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New criteria for Carathéodory functions

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Abstract

The object of the present paper is to investigate various conditions for Carathéodory functions in the open unit disk. Also we give some applications to univalent functions as special cases.

MSC: 30C45

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1 Introduction

For given r $(0 < r \leq 1)$, let $\mathbb{U}_r = \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{U} \equiv \mathbb{U}_1$ be the open unit disk, and let us denote by $\mathbb{T} = \partial\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}$ the boundary of $\mathbb{U}$. An analytic function p in $\mathbb{U}$ with $p(0) = 1$ is said to be a Carathéodory function of order α if it satisfies

$$\operatorname{Re}\{p(z)\} > \alpha \quad (0 \leq \alpha < 1, z \in \mathbb{U}).$$

We denote by $\mathcal{P}(\alpha)$ the class of all Carathéodory functions of order α in $\mathbb{U}$ and $\mathcal{P} \equiv \mathcal{P}(0)$ [4]. Let $\mathcal{A}$ denote the class of analytic functions f defined in $\mathbb{U}$ normalized by $f(0) = 0$ and $f'(0) = 1$. Further, we denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclasses of $\mathcal{A}$ consisting of starlike and convex functions of order α in $\mathbb{U}$, respectively. That is, a function $f \in \mathcal{A}$ belongs to the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ if f satisfies $\operatorname{Re}\{zf'(z)/f(z)\} > \alpha$ and $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > \alpha$, respectively, in $\mathbb{U}$.

For analytic functions f and g, we say that f is subordinate to g, denoted by $f \prec g$, if there is an analytic function $w : \mathbb{U} \to \mathbb{U}$ with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. Further, if g is univalent, then the definition of subordination $f \prec g$ simplifies to the conditions $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$ (see [10, p. 36]).

Let us denote by $\mathcal{Q}$ the set of functions q that are analytic and injective on $\overline{\mathbb{U}} \setminus \mathbf{E}(q)$, where

$$\mathbf{E}(q) = \left\{\zeta \in \mathbb{T} : \lim_{z\to\zeta} q(z) = \infty\right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \mathbb{T} \setminus \mathbf{E}(q)$.

Marx [3] and Strohhäcker [12] showed that if $f \in \mathcal{K} \equiv \mathcal{K}(0)$ then $f \in \mathcal{S}^*(1/2)$, that is, $\mathcal{K} \subset \mathcal{S}^*(1/2)$. Later, Miller [4] and Miller, Mocanu and Reade [7] proved the following results, respectively. If p is analytic in $\mathbb{U}$, then

$$\mathrm{Re}\{p(z)+\beta zp'(z)\}>0 \quad (\beta \geq 0) \quad \Longrightarrow \quad p\in\mathcal{P} \tag{1}$$

and

$$\mathrm{Re}\left\{p(z)+\beta\frac{zp'(z)}{p(z)}\right\}>0 \quad \left(\beta\in\mathbb{R}, p(z)\neq 0\right) \quad \Longrightarrow \quad p\in\mathcal{P}. \tag{2}$$

The result given in (2) clearly reduces the earlier works due to Marx and Strohhäcker. Many kinds of functions with geometric properties, such as starlikeness, convexity, close-to-convexity, and so on, are closely related to the class of Carathéodory functions and play a really important role in the study of univalent functions.

In the present paper, we show several new sufficient conditions, which are not connected to some recent results for Carathéodory functions of order α, which incorporate the implications given by (1) and (2). In addition to applying the well known Jack's Lemma, we approach the results in a quite different way than methods used in other papers. Moreover, we obtain other criteria for Carathéodory functions of order α. Many of the earlier results given by Marx [3], Strohhäcker [12] and others are shown here to follow as special cases of the results presented in this paper. Thus the various properties associated with the class $\mathcal{P}(\alpha)$ obtained here can be viewed as extensions and generalizations of numerous previously-obtained results in Geometric Function Theory.

2 Main results

In proving our results, we need the following lemmas due to Jack [2], and Miller and Mocanu [5] (see also [6, p. 24, Lemma 2.2d]).

Lemma 2.1 *Suppose that function w is analytic for $|z|\leq r$, $w(0)=0$ and $|w(z_0)|=\max_{|z|=r}|w(z)|$. Then $z_0w'(z_0)=kw(z_0)$, where k is a real number with $k\geq 1$.*

Lemma 2.2 *Let $q\in\mathcal{Q}$, with $q(0)=a$, and let $p(z)=a+a_nz^n+\cdots$ be analytic in $\mathbb{U}$ with $p(z)\not\equiv a$ and $n\geq 1$. If p is not subordinate to q, then there exist points $z_0=r_0e^{i\theta_0}\in\mathbb{U}$ and $\zeta_0\in\mathbb{T}\setminus\mathbf{E}(q)$, and an $m\geq n\geq 1$ for which $p(\mathbb{U}_{r_0})\subset q(\mathbb{U})$,*

(i) $p(z_0)=q(\zeta_0)$,

(ii) $z_0p'(z_0)=m\zeta_0q'(\zeta_0)$ *and*

(iii) $\mathrm{Re}\{1+\frac{z_0p''(z_0)}{p'(z_0)}\}\geq m\,\mathrm{Re}\{1+\frac{\zeta_0q''(\zeta_0)}{q'(\zeta_0)}\}$.

By using Lemma 2.1, we now derive the following theorem.

Theorem 2.3 *Let p be analytic in $\mathbb{U}$ with $p(0)=1$. If*

$$\mathrm{Re}\{p(z)+\beta zp'(z)\}>\alpha-\frac{\beta}{2(1-\alpha)}\left(1-2\alpha+|p(z)|^2\right) \quad (0\leq\alpha<1, \beta\geq 0), \tag{3}$$

then $p\in\mathcal{P}(\alpha)$.

