}{n(\log n)^{\alpha_2}(\log\log n)^{\alpha_3}} \sup_x \left| P\left\{ \frac{S_n}{\sigma\sqrt{n}} \leq x \right\} - P\{\mathcal{N} \leq x\} \right| \\
&\quad \leq \sum_{n=n_0}^{\infty} \frac{C(\log n)^{-\alpha_2}}{n} \sup_x \left| P\left\{ \frac{S_n}{\sigma\sqrt{n}} \leq x \right\} - P\{\mathcal{N} \leq x\} \right| < \infty.
\end{aligned}$$

If $1/2 < \alpha_1 < 1$, $\alpha_2 \geq 0$, $\alpha_3 \geq 0$, we know that $E|X|^{4-2\alpha_1} = E|X|^{2+\delta} < \infty$ (with $\delta = 2(1-\alpha_1) \in (0,1)$), and then, by the first part of Lemma 2.1,

$$\begin{aligned}
&\sum_{n=n_0}^{\infty} g'(n) \sup_x \left| P\left\{ \frac{S_n}{\sigma\sqrt{n}} \leq x \right\} - P\{\mathcal{N} \leq x\} \right| \\
&\quad \leq \sum_{n=n_0}^{\infty} \frac{1}{n^{\alpha_1}(\log n)^{\alpha_2}(\log\log n)^{\alpha_3}} \sup_x \left| P\left\{ \frac{S_n}{\sigma\sqrt{n}} \leq x \right\} - P\{\mathcal{N} \leq x\} \right| \\
&\quad \leq \sum_{n=n_0}^{\infty} \frac{C}{n^{\alpha_1}} \sup_x \left| P\left\{ \frac{S_n}{\sigma\sqrt{n}} \leq x \right\} - P\{\mathcal{N} \leq x\} \right| \\
&\quad \leq \sum_{n=n_0}^{\infty} \frac{C}{n^{1-\delta/2}} \sup_x \left| P\left\{ \frac{S_n}{\sigma\sqrt{n}} \leq x \right\} - P\{\mathcal{N} \leq x\} \right| < \infty.
\end{aligned}$$

The proof of Lemma 2.2 is completed. □

The existence and finiteness of γ_g and η_g can be obtained by the following two lemmas.

Lemma 2.3 *Under the conditions of Theorem 1.1, we have*

$$\gamma_{n,g} = \gamma_g + O\big(g'(n)\big),$$

where $\gamma_{n,g} = \sum_{k=n_0}^{n} g'(k) - g(n)$ *and* γ_g *is a constant depending only on function g satisfying*

$$-g(n_0) \leq \gamma_g \leq g'(n_0) - g(n_0).$$

Proof Note that $g(x)$ is a positive and twice differentiable function defined on $[n_0, \infty)$, which is strictly increasing to ∞; $g'(x)$ is strictly decreasing to 0. Then, by the mean value theorem for $g(x)$, we know

$$\gamma_{n+1,g} - \gamma_{n,g} = g'(n+1) - \big(g(n+1) - g(n)\big) = g'(n+1) - g'(\xi_n) \leq 0,$$

where $n < \xi_n < n+1$, then we obtain that $\{\gamma_{n,g}, n \geq n_0\}$ is a decreasing sequence. Note that $g'(x)$ is strictly decreasing to 0, and then we have

$$\gamma_{n,g} = \sum_{k=n_0+1}^{n} \int_{k-1}^{k} \big[g'(k) - g'(x)\big]\,dx + g'(n_0) - g(n_0) \leq g'(n_0) - g(n_0),$$

and

$$\begin{aligned}
\gamma_{n,g} &= \sum_{k=n_0+1}^{n} \int_{k-1}^{k} \big[g'(k) - g'(x)\big]\,dx + g'(n_0) - g(n_0) \\
&\geq \sum_{k=n_0+1}^{n} \big[g'(k) - g'(k-1)\big] + g'(n_0) - g(n_0) \\
&= g'(n) - g'(n_0) + g'(n_0) - g(n_0) \geq -g(n_0), \quad \text{as } n \to \infty.
\end{aligned}$$

The above two inequalities mean that $\{\gamma_{n,g}, n \geq n_0\}$ is a bounded sequence, so, by the monotone bounded sequence theorem, also that $\{\gamma_{n,g}, n \geq n_0\}$ is a convergent sequence, and therefore $-g(n_0) \leq \gamma_g \leq g'(n_0) - g(n_0)$.

