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A note on modified degenerate q-Daehee polynomials and numbers

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Abstract

We consider the modified degenerate q-Daehee polynomials and numbers of the second kind which can be represented as the p-adic q-integral. Furthermore, we investigate some properties of those polynomials and numbers.

Keywords: Modified q-Daehee polynomials and numbers; Modified degenerate q-Daehee polynomials and numbers

1 Introduction

Throughout this paper, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of integers, the field of rational numbers, the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. If $q \in \mathbb{C}_p$, we normally assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. The q-extension of x is defined as $[x]_q = \frac{1-q^x}{1-q}$ for $q \ne 1$ and x for $q = 1$ (see [3–6, 12, 17, 18, 20, 21, 25, 27, 29–31, 33–35, 41, 45, 46]). Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, Volkenborn integral (or p-adic bosonic integral) on $\mathbb{Z}_p$ is given by

$$I_1(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{1.1}$$

where $\mu_1(x) = \mu_1(x + p^N\mathbb{Z}_p)$ denotes the Haar distribution defined by $\mu_1(x + p^N\mathbb{Z}_p) = \frac{1}{p^N}$ (see [1, 2, 8–14, 16, 19, 24, 32, 35, 37–44, 46, 47]). Then, by (1.1), we get $I(f_1) - I_1(f) = f'(0)$, where $f_1(x) = f(x+1)$ and $\frac{d}{dx}f(x)|_{x=0} = f'(0)$.

For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x \tag{1.2}$$

(see [12, 17–20, 25, 29, 31, 33, 34, 47]). Note that

$$\lim_{q\to 1} I_q(f) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) = I_1(f)$$

(see [6, 9, 18, 19, 21, 25, 28, 29, 32–34, 36, 38, 42, 43, 47]). Let $f_1(x) = f(x+1)$. Then, by (1.2), we get

$$qI_q(f_1) - I_q(f) = q(q-1)f(0) + \frac{q(q-1)}{\log q}f'(0), \tag{1.3}$$

where $f'(0) = \frac{d}{dx}f(x)|_{x=0}$ (see [6, 9, 18, 19, 21, 25, 28, 29, 32–34, 36, 38, 42, 43, 47]).

Carlitz considered q-Bernoulli numbers which are recursively given by

$$\beta_{0,q} = 1, \qquad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$ (see [3–5]). He also defined q-Bernoulli polynomials as

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q}, \quad (n \geq 0) \quad (\text{see [3]})$$

(see [3–5]). In [19], Kim proved that the Carlitz q-Bernoulli polynomials are represented by p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} [x+y]_q^n \, dmu_q(y) = \beta_{n,q}(x) \quad (n \geq 0). \tag{1.4}$$

In [17], Kim considered the modified q-Bernoulli polynomials which are different from Carlitz to be

$$B_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n \, dmu_1(y) \quad (n \geq 0).$$

When $x = 0$, $B_{n,q} = B_{n,q}(0)$ are called the modified q-Bernoulli numbers (see [17, 18]). Thus, we note that

$$B_{0,q} = 1, \qquad (qB_q + 1)^n - B_{n,q} = \begin{cases} \frac{\log q}{q-1}, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention about replacing B_q^n by $B_{n,q}$ (see [17, 18, 21, 25, 34]).

In [33, 35, 46], the authors studied the q-Daehee polynomials which are defined by the generating function to be

$$\int_{\mathbb{Z}_p} (1+t)^{x+y} \, d\mu_q(y) = \frac{q - 1 + \frac{q-1}{\log q}\log(1+t)}{qt + q - 1}(1+t)^x = \sum_{n=0}^{\infty} D_{n,q}(x)\frac{t^n}{n!}. \tag{1.5}$$

In [12], the authors studied the degenerate λ-q-Daehee polynomials as follows:

$$\frac{q - 1 + \frac{q-1}{\log q}\lambda \log(1 + \frac{1}{u}\log(1+ut))}{q(1 + \frac{1}{u}\log(1+ut))^{\lambda} - 1}\left(1 + \frac{1}{u}\log(1+ut)\right)^x$$

$$= \int_{\mathbb{Z}_p} \left(1 + \frac{1}{u}\log(1+ut)\right)^{\lambda y + x} d\mu_q(y)$$
$$= \sum_{n=0}^{\infty} D_{n,\lambda,q}(x|u)\frac{t^n}{n!}. \tag{1.6}$$

Like this idea of the Carlitz q-Bernoulli polynomials (1.4), we will define the modified q-Daehee polynomials of the second kind which are different from the modified q-Daehee numbers and polynomials in [31].

As is well known, the Stirling number of the first kind is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \tag{1.7}$$

and the Stirling number of the second kind is given by the generating function,

$$\left(e^t - 1\right)^m = m!\sum_{l=m}^{\infty} S_2(l,m)\frac{t^l}{l!}. \tag{1.8}$$

We also have

$$\left(\log(1+t)\right)^m = m!\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} \tag{1.9}$$

and

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k \tag{1.10}$$

(see [7, 14, 15, 22, 23, 26, 28, 48]).

