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A new proof of the Orlicz–Lorentz centroid inequality

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Abstract

In this paper, we give another proof of the Orlicz–Lorentz centroid inequality which is obtained by Nguyen (Adv. Appl. Math. 92:99–121, 2018). We prove that a family of parallel chord movement under the Orlicz–Lorentz centroid operator is a shadow system along the same direction.

Keywords: Orlicz–Lorentz centroid body; Orlicz–Lorentz centroid inequality; Steiner's symmetrization; Shadow system

1 Introduction

Let K be an origin-symmetric convex body in Euclidean n-space, $\mathbb{R}^n$, the centroid body of K is the body whose boundary consists of the locus of the centroids of the halves of K formed when K is cut by codimension 1 subspaces. The concept of centroid body plays an important role in convex geometry. The most important affine isoperimetric inequalities that relate the volume of a convex body and of its centroid body or its projection body were established in the 1960s by Petty (see [18]), and nowadays are known as the Busemann–Petty centroid inequality or the Busemann-projection inequality. With the development of convex geometry, the Busemann-centroid inequality (or the Busemann-projection inequality) has gone through the L_p Busemann-centroid inequality (or L_p Busemann-projection inequality), and the Orlicz Busemann-centroid inequality (or the Orlicz Busemann-projection inequality). The Orlicz Busemann-centroid inequality and the Orlicz Busemann-projection inequality were given by Lutwak, Yang and Zhang in 2010 (see [15, 16]), which extend the L_p Brunn–Minkowski theory to Orlicz–Brunn–Minkowski theory. For more about the L_p Brunn–Minkowski theory and the Orlicz Brunn–Minkowski theory see, e.g., [1–14, 20, 22–26] and the references therein. Recently, Nguyen (see [17]) used the methods of [14, 15] to extend the Orlicz centroid bodies to the Orlicz–Lorentz centroid bodies and establishes the Orlicz–Lorentz centroid inequality. He conjectures that the shadow system approach would give another proof of the Orlicz–Lorentz centroid inequality. In this paper, we conform his assertion and give a proof of that a family of parallel chord movement under the behavior of the Orlicz–Lorentz centroid operator $\Gamma_{\phi,\omega}$ is a shadow system along the same direction.

In the next section, we follow the notation of [17]. Let (Ω, Σ, μ) be a measure space with an σ-finite, non-atom measure of these space. For any measurable function $f : \Omega \to \mathbb{R}$, we define the distribution function of f by

$$\mu_f(s) = \mu\bigl(\bigl\{x : \bigl|f(x)\bigr| > s, x \in \Omega\bigr\}\bigr), \quad \forall s > 0,$$

and the decreasing rearrangement of f by

$$f^*(t) = \inf\bigl\{\lambda > 0 : \mu_f(\lambda) \le t\bigr\},$$

for any $t > 0$.

We denote $I = (0, \mu(\Omega))$. A function $\phi : [0,\infty) \to [0,\infty)$ is called an Orlicz function if ϕ is a convex function such that $\phi(t) > 0$ if $t > 0$, $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. A weight function $\omega : I \to (0,\infty)$ is non-increasing function which is locally integrable with respect to the Lebesgue measure on I such that $\int_I \omega(t)\,dt = \infty$ if $I = (0,\infty)$. The Orlicz–Lorentz space $\Lambda_{\phi,\omega}$ on (Ω, Σ, μ) is the set of all measurable functions f on Σ such that

$$\int_I \phi\left(\frac{f^*(t)}{\lambda}\right)\omega(t)\,dt < \infty,$$

for some $\lambda > 0$. If the function $f \in \Lambda_{\phi,\omega}$, its Orlicz norm is defined by

$$\|f\|_{\Lambda_{\phi,\omega}} = \inf\left\{\lambda > 0 : \int_I \phi\left(\frac{f^*(t)}{\lambda}\right)\omega(t)\,dt \le 1\right\}. \tag{1.1}$$

By the definition of the Orlicz norm, it is obvious that if f and g have the same distribution function then $\|f\|_{\Lambda_{\phi,\omega}} = \|g\|_{\Lambda_{\phi,\omega}}$. Specially, when $\omega \equiv 1$, the Orlicz–Lorentz space $\Lambda_{\phi,\omega}$ is the Orlicz space. When $\phi(t) = t^p$, and $\omega \equiv 1$, it is the Lebesgue space $L_p(\Omega,\mu)$. When $\phi(t) = t$, we obtain the Lorentz space Λ_ω.

Let ϕ be a convex function and a weight function ω on $I = (0,1)$, consider the measure space $(K, \mathcal{B}_K, \mu^K)$, here $\mathcal{B}_K$ denotes the σ-algebra of all Lebesgue measurable subset of K, and μ^K denotes the normalized measure on K. For any vector $x \in \mathbb{R}^n$, we define the function $f_{x,K}$ on K by

$$f_{x,K} = \langle x, y\rangle, \quad y \in K.$$

The Orlicz–Lorentz centroid body $\Gamma_{\phi,\omega}K$ of K is defined by whose support function is given by

$$h(\Gamma_{\phi,\omega}K, x) = \|f_{x,K}\|_{\Lambda_{\phi,\omega}} = \inf\left\{\lambda > 0 : \int_0^1 \phi\left(\frac{f^*_{x,K}(t)}{\lambda}\right)\omega(t)\,dt \le 1\right\}. \tag{1.2}$$

Specially, when $\omega \equiv 1$, the definition of (1.2) coincides with the definition of Orlicz centroid body given by Lutwak, Yang and Zhang [15] for even convex function ϕ in $\mathbb{R}^n$. The following Orlicz–Lorentz centroid inequality was established by Nguyen.