Finally, for any $m > n$, by the monotonicity of $g'(x)$, we have

$$\begin{aligned}
\gamma_g - \gamma_{n,g} &= \lim_{m\to\infty} [\gamma_{m,g} - \gamma_{n,g}] = \lim_{m\to\infty} \left[\sum_{k=n+1}^{m} g'(k) - \int_n^m g'(x)\,dx \right] \\
&= \lim_{m\to\infty} \left[\sum_{k=n+1}^{m} \int_{k-1}^{k} \bigl[g'(k) - g'(x)\bigr]\,dx \right] \\
&\geq \lim_{m\to\infty} \left[\sum_{k=n+1}^{m} \bigl[g'(k) - g'(k-1)\bigr] \right] \\
&= \lim_{m\to\infty} \bigl[g'(m) - g'(n)\bigr] = -g'(n),
\end{aligned}$$

which means that Lemma 2.3 holds since the sequence $\{\gamma_{n,g}, n \geq n_0\}$ is decreasing. □

Lemma 2.4 *Under the conditions of Theorem 1.1, we have*

$$\eta_g = \sum_{n=n_0}^{\infty} g'(n) P\{S_n = 0\} < \infty.$$

Proof Note that

$$P\left\{ |\mathcal{N}| < \frac{1}{n^2} \right\} = \sqrt{\frac{2}{\pi}} \int_0^{\frac{1}{n^2}} e^{-t^2/2}\,dt \leq C\frac{1}{n^2},$$

so, by Lemma 2.2 and the monotonicity of $g'(x)$, we know

$$\begin{aligned}
\eta_g &= \sum_{n=n_0}^{\infty} g'(n) P\{S_n = 0\} \leq \sum_{n=n_0}^{\infty} g'(n) P\left\{ \frac{|S_n|}{\sigma\sqrt{n}} < \frac{1}{n^2} \right\} \\
&\leq \sum_{n=n_0}^{\infty} g'(n) \left| P\left\{ \frac{|S_n|}{\sigma\sqrt{n}} < \frac{1}{n^2} \right\} - P\left\{ |\mathcal{N}| < \frac{1}{n^2} \right\} \right| + \sum_{n=n_0}^{\infty} g'(n) P\left\{ |\mathcal{N}| < \frac{1}{n^2} \right\} \\
&\leq \sum_{n=n_0}^{\infty} g'(n) \sup_x \left| P\left\{ \frac{|S_n|}{\sigma\sqrt{n}} \leq x \right\} - P\bigl\{ |\mathcal{N}| \leq x \bigr\} \right| + \sum_{n=n_0}^{\infty} C g'(n) \frac{1}{n^2} < \infty.
\end{aligned}$$

□

Remark 2.5 Obviously, if X is a continuous random variable, then $\eta_g = 0$. For the simplest discrete case, if we take $P(X = 1) = P(X = -1) = \frac{1}{2}$, it is easy to check that $P\{S_{2n+1} = 0\} = 0$ and $P\{S_{2n} = 0\} = C_{2n}^{n} \frac{1}{2^{2n}} \sim \frac{1}{\sqrt{\pi}} \frac{1}{n^{1/2}}$; therefore, $0 < \eta_g < \infty$.

Lemma 2.6 *Under the conditions of Theorem 1.1, we have*

$$\lim_{\varepsilon \searrow 0} \left[\frac{2}{\sqrt{2\pi}} \sum_{k=n_0}^{\infty} g(k) \int_{\varepsilon g^s(k)}^{\varepsilon g^s(k+1)} e^{-t^2/2}\,dt - \varepsilon^{-1/s} E|\mathcal{N}|^{1/s} \right] = 0, \tag{2.1}$$

$$\lim_{\varepsilon \searrow 0} \frac{2\gamma_g}{\sqrt{2\pi}} \sum_{k=n_0}^{\infty} \int_{\varepsilon g^s(k)}^{\varepsilon g^s(k+1)} e^{-t^2/2}\,dt = \gamma_g, \tag{2.2}$$

$$\lim_{\varepsilon\searrow 0}\frac{2}{\sqrt{2\pi}}\sum_{k=n_0}^{\infty}g'(k)\int_{\varepsilon g^s(k)}^{\varepsilon g^s(k+1)}e^{-t^2/2}\,dt=0. \tag{2.3}$$