Proof Define function w by

$$p(z)=\frac{1+(1-2\alpha)w(z)}{1-w(z)} \quad (z\in\mathbb{U}). \tag{4}$$

We know that w is analytic in $\mathbb{U}$ with $w(0) = 0$. Suppose that there exists a point z_0 in $\mathbb{U}$ such that

$$\operatorname{Re}\{p(z)\} > \alpha \quad \text{for } |z| < |z_0| \quad \text{and} \quad \operatorname{Re}\{p(z_0)\} = \alpha. \tag{5}$$

Then we have

$$|w(z)| < 1 \quad \text{for } |z| < |z_0| \quad \text{and} \quad |w(z_0)| = 1. \tag{6}$$

By using Lemma 2.1, we get

$$z_0 w'(z_0) = k w(z_0), \tag{7}$$

where k is a real number with $k \geq 1$. We note that $z_0 p'(z_0)$ is a nonpositive real number, since

$$\frac{1}{2k(1-\alpha)} z_0 p'(z_0) = \frac{w(z_0)}{(1-w(z_0))^2} = \frac{2(\operatorname{Re}\{w(z_0)\} - 1)}{|1-w(z_0)|^4} \tag{8}$$

and, by (6), $\operatorname{Re}\{w(z_0)\} \leq 1$. Moreover, by putting

$$p(z_0) = \alpha + \mathrm{i}y \quad (y \in \mathbb{R}), \tag{9}$$

we obtain

$$w(z_0) = 1 - \frac{2(1-\alpha)^2}{(1-\alpha)^2 + y^2} + \mathrm{i}\frac{2(1-\alpha)y}{(1-\alpha)^2 + y^2} \tag{10}$$

and

$$z_0 p'(z_0) = -k \frac{(1-\alpha)^2 + y^2}{2(1-\alpha)}. \tag{11}$$

Therefore, from equations (9) and (11), we have

$$\begin{aligned} \operatorname{Re}\{p(z_0) + \beta z_0 p'(z_0)\} &= \alpha - \beta k \frac{(1-\alpha)^2 + y^2}{2(1-\alpha)} \\ &\leq \alpha - \frac{\beta}{2(1-\alpha)} \left(1 - 2\alpha + |p(z_0)|^2\right). \end{aligned} \tag{12}$$

This contradicts assumption (3). Therefore we complete the proof of Theorem 2.3. □

Taking $\alpha = 0$ and $\beta = 1$ in Theorem 2.3, we have the following result by Nunokawa et al. [9].

Corollary 2.4 *Let p be analytic in $\mathbb{U}$ with $p(0) = 1$. If*

$$\operatorname{Re}\{p(z) + zp'(z)\} > -\frac{1 + |p(z)|^2}{2},$$

then $p \in \mathcal{P}$.

Remark 2.5 Corollary 2.4 is an improvement of the result by Miller [4].

The right-hand side of assumption (3) in Theorem 2.3 depends on $|p(z)|$. But applying the same method as in the proof of Theorem 2.3 and using the new formula (12) where y is ignored, we can derive a similar result (Theorem 2.6 below) without requiring $|p(z)|$ in assumption (3) of Theorem 2.3.

Theorem 2.6 *Let p be analytic in $\mathbb{U}$ with $p(0) = 1$. If*

$$\operatorname{Re}\{p(z) + \beta zp'(z)\} > \alpha - \frac{\beta(1-\alpha)}{2} \quad (0 \le \alpha < 1, \beta \ge 0),$$

then $p \in \mathcal{P}(\alpha)$.

Letting $\beta = 1$ in Theorem 2.6, we have the following corollary.

Corollary 2.7 *Let p be analytic in $\mathbb{U}$ with $p(0) = 1$. If*

$$\operatorname{Re}\{p(z) + zp'(z)\} > \frac{3\alpha - 1}{2} \quad (0 \le \alpha < 1),$$

then $p \in \mathcal{P}(\alpha)$.

Remark 2.8 Corollary 2.7 is an improvement of the result by Nunokawa [8].

For given γ and c satisfying $\gamma > 0$ and $c > -\gamma$, let us consider an integral operator $I_{c,\gamma} : \mathcal{A} \to \mathcal{A}$ defined by

$$F(z) := I_{c,\gamma}[f](z) = \left(\frac{c+\gamma}{z^c}\int_0^z \xi^{c-1} f^{\gamma}(\xi)\,\mathrm{d}\xi\right)^{1/\gamma}. \tag{13}$$

By taking $p(z) = F'(z)(F(z)/z)^{\gamma-1}$, we have

$$\gamma p(z) + c\left(\frac{F(z)}{z}\right)^{\gamma} = (c+\gamma)\left(\frac{f(z)}{z}\right)^{\gamma}. \tag{14}$$

Moreover, taking derivatives of both sides of (14) leads to the equality

$$f'(z)\left(\frac{f(z)}{z}\right)^{\gamma-1} = p(z) + \frac{1}{c+\gamma} zp'(z).$$

Example 2.9 Taking $p(z) = f'(z)$ in Theorem 2.3 with $\alpha = 0$, $p(z) = f'(z)$ in Theorem 2.6 with $\alpha = 0$ and $\beta = 1$, $p(z) = f(z)/z$ in Theorem 2.3 with $\alpha = 0$ and $\beta = 1$ and $p(z) = F'(z)/(F(z)/z)^{\gamma-1}$, where F is defined in (13), in Theorem 2.6 with $\beta = 1/(c+\gamma)$, respectively, we have the following results: If $f \in \mathcal{A}$, then

(i) $\operatorname{Re}\{f'(z) + \beta zf''(z)\} > -\frac{\beta}{2}(1 + |f'(z)|^2)$ $(\beta > 0)$ implies $\operatorname{Re}\{f'(z)\} > 0$ (cf. [1]);
(ii) $\operatorname{Re}\{f'(z) + zf''(z)\} > -1/2$ implies $\operatorname{Re}\{f'(z)\} > 0$;
(iii) $\operatorname{Re}\{f'(z)\} > -\frac{1}{2}(1 + |f(z)/z|^2)$ implies $\operatorname{Re}\{f(z)/z\} > 0$;
(iv) $\operatorname{Re}\{f'(z)(f(z)/z)^{\gamma-1}\} > \alpha - \frac{1-\alpha}{2(c+\gamma)}$ $(0 \le \alpha < 1, \gamma > 0, c > -\gamma)$ implies $\operatorname{Re}\{F'(z)(F(z)/z)^{\gamma-1}\} > \alpha$, where F is defined as in (13) (cf. [11]).