In this paper, we consider the modified q-Daehee polynomials of the second kind and investigate their properties. Furthermore, we consider the modified degenerate q-Daehee polynomials of the second kind and investigate their properties.

2 The modified q-Daehee polynomials and numbers of the second kind

Let p be a fixed prime number. We assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$ and $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$.

The modified q-Daehee polynomials of the second kind are defined by

$$\int_{\mathbb{Z}_p} (1+t)^{[x+y]_q}\, d\mu_0(y) = \sum_{n=0}^{\infty} D_{n,q}^{*}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $D_{n,q}^{*} = D_{n,q}^{*}(0)$ are called the nth modified q-Daehee numbers of the second kind. By using the binomial theorem in (2.1), we observe that

$$\int_{\mathbb{Z}_p} (1+t)^{[x+y]_q}\, d\mu_0(y) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \left([x+y]_q\right)_n d\mu_0(y)\frac{t^n}{n!}. \tag{2.2}$$

Note that the modified q-Daehee polynomials were defined by Lim in [31] as follows:

$$D_n(x|q) = \int_{\mathbb{Z}_p} q^{-y}(x+y)_n \, d\mu_q(y). \tag{2.3}$$

From (2.1) and (2.2), we obtain the following theorem.

Theorem 2.1 *For $n \geq 0$, we have*

$$D^*_{n,q}(x) = \int_{\mathbb{Z}_p} \big([x+y]_q\big)_n \, d\mu_0(y). \tag{2.4}$$

From (2.1), we derive that

$$\begin{aligned}\int_{\mathbb{Z}_p} (1+t)^{[x+y]_q} \, d\mu_0(y) &= \int_{\mathbb{Z}_p} e^{[x+y]_q \log(1+t)} \, d\mu_0(y) \\ &= \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} [x+y]_q^m \, d\mu_0(y) \frac{1}{m!} \big(\log(1+t)\big)^m. \end{aligned} \tag{2.5}$$

By using (1.9) and (1.10) in Eq. (2.4), we have

$$\begin{aligned} &\sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} [x+y]_q^m \, d\mu_0(y) \frac{1}{m!} \big(\log(1+t)\big)^m \\ &\quad = \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} \sum_{k=0}^{m} S_2(m,k) \big([x+y]_q\big)_k \, d\mu_0(y) \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} \\ &\quad = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{k=0}^{m} S_2(m,k) S_1(n,m) \int_{\mathbb{Z}_p} \big([x+y]_q\big)_k \, d\mu_0(y) \right) \frac{t^n}{n!} \\ &\quad = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{k=0}^{m} S_2(m,k) S_1(n,m) D^*_{k,q}(x) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.6}$$

Thus, by (2.1), (2.5), and (2.6), we obtain the following theorem.

Theorem 2.2 *For $n \geq 0$, we have*

$$D^*_{n,q}(x) = \sum_{m=0}^{n} \sum_{k=0}^{m} S_2(m,k) S_1(n,m) D^*_{k,q}(x). \tag{2.7}$$

From (2.1), by replacing t by $e^t - 1$ and using (1.8), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} e^{[x+y]_q t} \, d\mu_0(y) &= \sum_{m=0}^{\infty} D^*_{m,q}(x) \frac{(e^t-1)^m}{m!} \\ &= \sum_{m=0}^{\infty} D^*_{m,q}(x) \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} D^*_{m,q}(x) S_2(n,m) \right) \frac{t^n}{n!}, \end{aligned} \tag{2.8}$$

and by using (1.10) and (2.3), we have

$$
\begin{aligned}
\int_{\mathbb{Z}_p} e^{[x+y]_q t}\, d\mu_0(y) &= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} [x+y]_q^n \frac{t^n}{n!}\, d\mu_0(y) \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} [x+y]_q^n\, d\mu_0(y) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \bigl([x]_q + q^x [y]_q\bigr)^n\, d\mu_0(y) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} \int_{\mathbb{Z}_p} [y]_q^k\, d\mu_0(y) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} \int_{\mathbb{Z}_p} \sum_{l=0}^{k} S_2(k,l) \bigl([y]_q\bigr)_l\, d\mu_0(y) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k} [x]_q^{n-k} q^{kx} S_2(k,l) D_{l,q}^{*} \right) \frac{t^n}{n!}.
\end{aligned}
\tag{2.9}
$$

From (2.8) and (2.9), we obtain the following theorem.

Theorem 2.3 *For $n \geq 0$, we have*

$$
\sum_{m=0}^{n} D_{m,q}^{*}(x) S_2(n,m) = \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k} [x]_q^{n-k} q^{kx} S_2(k,l) D_{l,q}^{*}. \tag{2.10}
$$

3 The modified degenerate q-Daehee polynomials of the second kind

Let p be a fixed prime number. We assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$.

The modified degenerate q-Daehee polynomials of the second kind are defined by

$$
\int_{\mathbb{Z}_p} \left(1 + \frac{1}{\lambda} \log(1+\lambda t) \right)^{[x+y]_q} d\mu_0(y) = \sum_{n=0}^{\infty} D_{n,\lambda,q}^{*}(x) \frac{t^n}{n!}. \tag{3.1}
$$

When $x = 0$, $D_{n,\lambda,q}^{*} = D_{n,\lambda,q}^{*}(0)$ are called the modified degenerate q-Daehee numbers of the second kind.