Proof By the mean value theorem for integrals, there exists a constant $\xi_k\in(k,k+1)$ such that

$$\int_{\varepsilon g^s(k)}^{\varepsilon g^s(k+1)}e^{-t^2/2}\,dt=\varepsilon\bigl[g^s(k+1)-g^s(k)\bigr]e^{-\varepsilon^2g^{2s}(\xi_k)/2}. \tag{2.4}$$

Using Taylor expansion, we know

$$\begin{aligned}&e^{-\varepsilon^2g^{2s}(\xi_k)/2}=e^{-\varepsilon^2g^{2s}(k)/2}+\varepsilon^2O\bigl(g^{2s-1}(k)g'(k)e^{-\varepsilon^2g^{2s}(k)/2}\bigr),\\&g^s(k+1)-g^s(k)=sg^{s-1}(k)g'(k)+O\bigl(g^{s-2}(k)\bigl(g'(k)\bigr)^2-g^{s-1}(k)g''(k)\bigr).\end{aligned}$$

Therefore we have

$$\begin{aligned}\int_{\varepsilon g^s(k)}^{\varepsilon g^s(k+1)}e^{-t^2/2}\,dt={}&s\varepsilon g^{s-1}(k)g'(k)e^{-\varepsilon^2g^{2s}(k)/2}\\&+\varepsilon O\bigl(\bigl(g^{s-2}(k)\bigl(g'(k)\bigr)^2-g^{s-1}(k)g''(k)\bigr)e^{-\varepsilon^2g^{2s}(k)/2}\bigr)\\&+\varepsilon^2O\bigl(g^{3s-2}(k)\bigl[g'(k)\bigr]^2e^{-\varepsilon^2g^{2s}(k)/2}\bigr).\end{aligned} \tag{2.5}$$

By the monotonicity of $g(x)$ and $g'(x)$, we have

$$\begin{aligned}&\frac{2s\varepsilon}{\sqrt{2\pi}}\sum_{k=n_0}^{\infty}g(k)g^{s-1}(k)g'(k)e^{-\varepsilon^2g^{2s}(k)/2}\\&\quad=\frac{2s\varepsilon}{\sqrt{2\pi}}\int_{n_0}^{\infty}g^s(x)g'(x)e^{-\varepsilon^2g^{2s}(x)/2}\,dx+O(\varepsilon)\\&\quad=\frac{2s\varepsilon}{\sqrt{2\pi}}\int_{g(n_0)}^{\infty}y^s(x)e^{-\varepsilon^2y^{2s}/2}\,dy+O(\varepsilon)\\&\quad=\varepsilon^{-1/s}\frac{2^{\frac{1}{2s}}}{\sqrt{\pi}}\int_{\varepsilon^2g^{2s}(n_0)/2}^{\infty}t^{\frac{1}{2s}-\frac{1}{2}}e^{-t}\,dt+O(\varepsilon)\\&\quad=\varepsilon^{-1/s}\frac{2^{\frac{1}{2s}}}{\sqrt{\pi}}\int_0^{\infty}t^{\frac{1}{2s}-\frac{1}{2}}e^{-t}\,dt-\varepsilon^{-1/s}\frac{2^{\frac{1}{2s}}}{\sqrt{\pi}}\int_0^{\varepsilon^2g^{2s}(n_0)/2}t^{\frac{1}{2s}-\frac{1}{2}}e^{-t}\,dt+O(\varepsilon)\\&\quad=\varepsilon^{-1/s}\frac{2^{\frac{1}{2s}}}{\sqrt{\pi}}\Gamma\biggl(\frac{1}{2s}+\frac{1}{2}\biggr)+O(\varepsilon)=\varepsilon^{-1/s}E|\mathcal{N}|^{1/s}+O(\varepsilon).\end{aligned} \tag{2.6}$$