Theorem 2.10 *Let p be analytic in $\mathbb{U}$ with $p(0) = 1$. If*

$$\operatorname{Re}\left\{p(z) + \frac{zp'(z)}{\beta p(z) + \gamma}\right\} > \delta\big(\alpha, \beta, \gamma, \big|p(z)\big|\big), \tag{15}$$

where

$$\delta\big(\alpha, \beta, \gamma, \big|p(z)\big|\big) = \alpha - \frac{(\alpha\beta + \gamma)(1 - 2\alpha + |p(z)|^2)}{2(1-\alpha)(\gamma^2 + 2\alpha\beta\gamma + \beta^2|p(z)|^2)}$$
$$(0 \le \alpha < 1, \beta \neq 0, \alpha\beta + \gamma > 0), \tag{16}$$

then $p \in \mathcal{P}(\alpha)$.

Proof At first, we note that $p(z) \neq -\gamma/\beta$ for $z \in \mathbb{U}$. In fact, if $\beta p(z) + \gamma$ has a zero of order m at $z = z_1 \in \mathbb{U}$, then we can write

$$\beta p(z) + \gamma = (z - z_1)^m p_1(z) \quad (m \in \mathbb{N}),$$

where p_1 is analytic in $\mathbb{U}$ and $p_1(z_1) \neq 0$. Then we have

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} = \frac{1}{\beta}\left\{(z - z_1)^m p_1(z) - \gamma + \frac{mz}{z - z_1} + \frac{zp_1'(z)}{p_1(z)}\right\}. \tag{17}$$

Thus choosing $z \to z_1$ suitably, the real part of the right-hand side of (17) can take any negative infinite values, which contradicts hypothesis (15). Defining w by (4), we see that function w is analytic in $\mathbb{U}$ with $w(0) = 0$. Suppose that there exists a point $z_0 \in \mathbb{U}$ satisfying (5). Then we have (6). By Lemma 2.1, there exists a real number k with $k \ge 1$ satisfying (7). Using the fact that $z_0 p'(z_0)$ is a real number, from (4) and (8), we can obtain

$$\begin{aligned}
&\operatorname{Re}\left\{p(z_0) + \frac{z_0 p'(z_0)}{\beta p(z_0) + \gamma}\right\} \\
&= \operatorname{Re}\big\{p(z_0)\big\} + z_0 p'(z_0) \operatorname{Re}\left\{\frac{1}{\beta p(z_0) + \gamma}\right\} \\
&= \operatorname{Re}\big\{p(z_0)\big\} + 2(1-\alpha)k\left(\frac{w(z_0)}{(1 - w(z_0))^2}\right) \operatorname{Re}\left\{\frac{1 - w(z_0)}{\beta + \gamma + (\beta - 2\alpha\beta - \gamma)w(z_0)}\right\}.
\end{aligned} \tag{18}$$

We now set $p(z_0)$ as in (9). Then we have the same function value of $w(z_0)$ which satisfies formula (10), and it follows from (18) with (10) and $k \ge 1$ that

$$\begin{aligned}
&\operatorname{Re}\left\{p(z_0) + \frac{z_0 p'(z_0)}{\beta p(z_0) + \gamma}\right\} \\
&= \alpha - 2(1-\alpha)k\left(\frac{(1-\alpha)^2 + y^2}{4(1-\alpha)^2}\right)\left(\frac{\alpha\beta + \gamma}{\gamma^2 + 2\alpha\beta\gamma + \beta^2(\alpha^2 + y^2)}\right) \\
&\le \alpha - \frac{(\alpha\beta + \gamma)(1 - 2\alpha + \alpha^2 + y^2)}{2(1-\alpha)(\gamma^2 + 2\alpha\beta\gamma + \beta^2(\alpha^2 + y^2))} \\
&= \delta\big(\alpha, \beta, \gamma, \big|p(z_0)\big|\big),
\end{aligned}$$

where $\delta(\alpha,\beta,\gamma,|p(z_0)|)$ is given by (16), which contradicts assumption (15). Therefore we complete the proof of Theorem 2.10. □

Remark 2.11 For $\gamma = 0$, Theorem 2.10 is an improvement of the result by Miller et al. [7].

Taking $\beta = 1$ and $\gamma = 0$ in Theorem 2.10, we have the following result.

Corollary 2.12 *Let p be analytic in $\mathbb{U}$ with $p(0) = 1$ and $0 \leq \alpha < 1$. If*

$$\operatorname{Re}\left\{p(z) + \frac{zp'(z)}{p(z)}\right\} > \frac{\alpha(1-2\alpha)(|p(z)|^2 - 1)}{2(1-\alpha)|p(z)|^2},$$

then $p \in \mathcal{P}(\alpha)$.

Applying Theorem 2.10 leads us to get the following theorem which doesn't depend on $|p(z)|$.

Theorem 2.13 *Let p be analytic in $\mathbb{U}$ with $p(0) = 1$ and $0 \leq \alpha < 1$. If p satisfies one of the following conditions*:

(i) $\operatorname{Re}\{p(z) + \frac{zp'(z)}{\beta p(z)+\gamma}\} > \alpha - \frac{\alpha\beta+\gamma}{2\beta^2(1-\alpha)}$ ($-\alpha\beta < \gamma < \beta(1-2\alpha)$ *for* $\beta > 0$ *or* $-\alpha\beta < \gamma < -\beta$ *for* $\beta < 0$),

(ii) $\operatorname{Re}\{p(z) + \frac{zp'(z)}{\beta p(z)+\gamma}\} > \alpha - \frac{1-\alpha}{2(\alpha\beta+\gamma)}$ ($\gamma \geq \beta(1-2\alpha)$ *for* $\beta > 0$ *or* $\gamma \geq -\beta$ *for* $\beta < 0$),

then $p \in \mathcal{P}(\alpha)$.