Orlicz–Lorentz centroid inequality *If ϕ is an Orlicz function, ω is a weight function on $(0,1)$ and K is a convex body in $\mathbb{R}^n$ containing the origin in its interior, then the volume ratio*

$$\frac{|\Gamma_{\phi,\omega}K|}{|K|} \tag{1.3}$$

is minimized if and only if K is an origin-centered ellipsoid.

The method used by Nguyen in [17] is the Steiner symmetrization and the trouble of the proof of the Orlicz–Lorentz centroid inequality is the decreasing rearrangement function of ω. In this paper, we prove the following.

Theorem 1.1 *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement along the direction v, then $\Gamma_{\phi,\omega}K_t$ is a shadow system along the same direction v.*

The paper is organized as follows. In Sect. 2, we give some basic facts regarding convex bodies, shadow system and properties of shadow system. Section 3 contains the proof of the main theorem.

2 Shadow system of convex body

Let S^{n-1} and B denote the unit sphere and the unit ball in $\mathbb{R}^n$, write ω_n for the n-dimensional volume of B, and where $\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$, $\Gamma(\cdot)$ is the Gamma function. We write $\mathcal{K}^n$ for the set of convex bodies (compact convex subsets) of $\mathbb{R}^n$, and denote $\mathcal{K}_o^n$ by the set of convex bodies that contain the origin in their interiors. For K in $\mathbb{R}^n$, the support function h_K is the real-valued function defined by $h_K(u) = \max\{\langle u,y\rangle : y \in K\}$ for all $u \in S^{n-1}$. From the definition of the support function we know that, for $c > 0$, the support function of the convex body $cK = \{cx : x \in K\}$ is

$$h_{cK} = ch_K. \tag{2.1}$$

Moreover, for $A \in \mathrm{GL}(n)$ the support function of the image $AK = \{Ay : y \in K\}$ is given by

$$h_{AK}(x) = h_K\bigl(A^t x\bigr).$$

A shadow system (or a linear parameter system) along the direction v is a family of convex bodies $K_t \subset \mathbb{R}^n$ that can be defined by (see [19, 21])

$$K_t = \mathrm{conv}\bigl\{z + \alpha(z)tv : z \in A \subset \mathbb{R}^n\bigr\}, \tag{2.2}$$

where A is an arbitrary bounded set of points, $\alpha(z)$ is a real bounded function on A, and the parameter t runs in an interval of the real axis.

Note that the orthogonal projection $K_t|v^\perp$ of K_t onto $v^\perp = \{x \in \mathbb{R}^n : \langle v,x\rangle = 0\}$ is independent of t.

The following lemma related to the volumes of K_t is due to Shephard (see [21]).

Lemma 2.1 *Every mixed volume involving n shadow systems along the same direction is a convex function of the parameter. In particular, the volume $V(K_t)$ and all quermassintegrals $W_i(K_t)$, $i = 1,2,\ldots,n$, of a shadow system are convex functions of t.*

A parallel chord movement along the direction v is a family of convex bodies K_t in $\mathbb{R}^n$ defined by

$$K_t = \bigl\{z + \beta(x)tv : z \in K, x = z - \langle z,v\rangle v\bigr\}, \tag{2.3}$$

where K is a convex body in $\mathbb{R}^n$, $\beta(x)$ is a continuous real function on $v^\perp$ and the parameter t runs in an interval of the real axis, say $t \in [0,1]$. In other words, to each chord of K parallel to v we assign a speed vector $\beta(x)v$, where x is the projection of the chord onto $v^{\perp}$; then we let the chords move for a time t and denote by K_t their union. Such a union has to be convex, this is the only restriction we have on defining the speed function β.

Notice that if $\{K_t : t \in [0,1]\}$ is a parallel chord movement, then via Fubini's theorem one deduces that the volume of K_t is independent of t.

Another special instance is the movement related to Steiner's symmetrization. For a direction v and let

$$K = \{x + yv \in \mathbb{R}^n : x \in K|v^{\perp}, y \in \mathbb{R}, f(x) \le y \le g(x)\}, \tag{2.4}$$

here f and $-g$ are convex functions on $K|v^{\perp}$. If let $\beta(x) = -(f(x) + g(x))$ and $t \in [0,1]$ is such that $K_0 = K$ and $K_1 = K^v$, where K^v is the reflection of K in the hyperplane $v^{\perp}$, and $K_{1/2}$ is the Steiner symmetrization of K with respect to $v^{\perp}$.

3 Proof of Orlicz–Lorentz centroid inequality

Let $K \in \mathcal{S}_0^n$ be a star body with respect to the origin in $\mathbb{R}^n$, recall $\phi \in \mathcal{C}$ and the definition of $h_{\Gamma_{\phi,\omega}K}$, there is a lemma obtained by Nguyen [17].

Lemma 3.1 *Suppose $K \in \mathcal{S}_o^n$ and $u_0 \in S^{n-1}$. Then we have*

$$\int_0^1 \phi\left(\frac{f^*_{u_0,K}(t)}{\lambda_0}\right)\omega(t)\,dt = 1,$$

if and only if

$$h_{\Gamma_{\phi,\omega}K}(u_0) = \lambda_0.$$

Let $\{K_t : t \in [0,1]\}$ be a parallel chord movement along the direction v, for $x \in v^{\perp}$, we have

