Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On complexity of a new Mehrotra-type interior point algorithm for $P_*(\kappa)$ linear complementarity problems

Yiyuan Zhou[1,2*], Mingwang Zhang[1,2] and Zhengwei Huang[3]

*Correspondence: zyy@ctgu.edu.cn
[1]College of Science, China Three Gorges University, Yichang, P.R. China
[2]Three Gorges Mathematical Research Center, China Three Gorges University, Yichang, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, a variant of Mehrotra-type predictor–corrector algorithm is proposed for $P_*(\kappa)$ linear complementarity problems. In this algorithm, a safeguard step is used to avoid small step sizes and a new corrector direction is adopted. The algorithm has polynomial iteration complexity and the iteration bound is $O((14\kappa + 11)\sqrt{(1+4\kappa)(1+2\kappa)}\, n \log \frac{(x^0)^T s^0}{\varepsilon})$. Some numerical results are reported as well.

MSC: 90C51; 90C33

Keywords: Interior point algorithm; $P_*(\kappa)$ linear complementarity problem; Mehrotra-type algorithm; Polynomial complexity

1 Introduction

A $P_*(\kappa)$ linear complementarity problem (LCP) is to find vectors $x \in \mathbb{R}^n$ and $s \in \mathbb{R}^n$ such that

$$\begin{aligned} &Mx + q = s, \\ &x^T s = 0, \\ &x, s \ge 0, \end{aligned} \tag{1}$$

where $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$ matrix and $q \in \mathbb{R}^n$.

LCPs are closely associated with linear programming and quadratic programming. It is well known that a differentiable convex quadratic programming can be formulated as a monotone LCP by exploiting the first-order optimality conditions, and vice versa [1]. Transportation planning and game theory also have a close connection with LCPs [2, 3].

Interior point algorithms for LCPs have been widely studied in the last few decades [4]. In 1991, Kojima et al. [5] extended all the previously known results to $P_*(\kappa)$ LCPs and unified the theory of LCPs from the view point of interior point methods. Since then, many interior point algorithms for linear programming have been extended to $P_*(\kappa)$ LCPs. Illés and Nagy [6], and Miao [7] studied the Mizuno-Todd-Ye type interior point algorithms on $P_*(\kappa)$ LCPs. Cho [8], and Cho and Kim [9] proposed two interior point polynomial algorithms based on kernel functions for $P_*(\kappa)$ LCPs. Using a new updating strategy of the centering parameter, Liu et al. [10] extended two Mehrotra-type predictor–corrector algorithms to sufficient LCPs.

Predictor–corrector algorithms are practical interior point methods for linear programming, quadratic programming and LCPs, variants of which have become backbones of several optimization software packages [11, 12]. Mehrotra [13] proposed a predictor–corrector algorithm for linear programming, in which the coefficient matrices in both predictor steps and corrector steps are the same, and it needs less computational efforts than other methods. After that, several variants of this algorithm have been studied. Jarre and Wechs [14] studied a new primal–dual interior point method in which the search directions are based on corrector directions of Mehrotra-type algorithm. Zhang and Zhang [15] presented a second order Mehrotra-type predictor–corrector algorithm without updating the central path. Salahi et al. [16] found that in a variant of Mehrotra's algorithm, in order to keep iterates in a large neighborhood of the central path, some steps are very small. To avoid small or zero steps, Salahi et al. introduced some safeguards in the corrector steps [16] and a new criterion on predictor step sizes [17], moreover, they proved that the two algorithms have polynomial complexity and practical efficiency. Infeasible versions of Mehrotra-type algorithms are studied by Liu et al. [18], and Yang et al. [19].