Proof First of all, we consider a function $\psi : [0,\infty) \to \mathbb{R}$ defined by

$$\psi(x) = \frac{(1-\alpha)^2 + x}{(\alpha\beta+\gamma)^2 + \beta^2 x}. \tag{19}$$

By differentiating ψ, we obtain

$$\left[(\alpha\beta+\gamma)^2 + \beta^2 x\right]^2 \psi'(x) = (\beta+\gamma)\bigl(\gamma + \beta(2\alpha-1)\bigr).$$

Therefore the derivative of ψ is negative when $-\alpha\beta < \gamma < -\beta(2\alpha-1)$ for $\beta > 0$ and $-\alpha\beta < \gamma < -\beta$ for $\beta < 0$, which means that function ψ is decreasing. Hence

$$\psi(x) \geq \lim_{x\to\infty} \psi(x) = \frac{1}{\beta^2} \quad (x \geq 0). \tag{20}$$

On the other hand, the derivative of ψ is positive when $-\beta(2\alpha-1) < \gamma$ for $\beta > 0$ and $-\beta < \gamma$ for $\beta < 0$, which means that function ψ is increasing. In this case, the following inequality holds:

$$\psi(x) \geq \psi(0) = \left(\frac{1-\alpha}{\alpha\beta+\gamma}\right)^2 \quad (x \geq 0). \tag{21}$$

According to the same contradiction method as in Theorem 2.10, when $p(z_0)$ is defined by (9), we now have

$$\begin{aligned}\operatorname{Re}\left\{p(z_0)+\frac{z_0p'(z_0)}{\beta p(z_0)+\gamma}\right\} &\le \alpha-\frac{(\alpha\beta+\gamma)(1-2\alpha+\alpha^2+y^2)}{2(1-\alpha)(\gamma^2+2\alpha\beta\gamma+\beta^2(\alpha^2+y^2))}\\ &=\alpha-\frac{(\alpha\beta+\gamma)}{2(1-\alpha)}\psi\left(y^2\right),\end{aligned}\tag{22}$$

where ψ is the function defined by (19).

When $-\alpha\beta<\gamma<-\beta(2\alpha-1)$ for $\beta>0$ and $-\alpha\beta<\gamma<-\beta$ for $\beta<0$, by (22) and (20), we have

$$\operatorname{Re}\left\{p(z_0)+\frac{z_0p'(z_0)}{\beta p(z_0)+\gamma}\right\}\le\alpha-\frac{(\alpha\beta+\gamma)}{2\beta^2(1-\alpha)}.$$

This is a contradiction to the assumption. And, when $-\beta(2\alpha-1)<\gamma$ for $\beta>0$ and $-\beta<\gamma$ for $\beta<0$, by (22) and (21), we have

$$\operatorname{Re}\left\{p(z_0)+\frac{z_0p'(z_0)}{\beta p(z_0)+\gamma}\right\}\le\alpha-\frac{(1-\alpha)}{2(\alpha\beta+\gamma)}.$$

But this also contradicts our assumption. Hence the proof of Theorem 2.13 is completed. □

Letting $\beta=1$ and $\gamma=0$ in Theorem 2.13, we have the following corollary.

Corollary 2.14 *Let p be analytic in $\mathbb{U}$ with $p(0)=1$. If function p satisfies the following condition, then $p\in\mathcal{P}(\alpha)$:*

$$\operatorname{Re}\left\{p(z)+\frac{zp'(z)}{p(z)}\right\}>\begin{cases}\frac{\alpha-2\alpha^2}{2(1-\alpha)}, & \text{when } 0\le\alpha<1/2,\\ \frac{(1+\alpha)(2\alpha-1)}{2\alpha}, & \text{when } 1/2\le\alpha<1.\end{cases}$$

Remark 2.15 Taking $p(z)=zf'(z)/f(z)$ and $\alpha=1/2$ in Corollary 2.14, we have the classical result by Marx [3] and Strohhäcker [12], that is, $\mathcal{K}\subset\mathcal{S}^*(1/2)$.

Let α and β be real numbers such that $0\le\alpha<1$ and $\beta\ge(3\alpha-1)/2$. Then it can be easily shown that

$$\alpha-\beta-\frac{(1-\alpha)^2+y^2}{2(1-\alpha)}k\le\alpha-\beta-\frac{(1-\alpha)^2+y^2}{2(1-\alpha)}\le 0,\quad y\in\mathbb{R},k\ge1.$$

Hence it follows that the following inequality holds for $y\in\mathbb{R}$ and $k\ge1$:

$$\left(\alpha-\beta-\frac{(1-\alpha)^2+y^2}{2(1-\alpha)}k\right)^2+y^2\ge\left(\alpha-\beta-\frac{(1-\alpha)^2+y^2}{2(1-\alpha)}k\right)^2+y^2.\tag{23}$$

Now let $p(z_0)$ and $z_0p'(z_0)$ be given as in (9) and (11), respectively. Then, from (23) and replacing y^2 by $|p(z_0)|^2 - \alpha^2$, we have

$$\begin{aligned}\left|p(z_0) + z_0p'(z_0) - \beta\right|^2 &= \left(\alpha - \beta - \frac{(1-\alpha)^2 + y^2}{2(1-\alpha)}k\right)^2 + y^2 \\ &\geq \left(1 - \alpha + \beta + \frac{|p(z_0)|^2 - 1}{2(1-\alpha)}\right)^2 + \left|p(z_0)\right|^2 - \alpha^2. \qquad (24)\end{aligned}$$

Now, applying the same method as in the proof of Theorems 2.3 and 2.10 and inequality (24), we obtain the following result.