$$\begin{aligned}
\mu_{f_{x,K_t}}(\lambda) &= \mu\left(\left\{y' : \left|f_{x,K_t}(y')\right| > \lambda, y' \in K_t\right\}\right)\\
&= \mu\left(\left\{y : \left|\left\langle x, \left(y + \beta\left(y|v^{\perp}\right)tv\right)\right\rangle\right| > \lambda, y \in K_0\right\}\right)\\
&= \mu\left(\left\{y : \left|f_{x,K_0}(y)\right| > \lambda, y \in K_0\right\}\right)\\
&= \mu_{f_{x,K_0}}(\lambda)
\end{aligned}$$

and

$$\begin{aligned}
f^*_{x,K_t}(s) &= \inf\left\{\lambda > 0 : \mu_{f_{x,K_t}}(\lambda) \le s\right\}\\
&= \inf\left\{\lambda > 0 : \mu_{f_{x,K_0}}(\lambda) \le s\right\}\\
&= f^*_{x,K_0}(s),
\end{aligned}$$

which means that, for $x \in v^{\perp}$, we have $f^*_{x,K_t}(s) = f^*_{x,K_0}(s)$. Moreover, we have

$$\begin{aligned}
h_{\Gamma_{\phi,\omega}K_t}(x) &= \inf\left\{\lambda \ge 0 : \int_0^1 \phi\left(\frac{f^*_{x,K_t}(t)}{\lambda}\right)\omega(t)\,dt \le 1\right\}\\
&= \inf\left\{\lambda \ge 0 : \int_0^1 \phi\left(\frac{f^*_{x,K_0}(t)}{\lambda}\right)\omega(t)\,dt \le 1\right\},
\end{aligned} \tag{3.1}$$

for every $x \in v^{\perp}$. Then we have

$$h_{\Gamma_{\phi,\omega}K_t}(x) = h_{\Gamma_{\phi,\omega}K_0}(x), \tag{3.2}$$

which means that the orthogonal projection of $\Gamma_{\phi,\omega}K_t$ onto $v^{\perp}$ is independent of t. But this is not sufficient to say that $\Gamma_{\phi,\omega}K_t$ is a shadow system. The following lemma, given by Campi and Gronchi (see [2]), grants that a family of convex bodies having constant orthogonal projection onto a fixed hyperplane is actually a shadow system.

Lemma 3.2 *Let $\{K_t : t \in [0,1]\}$, be one parameter family of convex bodies such that $K_t|v^{\perp}$ is independent of t. If the bodies K_t have the following expression*:

$$K_t = \{x + y_t v : |x \in K_t|v^{\perp}, y_t \in \mathbb{R}, f_t(x) \le y_t \le g_t(x)\}, \quad \forall t \in [0,1],$$

for suitable functions $g_t(x), f_t(x)$, then $\{K_t : t \in [0,1]\}$ is a shadow system along the direction v if and only if for every $x \in K_0|v^{\perp}$,

1: *$g_t(x)$ and $-f_t(x)$ are convex functions of the parameter t in* $[0,1]$,

2: *$f_{\mu t_1+(1-\mu)t_2}(x) \le \mu g_{t_1}(x) + (1-\mu)f_{t_2}(x) \le g_{\mu t_1+(1-\mu)t_2}(x)$, for every $t_1, t_2, \mu \in [0,1]$.*

In the following we will prove that a parallel chord movement under the Orlicz–Lorentz centroid operator satisfies the above lemma. Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1 Let $\{K_t : t \in [0,1]\}$ be a parallel chord movement along the direction v. Since the orthogonal projection of $\Gamma_{\phi,\omega}K_t$ onto $v^{\perp}$ is independent of t, it is sufficient to show that the family $\Gamma_{\phi,\omega}K_t$ satisfies conditions 1 and 2 of Lemma 3.2.

As the projection of $\Gamma_{\phi,\omega}K_t$ onto $v^{\perp}$ is independent of t, then, for every $t \in [0,1]$, it can be represented as

$$\Gamma_{\phi,\omega}K_t = \{x + y_t v : x \in (\Gamma_{\phi,\omega}K_0)|v^{\perp}, f_t(x) \le y_t \le g_t(x)\}, \tag{3.3}$$

where $g_t(x)$ and $-f_t(x)$ are concave functions defined on $(\Gamma_{\phi,\omega}K_0)|v^{\perp}$.

On the other hand, by the definition of the support function, let $z \in \Gamma_{\phi,\omega}K_t$ if and only if

$$\langle z, u\rangle \le h_{\Gamma_{\phi,\omega}K_t}(u),$$

for all $u \in \mathbb{R}^n$. Then we obtain

$$\begin{aligned} g_t(x) &= \sup\{\mu \in \mathbb{R} : \langle x + \mu v, u\rangle \le h_{\Gamma_{\phi,\omega}K_t}(u), \forall u \in \mathbb{R}^n\} \\ &= \sup\{\mu \in \mathbb{R} : \mu\langle v, u\rangle \le h_{\Gamma_{\phi,\omega}K_t}(u) - \langle x, u\rangle, \forall u \in \mathbb{R}^n\}, \end{aligned} \tag{3.4}$$

for all $x \in (\Gamma_{\phi,\omega}K_0)|v^{\perp}$. Note that the inner product and support function are both homogeneous of degree 1. Thus in (3.4) we need consider only the vectors u such that $|\langle u, v\rangle| = 1$ and there exists a vector $\varpi \in v^{\perp}$ such that

$$\begin{aligned} g_t(x) &= \sup\{\mu \in \mathbb{R} : \mu \le h_{\Gamma_{\phi,\omega}K_t}(\varpi + v) - \langle x, \varpi + v\rangle, \forall \varpi \in v^{\perp}\} \\ &= \inf_{\varpi \in v^{\perp}}\{h_{\Gamma_{\phi,\omega}K_t}(\varpi + v) - \langle x, \varpi\rangle\}. \end{aligned} \tag{3.5}$$

Notice that $g_t(x)$ is in fact the minimum, as $\varpi \in \nu^\perp$, of $h_{\Gamma_{\phi,\omega}K_t}(\varpi + \nu) - \langle x, \varpi\rangle$, unless x belongs to the boundary of $(\Gamma_{\phi,\omega}K_t)|\nu^\perp$. Actually, the minimum is attained when $\varpi + \nu$ is directed as a normal vector to $\Gamma_{\phi,\omega}K_t$ at $x + g_t(x)\nu$.