We note that the reason for calling $D_{n,\lambda,q}^{*}$ the modified degenerate q-Daehee polynomials of the second kind is to distinguish it from the modified q-Daehee numbers and polynomials in [31]. From (3.1), we observe that

$$
\begin{aligned}
\int_{\mathbb{Z}_p} \left(1 + \frac{1}{\lambda} \log(1+\lambda t) \right)^{[x+y]_q} d\mu_0(y) &= \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} \binom{[x+y]_q}{m} d\mu_0(y) \left(\frac{1}{\lambda} \log(1+\lambda t) \right)^m \\
&= \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} \bigl([x+y]_q\bigr)_m\, d\mu_0(y) \lambda^{-m} \frac{1}{m!} \bigl(\log(1+\lambda t)\bigr)^m \\
&= \sum_{m=0}^{\infty} \bigl(D_{m,q}^{*}(x) \lambda^{-m}\bigr) \left(\sum_{n=m}^{\infty} \lambda^n S_1(n,m) \frac{t^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} D_{m,q}^{*}(x) \lambda^{n-m} S_1(n,m) \right) \frac{t^n}{n!}.
\end{aligned}
\tag{3.2}
$$

From (3.1) and (3.2), we obtain the following theorem.

Theorem 3.1 *For $n \geq 0$, we have*

$$D^*_{n,\lambda,q}(x) = \sum_{m=0}^{n} D^*_{m,q}(x)\lambda^{n-m} S_1(n,m). \tag{3.3}$$

From (3.1), by replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$, we derive

$$\begin{aligned}
\int_{\mathbb{Z}_p} (1+t)^{[x+y]_q}\, d\mu_0(y) &= \sum_{m=0}^{\infty} D^*_{m,\lambda,q}(x) \frac{(\frac{1}{\lambda}(e^{\lambda t}-1))^m}{m!} \\
&= \sum_{m=0}^{\infty} D^*_{m,\lambda,q}(x)\lambda^{-m} \sum_{n=m}^{\infty} S_2(n,m)\frac{\lambda^n t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n} D^*_{m,\lambda,q}(x)\lambda^{n-m} S_2(n,m)\frac{t^n}{n!}.
\end{aligned} \tag{3.4}$$

From (3.4) and (2.1), we obtain the following theorem.

Theorem 3.2 *For $n \geq 0$, we have*

$$D^*_{n,q}(x) = \sum_{m=0}^{n} D^*_{m,\lambda,q}(x)\lambda^{n-m} S_2(n,m). \tag{3.5}$$

From (3.1), we observe that

$$\begin{aligned}
\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{[x+y]_q} &= e^{[x+y]_q \log(1+\frac{1}{\lambda}\log(1+\lambda t))} \\
&= \sum_{m=0}^{\infty} [x+y]_q^m \left(\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)\right)^m \frac{1}{m!} \\
&= \sum_{m=0}^{\infty} [x+y]_q^m \sum_{l=m}^{\infty} S_1(l,m)\frac{(\frac{1}{\lambda}\log(1+\lambda t))^l}{l!} \\
&= \sum_{l=0}^{\infty}\sum_{m=0}^{l} [x+y]_q^m S_1(l,m)\lambda^{-l}\sum_{n=l}^{\infty} S_1(n,l)\lambda^n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l} [x+y]_q^m S_1(l,m)\lambda^{n-l} S_1(n,l)\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.6}$$

From (3.7), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{[x+y]_q} d\mu_0(y) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{k=0}^{m} S_2(m,k)S_1(l,m)\lambda^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\left([x+y]_q\right)_k d\mu_0(y)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{k=0}^{m} \lambda^{n-l}S_1(l,m)S_1(n,l)S_2(m,k)D^*_{k,q}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.7}$$

From (3.7) and (3.1), we obtain the following theorem.

Theorem 3.3 *For $n \geq 0$, we have*

$$D^*_{n,\lambda,q}(x) = \sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{k=0}^{m}\lambda^{n-l}S_1(l,m)S_1(n,l)S_2(m,k)D^*_{k,q}(x). \tag{3.8}$$

4 Conclusion

Many authors studied the q-Daehee polynomials (1.5), the degenerate λ-q-Daehee polynomials of the second kind in [12, 33, 46]. In this paper, we defined the modified q-Daehee polynomials of the second kind (2.1), which are different from the q-Daehee polynomials (1.5), and the modified degenerate q-Daehee polynomials of the second kind (3.1), which are different from the modified q-Daehee numbers and polynomials in [31]. We obtained the interesting results of Theorems 2.1, 2.2, and 2.3, which are some identity properties related with the modified degenerate q-Daehee polynomials of the second kind (3.1) and also we obtained the results of Theorems 3.1, 3.2, and 3.3, which are some identities related with the modified q-Daehee polynomials of the second kind.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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