Theorem 2.16 *Let α and β be real numbers such that $0 \leq \alpha < 1$ and $\beta \geq (3\alpha - 1)/2$. Let p be analytic in $\mathbb{U}$ with $p(0) = 1$. If*

$$\left|p(z) + zp'(z) - \beta\right| < \delta\left(\alpha, \beta, \left|p(z)\right|\right),$$

where

$$\delta\left(\alpha, \beta, \left|p(z)\right|\right) = \left\{\left(1 - \alpha + \beta + \frac{|p(z)|^2 - 1}{2(1-\alpha)}\right)^2 + \left|p(z)\right|^2 - \alpha^2\right\}^{1/2},$$

then $p \in \mathcal{P}(\alpha)$.

Corollary 2.17 *Let $0 \leq \alpha < 1$ and $\beta \geq (3\alpha - 1)/2$. And let p be an analytic function in $\mathbb{U}$ with $p(0) = 1$. If p satisfies*

$$\left|p(z) + zp'(z) - \beta\right| < \sqrt{\left(1 - \alpha + \beta - \frac{1}{2(1-\alpha)}\right)^2 - \alpha^2},$$

then $p \in \mathcal{P}(\alpha)$.

Taking $\alpha = 0$ and $\beta = 1$ in Theorem 2.16 and Corollary 2.17, we have Corollary 2.18 below.

Corollary 2.18 *Let p be analytic in $\mathbb{U}$ with $p(0) = 1$. If*

$$\left|p(z) + zp'(z) - 1\right| < \sqrt{\left(\frac{3 + |p(z)|^2}{2}\right)^2 + \left|p(z)\right|^2},$$

or

$$\left|p(z) + zp'(z) - 1\right| < \frac{3}{2},$$

then $p \in \mathcal{P}$.

With the aid of Lemma 2.2, we prove the following result.

Theorem 2.19 *Let α and A be real numbers with $0 \leq \alpha < 1$ and $A \geq 0$. And let B and C be functions defined in $\mathbb{U}$ such that $\operatorname{Re}\{B(z)\} > A$ for all $z \in \mathbb{U}$. If p is analytic in $\mathbb{U}$ with $p(0) = 1$ and*

$$\operatorname{Re}\{Az^2p''(z) + B(z)zp'(z) + C(z)p(z)\} > \delta(\alpha, A, B(z), C(z)),$$

where

$$\delta(\alpha, A, B(z), C(z)) = \frac{(1-\alpha)[(\operatorname{Im}\{C(z)\})^2 - (\operatorname{Re}\{B(z) - A\})^2]}{2(\operatorname{Re}\{B(z) - A\})} + \alpha \operatorname{Re}\{C(z)\},$$

then $p \in \mathcal{P}(\alpha)$.

Proof Define function w as in (4). We see that w is analytic in $\mathbb{U}$ with $w(0) = 0$. Suppose that there exists a point z_0 in $\mathbb{U}$ satisfying (5). Then we have (6). By Lemma 2.1, there exist a real number $k \geq 1$ satisfying (7). Moreover, by hypothesis (5), we have $p \not\prec h$, where $h : \mathbb{U} \to \mathbb{C}$ is the function defined by $h(z) = (1 + (1-2\alpha)z)/(1-z)$. Note that

$$\operatorname{Re}\left\{1 + \frac{\zeta h''(\zeta)}{h'(\zeta)}\right\} = 0$$

for $\zeta \in \mathbb{T}$. Lemma 2.2 with the equality above leads to the inequality

$$\operatorname{Re}\left\{1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right\} \geq 0. \tag{25}$$

Since $z_0 p'(z_0)$ is a nonpositive real number, from (25), we have

$$\operatorname{Re}\{z_0^2 p''(z_0)\} \leq -z_0 p'(z_0). \tag{26}$$

Putting

$$p(z_0) = \alpha + \mathrm{i}y \quad (y \in \mathbb{R}),$$

we obtain the same function value of $w(z_0)$ which satisfies equation (5). Then, by (26) and (11), we have the following inequalities:

$$\begin{aligned}
&\operatorname{Re}\{Az_0^2p''(z_0) + B(z_0)z_0p'(z_0) + C(z_0)p(z_0)\} \\
&\quad\leq (\operatorname{Re}\{B(z_0)\} - A)z_0p'(z_0) + \alpha \operatorname{Re}\{C(z_0)\} - y \operatorname{Im}\{C(z_0)\} \\
&\quad\leq \frac{1}{2}(A - \operatorname{Re}\{B(z_0)\})(1-\alpha) + \frac{A - \operatorname{Re}\{B(z_0)\}}{2(1-\alpha)}y^2 + \alpha \operatorname{Re}\{C(z_0)\} - y \operatorname{Im}\{C(z_0)\} \\
&\quad\leq \frac{(1-\alpha)[(\operatorname{Im}\{C(z_0)\})^2 - (\operatorname{Re}\{B(z_0) - A\})^2]}{2(\operatorname{Re}\{B(z_0) - A\})} + \alpha \operatorname{Re}\{C(z_0)\} \\
&\quad= \delta(\alpha, A, B(z_0), C(z_0)).
\end{aligned}$$

But this contradicts our assumption. Hence the proof is completed. □

Taking $A = 0$, $B(z) = C(z) \equiv 1$ and $\alpha = 0$ in Theorem 2.19, then we have the following result.

Corollary 2.20 *Let p be analytic in* $\mathbb{U}$ *with* $p(0) = 1$. *Then*

$$\operatorname{Re}\{p(z) + zp'(z)\} > -\frac{1}{2} \quad \Longrightarrow \quad \operatorname{Re}\{p(z)\} > 0.$$

Remark 2.21 Corollary 2.20 is the result obtained by Miller [4]. And this is also shown by Corollary 2.7.

Taking $A = 0$, $B(z) = C(z) \equiv 1$ and $\alpha = 1/2$ in Theorem 2.19, we have the following result.