Since $g'(x)$ is strictly decreasing to 0, there exists a constant k_0 such that $g'(k)<\delta$, for any $k>k_0$. Then by monotonicity of $g(x)$ and $g'(x)$, we obtain

$$\begin{aligned}&\lim_{\varepsilon\searrow 0}\biggl[\frac{2\varepsilon}{\sqrt{2\pi}}\sum_{k=n_0}^{\infty}g(k)g^{s-2}(k)\bigl[g'(k)\bigr]^2e^{-\varepsilon^2g^{2s}(k)/2}\biggr]\\&\quad\le\lim_{\varepsilon\searrow 0}\biggl[\frac{2\varepsilon}{\sqrt{2\pi}}\sum_{k=n_0}^{k_0}g^{s-1}(k)\bigl[g'(k)\bigr]^2e^{-\varepsilon^2g^{2s}(k)/2}\biggr]\end{aligned}$$

$$
\begin{aligned}
&\quad + \lim_{\varepsilon\searrow 0}\left[\frac{2\varepsilon\delta}{\sqrt{2\pi}}\sum_{k=k_0+1}^{\infty} g^{s-1}(k)\big[g'(k)\big]e^{-\varepsilon^2 g^{2s}(k)/2}\right] \\
&\leq \lim_{\varepsilon\searrow 0} C\varepsilon\delta\int_{k_0}^{\infty} g^{s-1}(x)g'(x)e^{-\varepsilon^2 g^{2s}(x)/2}\,dx \\
&= \lim_{\varepsilon\searrow 0} C\varepsilon\delta\int_{g(k_0)}^{\infty} y^{s-1}e^{-\varepsilon^2 y^{2s}/2}\,dy \\
&\leq \lim_{\varepsilon\searrow 0} C\delta\int_{\varepsilon^2 g^{2s}(k_0)/2}^{\infty} t^{-1/2}e^{-t}\,dy \\
&= C\delta\int_0^{\infty} t^{-1/2}e^{-t}\,dy = C\delta\Gamma\left(\frac{1}{2}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\varepsilon\searrow 0}\left[\frac{2\varepsilon^3}{\sqrt{2\pi}}\sum_{k=n_0}^{\infty} g(k)g^{3s-2}(k)\big[g'(k)\big]^2 e^{-\varepsilon^2 g^{2s}(k)/2}\right] \\
&\leq \lim_{\varepsilon\searrow 0} C\varepsilon^3\delta\int_{k_0}^{\infty} g^{3s-1}(x)g'(x)e^{-\varepsilon^2 g^{2s}(x)/2}\,dx \\
&= \lim_{\varepsilon\searrow 0} C\varepsilon^3\delta\int_{g(k_0)}^{\infty} y^{3s-1}e^{-\varepsilon^2 y^{2s}/2}\,dy \\
&\leq \lim_{\varepsilon\searrow 0} C\delta\int_{\varepsilon^2 g^{2s}(k_0)/2}^{\infty} t^{1/2}e^{-t}\,dy \\
&= C\delta\int_0^{\infty} t^{1/2}e^{-t}\,dy = C\delta\Gamma\left(\frac{3}{2}\right).
\end{aligned}
$$

Then, by the arbitrariness of δ and letting $\delta\to 0$, we derive

$$
\lim_{\varepsilon\searrow 0}\left[\frac{2\varepsilon}{\sqrt{2\pi}}\sum_{k=n_0}^{\infty} g(k)g^{s-2}(k)\big[g'(k)\big]^2 e^{-\varepsilon^2 g^{2s}(k)/2}\right] = 0, \tag{2.7}
$$

$$
\lim_{\varepsilon\searrow 0}\left[\frac{2\varepsilon^3}{\sqrt{2\pi}}\sum_{k=n_0}^{\infty} g(k)g^{3s-2}(k)\big[g'(k)\big]^2 e^{-\varepsilon^2 g^{2s}(k)/2}\right] = 0. \tag{2.8}
$$