In this paper, a new variant of Mehrotra-type predictor–corrector algorithm is proposed for $P_*(\kappa)$ LCPs. In this algorithm, the corrector step is different from other Mehrotra-type predictor–corrector algorithms [16, 17]. If $(\Delta x, \Delta s)$ is the search direction of a $P_*(\kappa)$ LCP, then $\Delta x^T\Delta s \neq 0$, while $\Delta x^T\Delta s = 0$ if $(\Delta x, \Delta s)$ is the search direction of linear programming. So the analysis is different from that in linear programming. If an iteration (x,s) takes a step along the corrector direction $(\Delta x, \Delta s)$, the parameter $\mu_g(\alpha) = (1-\alpha)\mu_g + \alpha\mu - \alpha\alpha_a^2\frac{\Delta x^{aT}\Delta s^a}{n} + \alpha^2\frac{\Delta x^T\Delta s}{n}$, where α_a is the predictor step size and $(\Delta x^a, \Delta s^a)$ is the predictor search direction. In order to reduce the dual gap x^Ts, the corrector step size α should be chosen such that $\mu_g(\alpha) \leq \mu_g$, that is to say, α should have an upper bound. If α is larger than a given threshold, then the threshold is chosen as the corrector step size. The iteration complexity of the new algorithm is $O((14\kappa + 11)\sqrt{(1+4\kappa)(1+2\kappa)}\, n\log\frac{(x^0)^Ts^0}{\varepsilon})$, which is analogous to that of linear programming.

This paper is organized as follows. In Sect. 2, a new Mehrotra-type predictor–corrector algorithm for $P_*(\kappa)$ LCPs is introduced. In Sect. 3, the polynomial iteration complexity is provided. Some illustrative numerical results are reported in Sect. 4. Finally, some concluding remarks are given in Sect. 5.

We use the following notations throughout the paper: $\|\cdot\|$ denotes the 2-norm of vectors, e is the n-dimensional vector of ones. For any two n-dimensional vectors x and s, xs is the componentwise product of the two vectors. We also use the following notations.

$$I = \{1,2,\ldots,n\}, \qquad I_+ = \left\{i \in I | \Delta x_i^a \Delta s_i^a \geq 0\right\}, \qquad I_- = \left\{i \in I | \Delta x_i^a \Delta s_i^a < 0\right\},$$
$$F_+ = \left\{(x,s) \in \mathbb{R}^n \times \mathbb{R}^n | s = Mx + q, (x,s) \geq 0\right\},$$
$$F_{++} = \left\{(x,s) \in \mathbb{R}^n \times \mathbb{R}^n | s = Mx + q, (x,s) > 0\right\}.$$

2 Mehrotra-type predictor–corrector algorithm

A matrix $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$ matrix [5] if

$$(1+4\kappa)\sum_{j\in J_+} x_j(Mx)_j + \sum_{j\in J_-} x_j(Mx)_j \geq 0, \quad \forall x \in \mathbb{R}^n, \tag{2}$$

or

$$x^T Mx \geq -4\kappa \sum_{j\in J_+} x_j(Mx)_j, \quad \forall x \in \mathbb{R}^n, \tag{3}$$

where $\kappa \geq 0$, $J_+ = \{j|j \in I, x_j(Mx)_j \geq 0\}$, and $J_- = \{j|j \in I, x_j(Mx)_j < 0\}$.

Noting that the positive semi-definite matrix is a $P_*(0)$ matrix, thus the class of $P_*(\kappa)$ matrices includes the class of positive semi-definite matrices. Other properties of $P_*(\kappa)$ matrices can be found in [5].

Without loss of generality [5], we assume that the $P_*(\kappa)$ LCP (1) satisfies the interior point condition, that is, there exists a point (x^0, s^0) such that

$$s^0 = Mx^0 + q, \qquad x^0 > 0, s^0 > 0.$$

To find an approximate solution of (1), the following parameterized system is established:

$$\begin{aligned} &Mx + q = s, \\ &xs = \mu e, \\ &x, s \geq 0, \end{aligned} \tag{4}$$

where $\mu > 0$.

If the $P_*(\kappa)$ LCP (1) satisfies the interior point condition, then the system (4) has a unique solution for any $\mu > 0$. For a given μ, the solution, denoted by $(x(\mu), s(\mu))$, is called a μ-center of (1). The set of all μ-centers gives the central path of (1). A primal–dual interior point algorithm follows the central path $\{(x(\mu), s(\mu))|\mu > 0\}$ approximately and approaches the solution of (1) as μ goes to zero.