By the similar method we have

$$f_t(x) = -\sup\{\lambda \in \mathbb{R} : \langle x - \lambda\nu, u\rangle \le h_{\Gamma_{\phi,\omega}K_t}(u), \forall u \in \mathbb{R}^n\},$$

which implies

$$f_t(x) = -\inf_{\varpi' \in \nu^\perp} \{h_{\Gamma_{\phi,\omega}K_t}(\varpi' - \nu) - \langle x, \varpi'\rangle\}. \tag{3.6}$$

In order to prove the convexity of $g_t(x)$, we only need to prove that, for $\forall t_1, t_2 \in [0, 1]$.

$$2g_{\frac{t_1+t_2}{2}}(x) \le g_{t_1}(x) + g_{t_2}(x).$$

Indeed by (3.4) and (3.5) we have

$$\begin{aligned}2g_{\frac{t_1+t_2}{2}}(x) &= 2\inf_{\varpi\in\nu^\perp}\{h_{\Gamma_{\phi,\omega}K_{\frac{t_1+t_2}{2}}}(\varpi + \nu) - \langle x, \varpi\rangle\}\\ &= \inf_{\varpi\in\nu^\perp}\{h_{\Gamma_{\phi,\omega}K_{\frac{t_1+t_2}{2}}}2(\varpi + \nu) - \langle x, 2\varpi\rangle\}.\end{aligned} \tag{3.7}$$

Let $h_{\Gamma_{\phi,\omega}K_{\frac{t_1}{2}}}(\varpi_1 + \nu) = \lambda_{t_1}$, $h_{\Gamma_{\phi,\omega}K_{\frac{t_2}{2}}}(\varpi_2 + \nu) = \lambda_{t_2}$, and $\varpi_1, \varpi_2 \in \nu^\perp$, then

$$1 = \int_0^1 \phi\left(\frac{f^*_{\frac{1}{2}(\varpi_1+\nu),K_{\frac{t_1}{2}}}(t)}{\frac{\lambda_{t_1}}{2}}\right)\omega(t)\,dt, \tag{3.8}$$

$$1 = \int_0^1 \phi\left(\frac{f^*_{\frac{1}{2}(\varpi_2+\nu),K_{\frac{t_2}{2}}}(t)}{\frac{\lambda_{t_2}}{2}}\right)\omega(t)\,dt. \tag{3.9}$$

And let $y \in K_{\frac{t_1+t_2}{2}}$, then $y = y' + \beta(y')(\frac{t_1+t_2}{2})\nu$, where $y' = y|_{\nu^\perp}$, we define the map $T_1 : K_{\frac{t_1+t_2}{2}} \to K_{\frac{t_1}{2}}$ and $T_2 : K_{\frac{t_1+t_2}{2}} \to K_{\frac{t_2}{2}}$ by

$$T_1 y = y' + \beta(y')\frac{t_1}{2}\nu, \qquad T_2 y = y' + \beta(y')\frac{t_2}{2}\nu.$$

Then we have

$$\begin{aligned}f_{\frac{1}{2}(\varpi_1+\varpi_2)+\nu,K_{\frac{t_1+t_2}{2}}}(y) &= \frac{1}{2}\langle\varpi_1 + \varpi_2, y'\rangle + \frac{1}{2}\beta(y')\left(\frac{t_1+t_2}{2}\right)\\ &= \frac{1}{2}\langle\varpi_1, y'\rangle + \frac{1}{2}\beta(y')\frac{t_1}{2} + \frac{1}{2}\langle\varpi_2, y'\rangle + \frac{1}{2}\beta(y')\frac{t_2}{2}\\ &= f_{\frac{1}{2}(\varpi_1+\nu),K_{\frac{t_1}{2}}}(T_1 y) + f_{\frac{1}{2}(\varpi_2+\nu),K_{\frac{t_2}{2}}}(T_2 y).\end{aligned}$$

Notice that T_1 and T_2 are preserving-measure maps and

$$(f_{\frac{1}{2}(\varpi_1+\nu),K_{\frac{t_1}{2}}} \circ T_1)^* = (f_{\frac{1}{2}(\varpi_1+\nu),K_{\frac{t_1}{2}}})^*,$$

$$(f_{\frac{1}{2}(\varpi_2+\nu),K_{\frac{t_2}{2}}} \circ T_2)^* = (f_{\frac{1}{2}(\varpi_2+\nu),K_{\frac{t_2}{2}}})^*.$$

Since the orthogonal projection of $\Gamma_{\varphi,\omega}K_t$ onto $v^{\perp}$ is independent of t, and for $\phi \in \mathcal{C}$, $\phi(f^*) = (\phi(|f|))^*$ holds for any measurable function f, which implies

$$\begin{aligned}
\phi\left(\frac{f^*_{\frac{1}{2}(\varpi_1+\varpi_2)+v,K_{\frac{t_1+t_2}{2}}}}{\frac{1}{2}\lambda_{t_1}+\frac{1}{2}\lambda_{t_2}}\right) &= \phi\left[\frac{(f_{\frac{1}{2}(\varpi_1+v),K_{\frac{t_1}{2}}}\circ T_1 + f_{\frac{1}{2}(\varpi_2+v),K_{\frac{t_2}{2}}}\circ T_2)^*}{\frac{1}{2}\lambda_{t_1}+\frac{1}{2}\lambda_{t_2}}\right] \\
&= \left[\phi\left(\frac{|f_{\frac{1}{2}(\varpi_1+v),K_{\frac{t_1}{2}}}+f_{\frac{1}{2}(\varpi_2+v),K_{\frac{t_2}{2}}}|}{\frac{1}{2}\lambda_{t_1}+\frac{1}{2}\lambda_{t_2}}\right)\right]^*.
\end{aligned}$$