From the fact that $g'(x)$ is strictly decreasing to 0, we know $g''(x)\leq 0$, and then, by using integration by parts, similar to the above discussion, one can get

$$
\begin{aligned}
&\lim_{\varepsilon\searrow 0} -\varepsilon\int_{n_0}^{\infty} g^{s}(x)g''(x)e^{-\varepsilon^2 g^{2s}(x)/2}\,dx \\
&= \lim_{\varepsilon\searrow 0} -\varepsilon\int_{n_0}^{\infty} g^{s}(x)e^{-\varepsilon^2 g^{2s}(x)/2}\,dg'(x) \\
&= \lim_{\varepsilon\searrow 0} -\varepsilon\big[g^{s}(x)e^{-\varepsilon^2 g^{2s}(x)/2}g'(x)\big]\Big|_{n_0}^{\infty} \\
&\quad + \lim_{\varepsilon\searrow 0}\varepsilon\int_{n_0}^{\infty} g'(x)\big[sg^{s-1}(x)g'(x) - s\varepsilon^2 g^{2s-1}(x)g'(x)\big]e^{-\varepsilon^2 g^{2s}(x)/2}\,dx \\
&= s\lim_{\varepsilon\searrow 0}\varepsilon\int_{n_0}^{\infty} g^{s-1}(x)\big[g'(x)\big]^2 e^{-\varepsilon^2 g^{2s}(x)/2}\,dx - s\lim_{\varepsilon\searrow 0}\varepsilon^3\int_{n_0}^{\infty} g^{2s-1}(x)\big[g'(x)\big]^2 e^{-\varepsilon^2 g^{2s}(x)/2}\,dx
\end{aligned}
$$

$$\begin{aligned}
&\leq s\delta \lim_{\varepsilon \searrow 0} \varepsilon \int_{k_0}^{\infty} g^{s-1}(x) g'(x) e^{-\varepsilon^2 g^{2s}(x)/2}\,dx + s\delta \lim_{\varepsilon \searrow 0} \varepsilon^3 \int_{k_0}^{\infty} g^{2s-1}(x) g'(x) e^{-\varepsilon^2 g^{2s}(x)/2}\,dx \\
&\leq C\delta.
\end{aligned}$$

Then, by letting $\delta \to 0$, we can derive

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \left[\frac{-2\varepsilon}{\sqrt{2\pi}} \sum_{k=n_0}^{\infty} g(k) g^{s-1}(k) g''(k) e^{-\varepsilon^2 g^{2s}(k)/2} \right] \\
&\quad \leq \lim_{\varepsilon \searrow 0} \left[-C\varepsilon \int_{n_0}^{\infty} g^{s}(x) g''(x) e^{-\varepsilon^2 g^{2s}(x)/2}\,dx \right] = 0.
\end{aligned} \tag{2.9}$$

Finally, (2.1) can be obtained by combining (2.4)–(2.9).

It is obvious that

$$\lim_{\varepsilon \searrow 0} \frac{2\gamma_g}{\sqrt{2\pi}} \sum_{k=n_0}^{\infty} \int_{\varepsilon g^s(k)}^{\varepsilon g^s(k+1)} e^{-t^2/2}\,dt = \lim_{\varepsilon \searrow 0} \frac{2\gamma_g}{\sqrt{2\pi}} \int_{\varepsilon g^s(n_0)}^{\infty} e^{-t^2/2}\,dt = \frac{2\gamma_g}{\sqrt{2\pi}} \int_0^{\infty} e^{-t^2/2}\,dt = \gamma_g,$$

thus (2.2) is proved.

Since $g'(x)$ is strictly decreasing to 0, there exists a constant k_0 such that $g'(k) < \delta$, for any $k > k_0$, and

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \frac{2}{\sqrt{2\pi}} \sum_{k=n_0}^{\infty} g'(k) \int_{\varepsilon g^s(k)}^{\varepsilon g^s(k+1)} e^{-t^2/2}\,dt \\
&\quad \leq \lim_{\varepsilon \searrow 0} CP\bigl\{|\mathcal{N}| \leq \varepsilon g^s(k_0+1)\bigr\} + \delta P\bigl\{|\mathcal{N}| \geq \varepsilon g^s(k_0+1)\bigr\} = \delta.
\end{aligned}$$

Thus (2.3) holds in view of the arbitrariness of δ. This completes the proof. □

Lemma 2.7 *Under the conditions of Theorem 1.1, we have*

$$\lim_{\varepsilon \searrow 0} \left[\sum_{k=n_0}^{\infty} g'(k) P\bigl\{|\mathcal{N}| \geq \varepsilon g^s(k)\bigr\} - \varepsilon^{-1/s} E|\mathcal{N}|^{1/s} \right] = \gamma_g.$$