Most interior point algorithms work in the neighborhood $N_\infty^-(\gamma)$ defined by

$$N_\infty^-(\gamma) = \bigl\{(x, s) \in F_{++}|x_i s_i \geq \gamma \mu_g, \forall i \in I\bigr\},$$

where $\gamma \in (0, 1)$ is a constant independent of n and $\mu_g = \frac{x^T s}{n}$.

Now, based on [16], a new variant of Mehrotra-type predictor–corrector algorithm for $P_*(\kappa)$ LCPs will be described.

The predictor search direction $(\Delta x^a, \Delta s^a)$ is determined by the following equations:

$$\begin{aligned} &M\Delta x^a = \Delta s^a, \\ &s\Delta x^a + x\Delta s^a = -xs. \end{aligned} \tag{5}$$

The predictor step size α_a is defined by

$$\alpha_a = \max\bigl\{\alpha|0 \leq \bigl(x + \alpha_a \Delta x^a, s + \alpha_a \Delta s^a\bigr), 0 < \alpha \leq 1\bigr\}. \tag{6}$$

However, this algorithm does not take a step along the direction $(\Delta x^a, \Delta s^a)$. Using information of the predictor step, the algorithm computes the corrector search direction

$(\Delta x, \Delta s)$ by solving the following system:

$$\begin{aligned} &M\Delta x = \Delta s, \\ &s\Delta x + x\Delta s = \mu e - xs - \alpha_a^2 \Delta x^a \Delta s^a, \end{aligned} \tag{7}$$

where

$$\mu = \left(\frac{g_a}{g}\right)^2 \frac{g_a}{n} \tag{8}$$

with $g_a = (x + \alpha_a \Delta x^a)^T (s + \alpha_a \Delta s^a)$ and $g = x^T s$. The second equation of (7) is different from that in [16], where it is $s\Delta x + x\Delta s = \mu e - xs - \Delta x^a \Delta s^a$.

The next iterate is denoted by

$$x(\alpha) = x + \alpha \Delta x, s(\alpha) = s + \alpha \Delta s,$$

where α is the corrector step size defined by

$$\alpha = \max\left\{\alpha \,|\, \big(x(\alpha), s(\alpha)\big) \in N_\infty^-(\gamma), 0 < \alpha \leq 1\right\}. \tag{9}$$

In order to avoid small steps, we combine Mehrotra's updating strategy of the centering parameter with a safeguard step at each iteration. The new Mehrotra-type predictor–corrector algorithm for $P_*(\kappa)$ LCPs is stated as Algorithm 1.

Algorithm 1 Mehrotra-type predictor–corrector algorithm for $P_*(\kappa)$ LCPs

Input:
  A proximity parameter $\gamma \in (0, \frac{1}{4\kappa+5})$;
  an accuracy parameter $\varepsilon > 0$;
  a starting point $(x^0, s^0) \in N_\infty^-(\gamma)$.
while $x^T s \geq \varepsilon$ **do**
begin (**Predictor Step**)
  Solve (5) and compute the predictor step size α_a by (6).
end
begin (**Corrector Step**)
  If $\alpha_a \geq 0.3$, **then** solve (7) with $\mu = (\frac{g_a}{g})^2 \frac{g_a}{n}$ and compute the corrector step size α by (9).
  end
  If $\alpha_a < 0.3$ or $\alpha < \frac{7\gamma}{16pn}$, where $p = \frac{11+14\kappa}{16}\sqrt{(1+4\kappa)(2+4\kappa)}$, **then** solve (7) with $\mu = \frac{\gamma}{1-\gamma}\mu_g$ and compute the corrector step size α.
  end
  If the corrector step size $\alpha > \alpha_1$, **then** let $\alpha = \alpha_1$, where $\alpha_1 = \frac{1-2\gamma-(1-\gamma)\kappa\alpha_a^2}{2q(1-\gamma)}$ and $q = \frac{14\kappa+11}{16}$.
  end
  Set $(x, s) = (x(\alpha), s(\alpha))$.
end

3 Complexity analysis

The polynomial complexity of Algorithm 1 will be discussed in this section. Firstly, we present three important lemmas which will be used in convergence analysis.