Corollary 2.22 *Let p be analytic in* $\mathbb{U}$ *with* $p(0) = 1$. *Then*

$$\operatorname{Re}\{p(z) + zp'(z)\} > \frac{1}{4} \quad \Longrightarrow \quad \operatorname{Re}\{p(z)\} > \frac{1}{2}.$$

Letting $p(z) = (I_{\gamma,\beta}[f](z)/z)^{\beta}$ $(f \in \mathcal{A})$, where $I_{\gamma,\beta} : \mathcal{A} \to \mathcal{A}$ is the integral operator defined by (13), in Theorem 2.19 with $A = 0$, $B(z) \equiv 1$, $C(z) \equiv \gamma + \beta$ and $\alpha = 0$, we have the following result.

Corollary 2.23 *Let* $f \in \mathcal{A}$ *and let* β *and* γ *be complex numbers. If*

$$\operatorname{Re}\left\{(\gamma + \beta)\left(\frac{f(z)}{z}\right)^{\beta}\right\} > \frac{1}{2}\left[\left(\operatorname{Im}\{\gamma + \beta\}\right)^2 + 2\alpha \operatorname{Re}\{\gamma + \beta\} - 1\right],$$

then

$$\operatorname{Re}\left\{\left(\frac{I_{\gamma,\beta}[f](z)}{z}\right)^{\beta}\right\} > 0.$$

By a similar method as in the proof of Theorem 2.19, we can obtain the following result, which shows that the condition $\operatorname{Re}\{B(z)\} \geq A$ $(z \in \mathbb{U})$ can be established in Theorem 2.19 when $\operatorname{Im}\{C(z)\} = 0$ $(z \in \mathbb{U})$.

Theorem 2.24 *Let* α *and* A *be real numbers with* $0 \leq \alpha < 1$ *and* $A \geq 0$. *And let B and C be functions defined in* $\mathbb{U}$ *such that* $\operatorname{Re}\{B(z)\} = A$ *and* $\operatorname{Im}\{C(z)\} = 0$ *for all* $z \in \mathbb{U}$. *If p is analytic in* $\mathbb{U}$ *with* $p(0) = 1$ *and*

$$\operatorname{Re}\{Az^2p''(z) + B(z)zp'(z) + C(z)p(z)\} > \alpha \operatorname{Re}\{C(z)\},$$

then $p \in \mathcal{P}(\alpha)$.

Taking $A = 1$, $B(z) = C(z) \equiv 1$ in Theorem 2.19, then we have the following result.

Corollary 2.25 *Let p be analytic in* $\mathbb{U}$ *with* $p(0) = 1$. *Then*

$$\operatorname{Re}\{z^2p''(z) + zp'(z) + p(z)\} > \alpha \quad \Longrightarrow \quad \operatorname{Re}\{p(z)\} > \alpha.$$

Next, we derive another conditions for Carathéodory functions of order α in Theorems 2.26 and 2.27 below.

Theorem 2.26 *Let p be analytic in $\mathbb{U}$ with $p(0) = 1$ and $0 \leq \alpha < 1$. If p satisfies*

$$\frac{zp'(z)}{p(z) - \alpha} \neq \mathrm{i}\Lambda \tag{27}$$

for all $\Lambda \in \mathbb{R}$ with $|\Lambda| \geq 1$, then $p \in \mathcal{P}(\alpha)$.

Proof Let

$$q(z) = \frac{1}{1-\alpha}\big(p(z) - \alpha\big).$$

Then q is analytic in $\mathbb{U}$ with $q(0) = 1$. Here, we note that $p(z) \neq \alpha$ for $z \in \mathbb{U}$. In fact, if there exists a point $z_1 \in \mathbb{U}$ such that $p(z_1) = \alpha$ and hence $q(z_1) = 0$ then $q(z)$ can written by

$$q(z) = (z - z_1)^m q_1(z) \quad (m \in \mathbb{N}),$$

where q_1 is analytic in $\mathbb{U}$ and $q_1(z_1) \neq 0$. Hence we have

$$\frac{zp'(z)}{p(z) - \alpha} = \frac{zq'(z)}{q(z)} = \frac{mz}{z - z_1} + \frac{zq_1'(z)}{q_1(z)}. \tag{28}$$

But the imaginary part of the right-hand side of (28) can take any value when z approaches z_1. This contradicts our assumption (27). Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\mathrm{Re}\big\{q(z)\big\} > 0 \quad \text{for } |z| < |z_0| \quad \text{and} \quad \mathrm{Re}\big\{q(z_0)\big\} = 0 \quad \big(q(z_0) \neq 0\big).$$

Setting

$$\phi(z) = \frac{1 - q(z)}{1 + q(z)},$$

we have

$$\big|\phi(z)\big| < 1 \quad \text{for } |z| < |z_0| \quad \text{and} \quad \big|\phi(z_0)\big| = 1 \quad \big(\phi(0) = 0\big).$$

Let $q(z_0) = \mathrm{i}y$ ($y \in \mathbb{R} \setminus \{0\}$). Then, by Lemma 2.1, we obtain

$$\frac{z_0\phi'(z_0)}{\phi(z_0)} = \frac{-2z_0q'(z_0)}{1 - q^2(z_0)} = \frac{-2z_0q'(z_0)}{1 + y^2} = k,$$

where k is a real number with $k \geq 1$, and so

$$-z_0q'(z_0) \geq \frac{1 + y^2}{2}.$$

Therefore, $z_0q'(z_0)$ is a negative real number. At first, suppose that $y > 0$. Then we have

$$\frac{z_0p'(z_0)}{p(z_0) - \alpha} = \frac{z_0q'(z_0)}{q(z_0)} = \frac{-\mathrm{i}z_0q'(z_0)}{y} \equiv \mathrm{i}\Lambda.$$

Hence we obtain

$$\Lambda = \frac{-z_0 q'(z_0)}{y} \geq \frac{1+y^2}{2y} \geq 1,$$

which contradicts assumption (27). Next, for $y < 0$, we have

$$\frac{z_0 p'(z_0)}{p(z_0)-\alpha} = \frac{z_0 q'(z_0)}{q(z_0)} = \frac{\mathrm{i} z_0 q'(z_0)}{|q(z_0)|} = \frac{\mathrm{i} z_0 q'(z_0)}{|y|} \equiv \mathrm{i}\Lambda$$

and Λ is a real number with $\Lambda \leq -1$. This also contradicts assumption (27). Hence we complete the proof of Theorem 2.26. □

Theorem 2.27 *Let $0 \leq \alpha < 1$ and let p be analytic in $\mathbb{U}$ with $p(0) = 1$. If p satisfies*

$$\frac{zp'(z)}{p(z)-\alpha}\frac{(p(z)-\alpha)^2-(1-\alpha)^2}{(p(z)-\alpha)^2+(1-\alpha)^2} \neq \mathrm{i}\Delta \tag{29}$$

for all $\Delta \in \mathbb{R}$ with $|\Delta| \geq 2$, then $p \in \mathcal{P}(\alpha)$.