Proof By Fubini's theorem, Lemmas 2.3 and 2.6, we derive that

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \left[\sum_{k=n_0}^{\infty} g'(k) P\bigl\{|\mathcal{N}| \geq \varepsilon g^s(k)\bigr\} - \varepsilon^{-1/s} E|\mathcal{N}|^{1/s} \right] \\
&\quad = \lim_{\varepsilon \searrow 0} \left[\frac{2}{\sqrt{2\pi}} \sum_{k=n_0}^{\infty} g'(k) \sum_{j=k}^{\infty} \int_{\varepsilon g^s(j)}^{\varepsilon g^s(j+1)} e^{-t^2/2}\,dt - \varepsilon^{-1/s} E|\mathcal{N}|^{1/s} \right] \\
&\quad = \lim_{\varepsilon \searrow 0} \left[\frac{2}{\sqrt{2\pi}} \sum_{j=n_0}^{\infty} \sum_{k=n_0}^{j} g'(k) \int_{\varepsilon g^s(j)}^{\varepsilon g^s(j+1)} e^{-t^2/2}\,dt - \varepsilon^{-1/s} E|\mathcal{N}|^{1/s} \right] \\
&\quad = \lim_{\varepsilon \searrow 0} \left[\frac{2}{\sqrt{2\pi}} \sum_{j=n_0}^{\infty} g(j) \int_{\varepsilon g^s(j)}^{\varepsilon g^s(j+1)} e^{-t^2/2}\,dt - \varepsilon^{-1/s} E|\mathcal{N}|^{1/s} \right]
\end{aligned}$$

$$
\begin{aligned}
&+\lim_{\varepsilon\searrow 0}\frac{2}{\sqrt{2\pi}}\sum_{j=n_0}^{\infty}\gamma_g\int_{\varepsilon g^s(j)}^{\varepsilon g^s(j+1)}e^{-t^2/2}\,dt\\
&+\lim_{\varepsilon\searrow 0}\frac{2}{\sqrt{2\pi}}\sum_{j=n_0}^{\infty}O\bigl(g'(j)\bigr)\int_{\varepsilon g^s(j)}^{\varepsilon g^s(j+1)}e^{-t^2/2}\,dt\\
=\ &\gamma_g. \qquad\square
\end{aligned}
$$

Lemma 2.8 *Under the conditions of Theorem* 1.1, *we have*

$$
\lim_{\varepsilon\searrow 0}\sum_{k=n_0}^{\infty}g'(k)\left[P\left\{\frac{|S_k|}{\sigma\sqrt{k}}\geq\varepsilon g^s(k)\right\}-P\bigl\{|\mathcal{N}|\geq\varepsilon g^s(k)\bigr\}\right]=-\eta_g.
$$

Proof By Lemma 2.2, we have

$$
\begin{aligned}
&\sum_{k=n_0}^{\infty}g'(k)\left|P\left\{\frac{|S_k|}{\sigma\sqrt{k}}\geq\varepsilon g^s(k)\right\}-P\bigl\{|\mathcal{N}|\geq\varepsilon g^s(k)\bigr\}\right|\\
&\quad\leq\sum_{k=n_0}^{\infty}g'(k)\sup_x\left|P\left\{\frac{|S_k|}{\sigma\sqrt{k}}\geq x\right\}-P\bigl\{|\mathcal{N}|\geq x\bigr\}\right|<\infty.
\end{aligned}
$$