Lemma 1 ([16]) *Let* $(\Delta x^a, \Delta s^a)$ *be the solution of* (5). *Then*

$$\begin{aligned}
&\Delta x_i^a \Delta s_i^a \leq \frac{1}{4} x_i s_i, \quad \forall i \in I_+,\\
&-\Delta x_i^a \Delta s_i^a \leq \frac{1}{\alpha_a}\left(\frac{1}{\alpha_a} - 1\right) x_i s_i, \quad \forall i \in I_-,\\
&\sum_{i \in I_+} \Delta x_i^a \Delta s_i^a \leq \frac{x^T s}{4}.
\end{aligned}$$

Lemma 2 *Let* M *be a* $P_*(\kappa)$ *matrix and* $(\Delta x^a, \Delta s^a)$ *be the solution of* (5). *Then*

$$\begin{aligned}
&\sum_{i \in I_-} \left|\Delta x_i^a \Delta s_i^a\right| \leq \frac{4\kappa + 1}{4} x^T s,\\
&-\kappa x^T s \leq \Delta x^{aT} \Delta s^a \leq \frac{x^T s}{4}.
\end{aligned}$$

Proof Since M is a $P_*(\kappa)$ matrix, we have

$$0 \geq \sum_{i \in I_-} \Delta x_i^a \Delta s_i^a \geq -(4\kappa + 1) \sum_{i \in I_+} \Delta x_i^a \Delta s_i^a \geq -\frac{4\kappa + 1}{4} x^T s,$$

where the last inequality is due to the third conclusion of Lemma 1. Furthermore, from (3), we get

$$\left(\Delta x^a\right)^T \Delta s^a \geq -4\kappa \sum_{i \in I_+} \Delta x_i^a \Delta s_i^a \geq -4\kappa \frac{x^T s}{4} = -\kappa x^T s.$$

This completes the proof. □

Lemma 3 *Let* M *be a* $P_*(\kappa)$ *matrix and* $(\Delta x, \Delta s)$ *be the solution of* (7) *with* $\mu > 0$. *Then*

$$\begin{aligned}
&\|\Delta x \Delta s\| \leq \sqrt{\left(\frac{1}{4} + \kappa\right)\left(\frac{1}{2} + \kappa\right)} \left\|\mu (xs)^{-\frac{1}{2}} - (xs)^{\frac{1}{2}} - \alpha_a^2 (xs)^{-\frac{1}{2}} \Delta x^a \Delta s^a\right\|^2,\\
&\sum_{i \in I_+} \Delta x_i \Delta s_i \leq \frac{1}{4} \left\|\mu (xs)^{-\frac{1}{2}} - (xs)^{\frac{1}{2}} - \alpha_a^2 (xs)^{-\frac{1}{2}} \Delta x^a \Delta s^a\right\|^2.
\end{aligned}$$

Proof The proof is similar to that of Lemma 8 in [6], and it is omitted. □

According to (8) and Lemma 2, it can be found that

$$\begin{aligned}
\left(\frac{g_a}{g}\right)^2 \frac{g_a}{n} &= \frac{[(1-\alpha_a) x^T s + \alpha_a^2 (\Delta x^a)^T \Delta s^a]^3}{n (x^T s)^2}\\
&\leq \frac{[(1-\alpha_a) x^T s + \frac{1}{4} \alpha_a^2 x^T s]^3}{n (x^T s)^2}
\end{aligned}$$

$$= \left(1 - \alpha_a + \frac{1}{4}\alpha_a^2\right)^3 \mu_g$$
$$\leq \left(1 - \frac{3}{4}\alpha_a\right)^3 \mu_g. \tag{10}$$

Consequently, $\frac{\mu}{\mu_g} \leq (1 - \frac{3}{4}\alpha_a)^3$ if $\mu = (\frac{g_a}{g})^2 \frac{g_a}{n}$.

The following theorem shows that the predictor step size has a lower bound.