It is obvious that

$$\frac{|f_{\frac{1}{2}(\varpi_1+v),K_{\frac{t_1}{2}}}+f_{\frac{1}{2}(\varpi_2+v),K_{\frac{t_2}{2}}}|}{\frac{1}{2}\lambda_{t_1}+\frac{1}{2}\lambda_{t_2}} \le \frac{\lambda_{t_1}}{\lambda_{t_1}+\lambda_{t_2}}\frac{|f_{\frac{1}{2}(\varpi_1+v),K_{\frac{t}{2}}}|}{\frac{1}{2}\lambda_{t_1}} + \frac{\lambda_{t_2}}{\lambda_{t_1}+\lambda_{t_2}}\frac{|f_{(\varpi_2+v),K_{\frac{t_2}{2}}}|}{\frac{1}{2}\lambda_{t_2}}. \tag{3.10}$$

The increasing monotonicity and convexity of ϕ together imply

$$\begin{aligned}
&\phi\left(\frac{f^*_{\frac{1}{2}(\varpi_1+\varpi_2)+v,K_{\frac{t_1+t_2}{2}}}}{\frac{1}{2}\lambda_{t_1}+\frac{1}{2}\lambda_{t_2}}\right) \\
&\le \left[\frac{\lambda_{t_1}}{\lambda_{t_1}+\lambda_{t_2}}\phi\left(\frac{|f_{\frac{1}{2}(\varpi_1+v),K_{\frac{t}{2}}}|}{\frac{1}{2}\lambda_{t_1}}\right) + \frac{\lambda_{t_2}}{\lambda_{t_1}+\lambda_{t_2}}\phi\left(\frac{|f_{\frac{1}{2}(\varpi_2+v),K_{\frac{t_2}{2}}}|}{\frac{1}{2}\lambda_{t_2}}\right)\right]^*.
\end{aligned} \tag{3.11}$$

Multiplying both sides of (3.11) by ω, then integrating the inequality on $(0,1)$ and using the fact

$$\int_0^1 (g_1+g_2)^*\omega(t)\,dt \le \int_0^1 (g_1)^*\omega(t)\,dt + \int_0^1 (g_2)^*\omega(t)\,dt$$

we obtain

$$\begin{aligned}
&\int_0^1 \phi\left(\frac{f^*_{\frac{1}{2}(\varpi_1+\varpi_2)+v,K_{\frac{t_1+t_2}{2}}}(t)}{\frac{1}{2}\lambda_{t_1}+\frac{1}{2}\lambda_{t_2}}\right)\omega(t)\,dt \\
&\le \frac{\lambda_{t_1}}{\lambda_{t_1}+\lambda_{t_2}}\int_0^1 \phi\left(\frac{f^*_{\frac{\varpi_1+v}{2},K_{\frac{t_1}{2}}}(t)}{\frac{1}{2}\lambda_{t_1}}\right)\omega(t)\,dt + \frac{\lambda_{t_2}}{\lambda_{t_1}+\lambda_{t_2}}\int_0^1 \phi\left(\frac{f^*_{\frac{\varpi_2+v}{2},K_{\frac{t_2}{2}}}(t)}{\frac{1}{2}\lambda_{t_2}}\right)\omega(t)\,dt \\
&= 1.
\end{aligned}$$

Then we have

$$h_{\Gamma_{\phi,\omega}K_{\frac{t_1+t_2}{2}}}\left(\frac{\varpi_1+\varpi_2}{2}+v\right) \le \frac{\lambda_{t_1}+\lambda_{t_2}}{2}. \tag{3.12}$$

Then we have

$$\begin{aligned}
g_{t_1}(x)+g_{t_2}(x) &= \inf_{\varpi_1\in v^{\perp}}\left\{h_{\Gamma_{\phi,\omega}K_{t_1}}(\varpi_1+v)-\langle x,\varpi_1\rangle\right\} + \inf_{\varpi_2\in v^{\perp}}\left\{h_{\Gamma_{\phi,\omega}K_{t_2}}(\varpi_2+v)-\langle x,\varpi_2\rangle\right\} \\
&= \inf_{\varpi_1\in v^{\perp}}\left\{\lambda_{t_1}-\langle x,\varpi_1\rangle\right\} + \inf_{\varpi_2\in v^{\perp}}\left\{\lambda_{t_2}-\langle x,\varpi_2\rangle\right\}
\end{aligned}$$

$$\begin{aligned} &= \inf_{\varpi\in\nu^{\perp}}\left\{\lambda_{t_1}+\lambda_{t_2}-\langle x,2\varpi\rangle\right\} \\ &\geq \inf_{\varpi\in\nu^{\perp}}\left\{2h_{\Gamma_{\phi,\omega}K_{\frac{t_1+t_2}{2}}}(\varpi+\nu)-\langle x,2\varpi\rangle\right\} \\ &= 2g_{\frac{t_1+t_2}{2}}(x). \end{aligned}$$

The third equality comes from the fact that the sets $\{\lambda_{t_1}-\langle x,\varpi_1\rangle\}$ and $\{\lambda_{t_2}-\langle x,\varpi_2\rangle\}$ are nonempty and bounded sets. Thus we prove the convexity of the $g_t(x)$. By the same method we can prove the convexity of $-f_t(x)$.