Proof Firstly, using a proof similar to that of Theorem 2.26 and assumption (29), we can derive easily that

$$p(z) \neq \alpha \quad \text{and} \quad p^2(z) - 2\alpha p(z) + 2\alpha^2 - 2\alpha + 1 \neq 0 \tag{30}$$

for all $z \in \mathbb{U}$. Let

$$q(z) = \frac{1}{1-\alpha}\bigl(p(z)-\alpha\bigr) = \frac{1+w(z)}{1-w(z)}.$$

Then we see that w is analytic in $\mathbb{U}$ with $w(0) = 0$. We claim that $|w(z)| < 1$ in $\mathbb{U}$. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that $\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1$. By Lemma 2.1, there exists a real number $k \geq 1$ satisfying (7). Writing $w(z_0) = \mathrm{e}^{\mathrm{i}\theta}$ with

$$-\pi < \theta < \pi \quad (\theta \neq 0, \pm\pi/2), \tag{31}$$

we obtain that

$$q(z_0) = \frac{1+\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}} = \mathrm{i}\cot(\theta/2)$$

and

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{2kw(z_0)}{1-w^2(z_0)} = \mathrm{i}\frac{k}{\sin\theta}.$$

Therefore we have the following identities:

$$\begin{aligned}\frac{z_0 p'(z_0)}{p(z_0)-\alpha}\frac{(p(z_0)-\alpha)^2-(1-\alpha)^2}{(p(z_0)-\alpha)^2+(1-\alpha)^2} &= \frac{z_0 q'(z_0)}{q(z_0)}\frac{q^2(z_0)-1}{q^2(z_0)+1} \\ &= -\mathrm{i}\frac{k}{\sin\theta}\frac{1+\cot^2(\theta/2)}{1-\cot^2(\theta/2)} \\ &\equiv -\mathrm{i}\Delta.\end{aligned}$$

It is now sufficient to show that

$$\Delta \geq 2 \quad \text{or} \quad \Delta \leq -2 \tag{32}$$

for all θ satisfying (31), since (32) contradicts the assumption (29). For this, let us define a function $\varphi : (0,1) \to \mathbb{R}$ by

$$\varphi(t) = \frac{(1+t^2)^2}{2t(1-t^2)}.$$

We can check $\varphi'(t) = 0$ occurs only at $t = \sqrt{2} - 1 =: t_0 \in (0,1)$. Moreover, we have $\varphi''(t_0) = 12 + 8\sqrt{2} > 0$. Therefore, on the interval $(0,1)$, function φ has its minimum at $t = t_0$. That is,

$$\varphi(t) \geq \varphi(t_0) = 2 \quad (0 < t < 1). \tag{33}$$

And, by (33), the following inequality holds for $t \in (1,\infty)$:

$$\varphi(t) = -\varphi(1/t) \leq -2 \quad (t > 1). \tag{34}$$

Consider the case $0 < \theta < \pi/2$. Then, we have $\cot(\theta/2) > 1$ and it follows from (34) that

$$\Delta = k\varphi\bigl(\cot(\theta/2)\bigr) \leq -2.$$

For the case $\pi/2 < \theta < \pi$, we have $0 < \cot(\theta/2) < 1$ and (33) gives us that

$$\Delta = k\varphi\bigl(\cot(\theta/2)\bigr) \geq 2.$$

A similar method as above leads us to inequality (32) for the case $-\pi < \theta < 0$ with $\theta \neq -\pi/2$ and the proof of Theorem 2.27 is now completed. □

Remark 2.28 Taking p to be appropriate analytic functions in Theorems 2.26 and 2.27, we can find conditions for univalence, starlikeness, convexity, and so on.

Theorem 2.29 *Let $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. If p is analytic in $\mathbb{U}$ with $p(0) = 1$ and*

$$\operatorname{Re}\left\{\bigl(p(z)-\alpha\bigr)^{\beta}\left(1+\frac{zp'(z)}{p(z)-\alpha}\right)\right\} > h\bigl(\delta(\alpha,\beta),\alpha,\beta\bigr),$$

where

$$\begin{aligned} h(x,\alpha,\beta) = \frac{1}{2(1-\alpha)}\biggl(&-x^{\beta+1}\sin\left(\frac{\pi}{2}\beta\right) + 2(1-\alpha)x^{\beta}\cos\left(\frac{\pi}{2}\beta\right) \\ &- (1-\alpha)^2 x^{\beta-1}\sin\left(\frac{\pi}{2}\beta\right)\biggr) \end{aligned}$$

and

$$\delta(\alpha,\beta) = \frac{1-\alpha}{(1+\beta)\sin(\frac{\pi}{2}\beta)}\left(\beta\cos\left(\frac{\pi}{2}\beta\right) + \sqrt{(1-2\beta^2)\sin^2\left(\frac{\pi}{2}\beta\right)+\beta^2}\right),$$

then $p \in \mathcal{P}(\alpha)$.