Theorem 4 *Let the current iterate $(x, s) \in N_\infty^-(\gamma)$, $(\Delta x^a, \Delta s^a)$ be the solution of* (5) *and α_a be the predictor step size. Then*

$$\alpha_a \geq \sqrt{\frac{\gamma}{(4\kappa + 1)n}}.$$

Proof According to (5), we get

$$\left(x_i + \alpha \Delta x_i^a\right)\left(s_i + \alpha \Delta s_i^a\right) = (1 - \alpha)x_i s_i + \alpha^2 \Delta x_i^a \Delta s_i^a.$$

Following from Lemma 2, we have $\Delta x_i^a \Delta s_i^a \geq 0$ if $i \in I_+$ and $\Delta x_i^a \Delta s_i^a \geq -\frac{4\kappa+1}{4} x^T s$ if $i \in I_-$. Therefore, for all $i \in I$,

$$\Delta x_i^a \Delta s_i^a \geq -\frac{4\kappa + 1}{4} x^T s.$$

Noting that $(x, s) \in N_\infty^-(\gamma)$ implies $x_i s_i \geq \gamma \mu_g$, we have

$$(1 - \alpha)x_i s_i + \alpha^2 \Delta x_i^a \Delta s_i^a \geq (1 - \alpha)\gamma \frac{x^T s}{n} - \frac{4\kappa + 1}{4}\alpha^2 x^T s.$$

Thus, to show $(x + \alpha \Delta x^a, s + \alpha \Delta s^a) \in F_+$, one has to prove the following inequality:

$$(4\kappa + 1)n\alpha^2 + 4\gamma\alpha - 4\gamma \leq 0. \tag{11}$$

Clearly, inequality (11) is true if

$$0 \leq \alpha \leq \frac{2\sqrt{\gamma^2 + (4\kappa + 1)n\gamma} - 2\gamma}{(4\kappa + 1)n}.$$

Therefore, the predictor step size α_a satisfies

$$\alpha_a \geq \frac{2\sqrt{\gamma^2 + (4\kappa + 1)n\gamma} - 2\gamma}{(4\kappa + 1)n}.$$

Since $0 < \frac{\gamma}{(4\kappa+1)n} < \frac{1}{4\kappa+1}\frac{1}{4\kappa+5}\frac{1}{n} < \frac{1}{2}$, it can be found that

$$\frac{2\sqrt{\gamma^2 + (4\kappa + 1)n\gamma} - 2\gamma}{(4\kappa + 1)n} = \frac{2}{1 + \sqrt{1 + (4\kappa + 1)\frac{n}{\gamma}}} \geq \sqrt{\frac{\gamma}{(4\kappa + 1)n}}.$$

This completes the proof. $\square$

In what follows, the lower bound of α_a is used as the default predictor step size with the notation

$$\alpha_a = \sqrt{\frac{\gamma}{(4\kappa+1)n}}. \tag{12}$$

Lemma 5 *If $(x,s) \in N_\infty^-(\gamma)$, and $(\Delta x, \Delta s)$ is the solution of (7) with $\mu \geq 0$, then*

$$\| \Delta x \Delta s \| \leq \omega \sqrt{\left(\frac{1}{4}+\kappa\right)\left(\frac{1}{2}+\kappa\right)}$$

and

$$\Delta x^T \Delta s \leq \frac{1}{4}\omega,$$

where $\omega = \frac{n\mu^2}{\gamma\mu_g} - 2n\mu + \frac{n\mu\alpha_a^2(4\kappa+1)}{2\gamma} + \frac{\alpha_a^4+8\alpha_a^2+4\alpha_a^2(4\kappa+1)(1-\alpha_a)+16}{16} n\mu_g$.