Now we need to prove condition 2.

First we prove

$$f_{\mu t_1+(1-\mu)t_2}(x)\leq \mu g_{t_1}(x)+(1-\mu)f_{t_2}(x). \tag{3.13}$$

Let $h_{\Gamma_{\phi,\omega}K_{t_1}}(-\mu\varpi_1+\mu\nu)=\lambda_{t_1}$, and $h_{\Gamma_{\phi,\omega}K_{\mu t_1+(1-\mu)t_2}}(\varpi_2-\nu)=\lambda_{\mu t_1+(1-\mu)t_2}$, we write $\lambda=\lambda_{t_1}+\lambda_{\mu t_1+(1-\mu)t_2}$ for short. We also define the map $T_1':K_{(1-\mu)t_2+\mu t_1}\to K_{t_1}$ and $T_2':K_{(1-\mu)t_2+\mu t_1}\to K_{t_2}$ by $T_1'y=y'+\beta(y')t_1\nu$, $T_2'y=y'+\beta(y')t_2\nu$. Note that

$$\begin{aligned} f_{\varpi_2-\mu\varpi_1-(1-\mu)\nu}K_{t_2}\left(T_2'y\right) &= \left\langle\varpi_2-\mu\varpi_1-(1-\mu)\nu,T_2'y\right\rangle \\ &= \left\langle\varpi_2-\mu\varpi_1-(1-\mu)\nu,y'+\beta\left(y'\right)t_2\nu\right\rangle \\ &= \left\langle\varpi_2-\nu,y'\right\rangle+\left\langle-\mu\varpi_1+\mu\nu,y'\right\rangle-\beta\left(y'\right)\left(t_2(1-\mu)+\mu t_1-\mu t_1\right) \\ &= \left\langle\varpi_2-\nu,y'\right\rangle+\beta\left(y'\right)\left((1-\mu)t_2+\mu t_1\right)\langle\varpi_2-\nu,\nu\rangle \\ &\quad+\left\langle-\mu\varpi_1+\mu\nu,y'\right\rangle+\beta\left(y'\right)t_1\langle-\mu\varpi_1+\mu\nu,\nu\rangle \\ &= f_{\varpi_2-\nu}K_{(1-\mu)t_2+\mu t_1}(y)+f_{-\mu\varpi_1+\mu\nu}K_{t_1}\left(T_1'y\right). \end{aligned}$$

So we have

$$\begin{aligned} \phi\left(\frac{f^*_{\varpi_2-\mu\varpi_1-(1-\mu)\nu}K_{t_2}\circ T_2'}{\lambda}\right) &= \phi\left(\frac{(f_{\varpi_2-\nu}K_{(1-\mu)t_2+\mu t_1}+f_{-\mu\varpi_1+\mu\nu}K_{t_1}\circ T_1')^*}{\lambda}\right) \\ &= \phi\left(\frac{|f_{\varpi_2-\nu}K_{(1-\mu)t_2+\mu t_1}+f_{-\mu\varpi_1+\mu\nu}K_{t_1}\circ T_1'|}{\lambda}\right)^*. \end{aligned}$$

The increasing monotonicity and convexity of ϕ and the preserving-measure maps T_1' and T_2', imply

$$\begin{aligned} &\phi\left(\frac{|f_{\varpi_2-\nu}K_{(1-\mu)t_2+\mu t_1}+f_{-\mu\varpi_1+\mu\nu}K_{t_1}\circ T_1'|}{\lambda}\right) \\ &\quad\leq \frac{\lambda_{t_1}}{\lambda}\phi\left(\frac{|f_{-\mu\varpi_1+\mu\nu}K_{t_1}\circ T_1'|}{\lambda_{t_1}}\right)+\frac{\lambda_{\mu t_1+(1-\mu)t_2}}{\lambda}\phi\left(\frac{|f_{\varpi_2-\nu}K_{(1-\mu)t_2+\mu t_1}|}{\lambda_{\mu t_1+(1-\mu)t_2}}\right). \end{aligned}$$

Then

$$\begin{aligned} &\phi\left(\frac{f^*_{\varpi_2-\mu\varpi_1-(1-\mu)\nu}K_{t_2}}{\lambda}\right) \\ &\quad\leq \left(\frac{\lambda_{t_1}}{\lambda}\phi\left(\frac{|f_{-\mu\varpi_1+\mu\nu}K_{t_1}|}{\lambda_{t_1}}\right)+\frac{\lambda_{\mu t_1+(1-\mu)t_2}}{\lambda}\phi\left(\frac{|f_{\varpi_2-\nu}K_{(1-\mu)t_2+\mu t_1}|}{\lambda_{\mu t_1+(1-\mu)t_2}}\right)\right)^*. \end{aligned}$$

Integrating both sides on $[0,1]$ we have

$$\begin{aligned}
&\int_0^1 \phi\left(\frac{f^*_{\varpi_2-\mu\varpi_1-(1-\mu)\nu}K_{t_2}}{\lambda}\right)\omega(t)\,dt \\
&\quad\le \int_0^1 \left(\frac{\lambda_{t_1}}{\lambda}\phi\left(\frac{|f_{-\mu\varpi_1+\mu\nu}K_{t_1}|}{\lambda_{t_1}}\right) + \frac{\lambda_{\mu t_1+(1-\mu)t_2}}{\lambda}\phi\left(\frac{|f_{\varpi_2-\nu}K_{(1-\mu)t_2+\mu t_1}|}{\lambda_{\mu t_1+(1-\mu)t_2}}\right)\right)^*\omega(t)\,dt \\
&\quad\le \frac{\lambda_{t_1}}{\lambda}\int_0^1 \phi\left(\frac{f^*_{-\mu\varpi_1+\mu\nu}K_{t_1}}{\lambda_{t_1}}\right)\omega(t)\,dt + \frac{\lambda_{\mu t_1+(1-\mu)t_2}}{\lambda}\int_0^1 \phi\left(\frac{f^*_{\varpi_2-\nu}K_{(1-\mu)t_2+\mu t_1}}{\lambda_{\mu t_1+(1-\mu)t_2}}\right)\omega(t)\,dt \\
&\quad= 1.
\end{aligned}$$