Then, by the dominated convergence theorem and continuity of $\mathcal{N}$, we have

$$
\begin{aligned}
&\lim_{\varepsilon\searrow 0}\sum_{k=n_0}^{\infty}g'(k)\left[P\left\{\frac{|S_k|}{\sigma\sqrt{k}}\geq\varepsilon g^s(k)\right\}-P\bigl\{|\mathcal{N}|\geq\varepsilon g^s(k)\bigr\}\right]\\
&\quad=\lim_{\varepsilon\searrow 0}\sum_{k=n_0}^{\infty}g'(k)\left[P\bigl\{|\mathcal{N}|<\varepsilon g^s(k)\bigr\}-P\left\{\frac{|S_k|}{\sigma\sqrt{k}}<\varepsilon g^s(k)\right\}\right]\\
&\quad=\sum_{k=n_0}^{\infty}g'(k)\lim_{\varepsilon\searrow 0}\left[P\bigl\{|\mathcal{N}|<\varepsilon g^s(k)\bigr\}-P\left\{\frac{|S_k|}{\sigma\sqrt{k}}<\varepsilon g^s(k)\right\}\right]\\
&\quad=-\sum_{k=n_0}^{\infty}g'(k)P\{S_k=0\}=-\eta_g. \qquad\square
\end{aligned}
$$

Proof of Theorem 1.1 By combining Lemmas 2.7 and 2.8, we obtain Theorem 1.1. □

3 Conclusions

In this paper, using the rate of convergence to the normal distribution and Fubini theorem, under some suitable conditions, the convergence rates in precise asymptotics for the complete convergence have been discussed with more general boundary functions. The result extends and generalizes the corresponding results of Gut and Steinebach [1], Kong [2], and Kong and Dai [3]. However, this paper has only studied the convergence rates for complete convergence. In the future research, we will discuss the convergence rates in precise asymptotics for complete moment convergence, which was first studied by Liu and Lin [13], as it is more difficult to handle the moment terms.

Acknowledgements
The author greatly appreciates both the Editors and the referees for their valuable comments and some helpful suggestions that improved the clarity and readability of this paper.

Funding
The paper was supported by National Natural Science Foundation of China (Grant No. 11771178); the Science and Technology Development Program of Jilin Province (Grant No. 20170101152JC) and Science and Technology Program of Jilin Educational Department during the "13th Five-Year" Plan Period (Grant No. JJKH20180110KJ).

Competing interests
The author declares to have no competing interests.

Authors' contributions
This is a single-authored paper. The author read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 20 August 2018 Accepted: 15 January 2019 Published online: 18 January 2019

References
1. Gut, A., Steinebach, J.: Convergence rates in precise asymptotics II. Ann. Univ. Sci. Budapest. Sect. Comput. **39**, 95–110 (2013)
2. Kong, L.T.: Convergence rate in precise asymptotics for the law of the iterated logarithm. Lith. Math. J. **56**(3), 318–324 (2016)
3. Kong, L.T., Dai, H.S.: Convergence rate in precise asymptotics for Davis law of large numbers. Stat. Probab. Lett. **119**(10), 295–300 (2016)
4. Hsu, P.L., Robbins, H.: Complete convergence and the law of large numbers. Proc. Natl. Acad. Sci. USA **33**(2), 25–31 (1947)
5. Heyde, C.C.: A supplement to the strong law of large numbers. J. Appl. Probab. **12**(1), 173–175 (1975)
6. Gut, A., Spătaru, A.: Precise asymptotics in the law of the iterated logarithm. Ann. Probab. **28**(4), 1870–1883 (2000)
7. Gut, A., Spătaru, A.: Precise asymptotics in the Baum–Katz and Davis laws of large numbers. J. Math. Anal. Appl. **248**(1), 233–246 (2000)
8. Gut, A., Steinebach, J.: Precise asymptotics—a general approach. Acta Math. Hung. **138**(4), 365–385 (2013)
9. Xie, T.F., He, J.J.: Rate of convergence in a theorem of Heyde. Stat. Probab. Lett. **82**(8), 1576–1582 (2012)
10. Klesov, O.I.: On the convergence rate in a theorem of Heyde. Theory Probab. Math. Stat. **49**, 83–87 (1994)
11. Heyde, C.C.: On the influence of moments on the rate of convergence to the normal distribution. Z. Wahrscheinlichkeitstheor. Verw. Geb. **8**(1), 12–18 (1967)
12. Heyde, C.C., Leslie, J.R.: On the influence of moments on approximations by portion of a Chebyshev series in central limit convergence. Z. Wahrscheinlichkeitstheor. Verw. Geb. **21**(4), 255–268 (1972)
13. Liu, W.D., Lin, Z.Y.: Precise asymptotics for a new kind of complete moment convergence. Stat. Probab. Lett. **76**(16), 1787–1799 (2006)