Proof From Lemma 3, we have

$$\|\Delta x \Delta s\| \leq \sqrt{\left(\frac{1}{4}+\kappa\right)\left(\frac{1}{2}+\kappa\right)} \left\| \mu(xs)^{-\frac{1}{2}} - (xs)^{\frac{1}{2}} - \alpha_a^2 (xs)^{-\frac{1}{2}} \Delta x^a \Delta s^a \right\|^2$$

and

$$\Delta x^T \Delta s \leq \frac{1}{4} \left\| \mu(xs)^{-\frac{1}{2}} - (xs)^{\frac{1}{2}} - \alpha_a^2 (xs)^{-\frac{1}{2}} \Delta x^a \Delta s^a \right\|^2,$$

where

$$\begin{aligned} &\left\| \mu(xs)^{-\frac{1}{2}} - (xs)^{\frac{1}{2}} - \alpha_a^2 (xs)^{-\frac{1}{2}} \Delta x^a \Delta s^a \right\|^2 \\ &\quad = \mu^2 \sum_{i\in I} \frac{1}{x_i s_i} + \sum_{i \in I} x_i s_i - 2n\mu + \alpha_a^4 \sum_{i\in I} \frac{(\Delta x_i^a \Delta s_i^a)^2}{x_i s_i} \\ &\qquad - 2\mu\alpha_a^2 \sum_{i\in I} \frac{\Delta x_i^a \Delta s_i^a}{x_i s_i} + 2\alpha_a^2 \sum_{i\in I} \Delta x_i^a \Delta s_i^a. \end{aligned}$$

Since $x_i s_i \geq \gamma\mu_g$ for all $i \in I$, we have

$$\mu^2 \sum_{i\in I} \frac{1}{x_i s_i} \leq \frac{n\mu^2}{\gamma\mu_g}.$$

From Lemma 1 and Lemma 2, it follows that

$$\begin{aligned} \sum_{i\in I} \frac{(\Delta x_i^a \Delta s_i^a)^2}{x_i s_i} &= \sum_{i\in I_+} \frac{(\Delta x_i^a \Delta s_i^a)^2}{x_i s_i} + \sum_{i\in I_-} \frac{(\Delta x_i^a \Delta s_i^a)^2}{x_i s_i} \\ &\leq \sum_{i\in I_+} \frac{(\frac{x_i s_i}{4})^2}{x_i s_i} + \sum_{i\in I_-} \frac{-\Delta x_i^a \Delta s_i^a}{x_i s_i} \left(-\Delta x_i^a \Delta s_i^a\right) \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{x^T s}{16} + \frac{1-\alpha_a}{\alpha_a^2} \sum_{i \in I_-} \left| \Delta x_i^a \Delta s_i^a \right| \\
&\leq \frac{n\mu_g}{16} + \frac{(1-\alpha_a)(4\kappa+1)n\mu_g}{4\alpha_a^2}.
\end{aligned}
$$

Since $(x,s) \in N_\infty^-(\gamma)$ and $\sum_{i\in I_-} |\Delta x_i^a \Delta s_i^a| \leq \frac{4\kappa+1}{4} x^T s$, we obtain

$$
-2\mu \sum_{i \in I} \frac{\Delta x_i^a \Delta s_i^a}{x_i s_i} \leq 2\mu \sum_{i \in I_-} \frac{|\Delta x_i^a \Delta s_i^a|}{x_i s_i} \leq \frac{2\mu}{\gamma \mu_g} \sum_{i \in I_-} \left| \Delta x_i^a \Delta s_i^a \right| \leq \frac{n\mu(4\kappa+1)}{2\gamma}.
$$

As $\sum_{i\in I_+} \Delta x_i^a \Delta s_i^a \leq \frac{x^T s}{4}$, we get

$$
2 \sum_{i \in I} \Delta x_i^a \Delta s_i^a \leq 2 \sum_{i \in I_+} \Delta x_i^a \Delta s_i^a \leq \frac{n\mu_g}{2}.
$$

Combining the above inequalities yields the result of this lemma. □

Corollary 6 *If* $(x,s) \in N_\infty^-(\gamma)$, $\gamma \in (0, \frac{1}{4\kappa+5})$, *and* $(\Delta x, \Delta s)$ *is the solution of* (7) *with* $\mu = \frac{\gamma}{1-\gamma}\mu_g$, *then*

$$
\|\Delta x \Delta s\| \leq pn\mu_g, \qquad \Delta x^T \Delta s \leq qn\mu_g,
$$

where $p = \frac{14\kappa+11}{16}\sqrt{(1+4\kappa)(2+4\kappa)}$ *and* $q = \frac{14\kappa+11}{16}$.