The definition of $h(\Gamma_{\phi,\omega}K,\cdot)$ gives

$$\begin{aligned}
&h_{\Gamma_{\phi,\omega}K_{t_2}}\bigl(\varpi_2-\mu\varpi_1-(1-\mu)\nu\bigr) \\
&\quad\le h_{\Gamma_{\phi,\omega}K_{t_1}}(-\mu\varpi_1+\mu\nu) + h_{\Gamma_{\phi,\omega}K_{\mu t_1+(1-\mu)t_2}}(\varpi_2-\nu).
\end{aligned} \tag{3.14}$$

Note that

$$\begin{aligned}
&(1-\mu)f_{t_2}(x) \\
&\quad= -\inf_{\varpi\in\nu^\perp}\bigl\{h_{\Gamma_{\phi,\omega K_{t_2}}}\bigl((1-\mu)(\varpi-\nu)\bigr) - \bigl\langle x,(1-\mu)\varpi\bigr\rangle\bigr\} \\
&\quad= -\inf_{\varpi_1,\varpi_2\in\nu^\perp}\bigl\{h_{\Gamma_{\phi,\omega K_{t_2}}}\bigl((\varpi_2-\mu\varpi_1)-(1-\mu)\nu\bigr)\bigr) - \bigl\langle x,(\varpi_2-\mu\varpi_1\bigr\rangle\bigr\} \\
&\quad\ge -\inf_{\varpi_1,\varpi_2\in\nu^\perp}\bigl\{h_{\Gamma_{\phi,\omega K_{t_1}}}(-\mu\varpi_1+\mu\nu) + h_{\Gamma_{\phi,\omega}K_{\mu t_1+(1-\mu)t_2}}(\varpi_2-\nu) - \langle x,(\varpi_2-\mu\varpi_1)\bigr\} \\
&\quad= -\inf_{\varpi_1\in\nu^\perp}\bigl\{h_{\Gamma_\phi K_{\mu t_1+(1-\mu)t_2}}(\varpi_2-\nu) - \langle x,\varpi_2\rangle\bigr\} - \inf_{\varpi_2\in\nu^\perp}\bigl\{h_{\Gamma_\phi K_{t_1}}(-\mu\varpi_1+\mu\nu) - \langle x,\mu\varpi_1\rangle\bigr\} \\
&\quad= g_{\mu t_1+(1-\mu)t_2}(x) - \mu g_{t_1}(x).
\end{aligned}$$

So we obtain

$$\mu g_{t_1}(x) + (1-\mu)f_{t_2}(x) \ge f_{\mu_{t_1}+(1-\mu)t_2}(x). \tag{3.15}$$

This prove that the left hand of 2 of Lemma 3.2. The same as the right hand inequality. So we complete the proof of Theorem 1.1. □

Specially, when taking $\phi=\phi_p=|\cdot|^p$ in Theorem 1.1, we obtain the following corollary, which was given by Campi and Gronchi.

Corollary 3.1 *Let $\{K_t : t\in[0,1]\}$ be a parallel chord movement along the direction ν, then $\Gamma_\phi K_t$ is a shadow system along the same direction ν.*

By Theorem 1.1 we can give another proof of the Orlicz centroid inequality. First we need the following lemma.

Lemma 3.3 *Let $K \in \mathcal{S}_o^n$ and $\phi \in \mathcal{C}$, denote by K^v the reflection of K in the hyperplane $v^\perp$, then we have*

$$\Gamma_{\phi,\omega}K^v = (\Gamma_{\phi,\omega}K)^v. \tag{3.16}$$

Proof In fact, note that

$$h_{K^v}\big(\mu w + (1-\mu)v\big) = h_K\big(\mu w - (1-\mu)v\big), \tag{3.17}$$

for all $w \in v^\perp$ and $\mu \in [0,1]$. We write $K = \{x : x + yv, x \in K|v^\perp, f(x) \le y \le g(x)\}$. In order to prove (3.16), by (3.17) we only need to prove that

$$h_{\Gamma_{\phi,\omega}K}\big(\mu w - (1-\mu)v\big) = h_{\Gamma_{\phi,\omega}K^v}\big(\mu w + (1-\mu)v\big).$$

Let $h_{\Gamma_{\phi,\omega}K^v}(\mu w + (1-\mu)v) = \lambda_0$, by Lemma 3.1 we obtain

$$1 = \int_0^1 \phi\left(\frac{f^*_{\mu w+(1-\mu)v,K^v}(t)}{\lambda_0}\right)\omega(t)\,dt = \int_0^1 \phi\left(\frac{f^*_{\mu w-(1-\mu)v,K}(t)}{\lambda_0}\right)\omega(t)\,dt.$$

This means that $h_{\Gamma_{\phi,\omega}K}(\mu w + (1-\mu)v) = \lambda_0$, we prove that $\Gamma_{\phi,\omega}K^v = (\Gamma_{\phi,\omega}K)^v$. □

If $\{K_t : t \in [0,1]\}$ is the parallel chord movement related to Steiner symmetrization along v, then

$$2|\Gamma_{\phi,\omega}K_{1/2}| \le |\Gamma_{\phi,\omega}K_0| + |\Gamma_{\phi,\omega}K_1| = |\Gamma_{\phi,\omega}K| + \big|(\Gamma_{\phi,\omega}K)^v\big| = 2|\Gamma_{\phi,\omega}K|,$$

that is, the volume of the Orlicz–Lorentz centroid body is not increased after a Steiner symmetrization. Note that after finite Steiner symmetrizations a convex body can be transformed into a ball. Thus we see that the ratio $|\Gamma_{\phi,\omega}K|/|K|$ attains its minimum value when K is a ball.