Proof First, we note that $p(z) \neq \alpha$ for $0 \leq \alpha < 1$. Defining function w by (4), we see that w is analytic in $\mathbb{U}$ with $w(0) = 0$. Suppose that there exists a point z_0 in $\mathbb{U}$ satisfying (5). Then we have (6). By using Lemma 2.1, we obtain

$$z_0 w'(z_0) = k w(z_0),$$

where k is a real number with $k \geq 1$. Putting $p(z_0) = \alpha + \mathrm{i}y$ with $y \in \mathbb{R} \setminus \{0\}$, we obtain (10). Then we have

$$\begin{aligned}
&\operatorname{Re}\left\{\left(p(z_0)-\alpha\right)^{\beta}\left(1+\frac{z_0 p'(z_0)}{p(z_0)-\alpha}\right)\right\} \\
&= \operatorname{Re}\left\{\left(p(z_0)-\alpha+z_0 p'(z_0)\right)\left(p(z_0)-\alpha\right)^{\beta-1}\right\} \\
&= \operatorname{Re}\left\{\left(\mathrm{i}y-\frac{k((1-\alpha)^2+y^2)}{2(1-\alpha)}\right)(\mathrm{i}y)^{\beta-1}\right\} \\
&= \operatorname{Re}\left\{\left(\mathrm{i}y-\frac{k((1-\alpha)^2+y^2)}{2(1-\alpha)}\right)|y|^{\beta-1}\left(\cos\left(\pm\frac{(\beta-1)\pi}{2}\right)+\mathrm{i}\sin\left(\pm\frac{(\beta-1)\pi}{2}\right)\right)\right\}.
\end{aligned}$$

At first, we consider the case $0 < \beta < 1$.

(i) For the case $y > 0$, we have

$$\begin{aligned}
&\operatorname{Re}\left\{\left(p(z_0)-\alpha\right)^{\beta}\left(1+\frac{z_0 p'(z_0)}{p(z_0)-\alpha}\right)\right\} \\
&= \operatorname{Re}\left\{\left(-\frac{k}{2(1-\alpha)}\left((1-\alpha)^2 y^{\beta-1}+y^{\beta+1}\right)+\mathrm{i}y^{\beta}\right)\left(\sin\left(\frac{\pi}{2}\beta\right)-\mathrm{i}\cos\left(\frac{\pi}{2}\beta\right)\right)\right\} \\
&= -\frac{k}{2(1-\alpha)}\left((1-\alpha)^2 y^{\beta-1}+y^{\beta+1}\right)\sin\left(\frac{\pi}{2}\beta\right)+y^{\beta}\cos\left(\frac{\pi}{2}\beta\right) \\
&\leq \frac{1}{2(1-\alpha)}\left(-y^{\beta+1}\sin\left(\frac{\pi}{2}\beta\right)+2(1-\alpha)y^{\beta}\cos\left(\frac{\pi}{2}\beta\right)-(1-\alpha)^2 y^{\beta-1}\sin\left(\frac{\pi}{2}\beta\right)\right) \\
&= h(y,\alpha,\beta).
\end{aligned}$$

Then, by a simple calculation, we obtain

$$h(y,\alpha,\beta) \leq h\left(\delta(\alpha,\beta),\alpha,\beta\right),$$

which is a contradiction to our assumption.

(ii) For the case $y < 0$, we have

$$\begin{aligned}
&\operatorname{Re}\left\{\left(p(z_0)-\alpha\right)^{\beta}\left(1+\frac{z_0 p'(z_0)}{p(z_0)-\alpha}\right)\right\} \\
&= \operatorname{Re}\left\{\left(-\frac{k}{2(1-\alpha)}\left((1-\alpha)^2 |y|^{\beta-1}+|y|^{\beta+1}\right)-\mathrm{i}|y|^{\beta}\right)\left(\sin\left(\frac{\pi}{2}\beta\right)+\mathrm{i}\cos\left(\frac{\pi}{2}\beta\right)\right)\right\} \\
&= -\frac{k}{2(1-\alpha)}\left((1-\alpha)^2 |y|^{\beta-1}+|y|^{\beta+1}\right)\sin\left(\frac{\pi}{2}\beta\right)+|y|^{\beta}\cos\left(\frac{\pi}{2}\beta\right) \\
&\leq h\left(|y|,\alpha,\beta\right) \leq h\left(\delta(\alpha,\beta),\alpha,\beta\right).
\end{aligned}$$

We also come up to the same contradiction to our assumption under $y < 0$ condition. Now, we consider the case $\beta = 1$ and obtain

$$\begin{aligned} \operatorname{Re}\{p(z_0) - \alpha + z_0 p'(z_0)\} &= -k\left(\frac{(1-\alpha)^2 + y^2}{2(1-\alpha)}\right) \\ &\le -\frac{(1-\alpha)^2 + y^2}{2(1-\alpha)} \\ &\le -\frac{1-\alpha}{2} = h\big(\delta(\alpha,1),\alpha,1\big). \end{aligned}$$

This contradicts our assumption. So, the proof is completed. □

Remark 2.30 Taking $\alpha = 0$ and $\beta = 1$ in Theorem 2.29, we obtain the same result of Corollary 2.22.

Taking $p(z) = f(z)/z$ and $\alpha = 0$ in Theorem 2.29, we have the following result.

Corollary 2.31 *Let $f \in \mathcal{A}$ and $0 < \beta \le 1$. If*

$$\operatorname{Re}\left\{f'(z)\left(\frac{f(z)}{z}\right)^{\beta-1}\right\} > h\big(\delta(0,\beta),0,\beta\big) \quad (z \in \mathbb{U}),$$

where h and $\delta(0,\beta)$ are given in Theorem 2.29, respectively, then

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\} > 0 \quad (z \in \mathbb{U}).$$

Example 2.32 Taking $\beta = 1/2$ in Corollary 2.31, we have $h(\delta(0,\beta),0,\beta) = 0$. Then

$$\operatorname{Re}\left\{f'(z)\left(\frac{z}{f(z)}\right)^{1/2}\right\} > 0 \quad \text{implies} \quad \operatorname{Re}\left\{\frac{f(z)}{z}\right\} > 0.$$

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