Moreover, by the definition of the Orlicz–Lorentz centroid body we know that it is origin symmetric, then we have the following.

Theorem 3.1 *Let $\phi \in \mathcal{C}$, $\{K_t : t \in [0,1]\}$ be a parallel chord movement with speed function β, then the volume of $\Gamma_{\phi,\omega}K_t$ is strictly convex function of t unless β is linear function defined on $v^\perp$, that is, $\beta(x) = \langle x, u\rangle$.*

Proof By definition, $\Gamma_{\phi,\omega}K_t$ is an origin-symmetric convex body, then $-f_t(x) = g_t(-x)$ for all $x \in (\Gamma_{\phi,\omega}K_t)|v^\perp$. Then the volume of $\Gamma_{\phi,\omega}K_t$ can be expressed as

$$\begin{aligned} V(\Gamma_{\phi,\omega}K_t) &= \int_{(\Gamma_{\phi,\omega}K_0)|v^\perp} \big[g_t(x) - f_t(x)\big]\,dx = \int_{(\Gamma_{\phi,\omega}K_0)|v^\perp} \big[g_t(x) + g_t(-x)\big]\,dx \\ &= 2\int_{(\Gamma_{\phi,\omega}K_0)|v^\perp} g_t(x)\,dx. \end{aligned} \tag{3.18}$$

Hence the convexity of the $g_t(x)$ with t implies the convexity of the volume $V(\Gamma_{\phi,\omega}K_t)$.

If $2V(\Gamma_{\phi,\omega}K_{\frac{t_1+t_2}{2}}) = V(\Gamma_{\phi,\omega}K_{t_1}) + V(\Gamma_{\phi,\omega}K_{t_2})$ for some $t_1, t_2 \in [0,1]$, then we deduce that

$$2g_{\frac{t_1+t_2}{2}}(x) = g_{t_1}(x) + g_{t_2}(x), \tag{3.19}$$

for almost every $x \in (\Gamma_{\phi,\omega}K_0)|v^{\perp}$. In fact, the function g_t is a minimum, for every t. Therefore there exist $\varpi_1, \varpi_2 \in v^{\perp}$ such that

$$g_{t_1}(x) + g_{t_2}(x) = h_{\Gamma_{\phi,\omega}K_{t_1}} - \langle x, \varpi_1\rangle + h_{\Gamma_{\phi,\omega}K_{t_2}} - \langle x, \varpi_2\rangle. \tag{3.20}$$

Let $h_{\Gamma_{\phi,\omega}K_{t_1}} = \lambda_{t_1}$, $h_{\Gamma_{\phi,\omega}K_{t_2}} = \lambda_{t_2}$, then we obtain

$$g_{t_1}(x) + g_{t_2}(x) = \lambda_{t_1} + \lambda_{t_2} - 2\left\langle x, \frac{\varpi_1 + \varpi_2}{2}\right\rangle.$$

Thus by (3.12) we have

$$g_{t_1}(x) + g_{t_2}(x) \geq 2h_{\Gamma_{\phi,\omega}K_{\frac{t_1+t_2}{2}}}\left(\frac{\varpi_1 + \varpi_2}{2} + v\right) - 2\left\langle x, \frac{\varpi_1 + \varpi_2}{2}\right\rangle \geq g_{\frac{t_1+t_2}{2}}(x). \tag{3.21}$$

Note that the first inequality in (3.21) holds if and only if $h_{\Gamma_{\phi,\omega}K_{\frac{t_1+t_2}{2}}}(\frac{\varpi_1+\varpi_2}{2} + v) = \frac{\lambda_{t_1}+\lambda_{t_2}}{2}$, which means

$$\int_0^1 \phi\left(\frac{f^*_{2(w+v),K_{\frac{\varpi_1+\varpi_2}{2}}}(t)}{\lambda_{t_1} + \lambda_{t_2}}\right)\omega(t)\,dt = 1.$$

By the convexity of ϕ and the continuity of β, we see that

$$\frac{\langle(\varpi_1 + v), z\rangle + \beta(y|v^{\perp})t_1}{\lambda_{t_1}} = \frac{\langle(\varpi_2 + v), y\rangle + \beta(y|v^{\perp})t_2}{\lambda_{t_2}}, \tag{3.22}$$

for every $y \in K_0$. This means that β is a linear function. Set $y = y' + sv$, $y' \in K_0|v^{\perp}$ in (3.22) and differentiating with respect to the parameter s, it turns out that $\lambda_{t_1}/\lambda_{t_2} = 1$, that is,

$$\big\langle(\varpi_1 + v), z\big\rangle + \beta\big(y|v^{\perp}\big)t_1 = \big\langle(\varpi_2 + v), y\big\rangle + \beta\big(y|v^{\perp}\big)t_2. \tag{3.23}$$

So we conclude that $\beta(x) = \langle x, u\rangle$ for some vector u. This completes the proof. □

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Competing interests
The authors declare that they have no competing interests.

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