Proof Since $0 < \gamma < \frac{1}{4\kappa+5} \leq \frac{1}{5}$, we have $\frac{\gamma}{(1-\gamma)^2} \leq \frac{5}{16}$ and $\frac{\alpha_a^2(4\kappa+1)-4\gamma}{2(1-\gamma)} \leq \frac{5\alpha_a^2(4\kappa+1)}{8}$.
If $\mu = \frac{\gamma}{1-\gamma}\mu_g$, then

$$
\begin{aligned}
\omega &= \left(\frac{\gamma}{(1-\gamma)^2} + \frac{\alpha_a^2(4\kappa+1)-4\gamma}{2(1-\gamma)} + \frac{\alpha_a^4 + 8\alpha_a^2 + 4\alpha_a^2(4\kappa+1)(1-\alpha_a)+16}{16} \right) n\mu_g \\
&\leq \left(\frac{5}{16} + \frac{5\alpha_a^2(4\kappa+1)}{8} + \frac{\alpha_a^4 + 8\alpha_a^2 + 4\alpha_a^2(4\kappa+1)(1-\alpha_a)+16}{16} \right) n\mu_g \\
&\leq \left(\frac{5}{16} + \frac{5(4\kappa+1)}{8} + \frac{1+8+4(4\kappa+1)+16}{16} \right) n\mu_g = \frac{56\kappa+44}{16} n\mu_g,
\end{aligned}
$$

where the second inequality follows from $0 < \alpha_a \leq 1$.
From Lemma 5, it follows that

$$
\|\Delta x \Delta s\| \leq \omega \sqrt{\left(\frac{1}{4}+\kappa\right)\left(\frac{1}{2}+\kappa\right)} \leq pn\mu_g.
$$

Similarly, the second result can easily be verified. □

For simplicity, the following notation is used in the rest of this paper:

$$
t = \max_{i \in I_+} \left\{ \frac{\Delta x_i^a \Delta s_i^a}{x_i s_i} \right\}.
$$

Obviously, $\Delta x_i^a \Delta s_i^a \leq t x_i s_i$ and $t \leq \frac{1}{4}$.

Theorem 7 *Let $(x,s) \in N_\infty^-(\gamma)$, $(\Delta x, \Delta s)$ be the solution of* (7) *with $\mu = \frac{\gamma}{1-\gamma}\mu_g$ and α be the corrector step size. Then*

$$\alpha \geq \frac{7\gamma}{16pn}.$$

Proof The corrector step size is the maximum α such that $\alpha \in (0,1]$ and

$$x_i(\alpha)s_i(\alpha) \geq \gamma\mu_g(\alpha), \quad \forall i \in I.$$

After a simple computation, we get

$$\begin{aligned} x_i(\alpha)s_i(\alpha) &= x_is_i + \alpha\bigl(\mu - x_is_i - \alpha_a^2\Delta x_i^a\Delta s_i^a\bigr) + \alpha^2\Delta x_i\Delta s_i \\ &\geq \bigl(1-\alpha-\alpha t\alpha_a^2\bigr)x_is_i + \alpha\mu - \alpha^2 pn\mu_g, \end{aligned}$$

where the inequality is due to $\Delta x_i^a\Delta s_i^a \leq tx_is_i$ and $\|\Delta x\Delta s\| \leq pn\mu_g$. Clearly, $1 + t\alpha_a^2 \leq 1 + \frac{1}{4}\alpha_a^2$. Thus, for $0 \leq \alpha \leq \frac{1}{1+\frac{1}{4}\alpha_a^2}$, we have

$$x_i(\alpha)s_i(\alpha) \geq \bigl(1-\alpha-\alpha t\alpha_a^2\bigr)\gamma\mu_g + \alpha\mu - \alpha^2 pn\mu_g.$$

Applying Lemma 2 and Corollary 6 yields

$$\begin{aligned} \mu_g(\alpha) &= \frac{(x+\alpha\Delta x)^T(s+\alpha\Delta s)}{n} \\ &= (1-\alpha)\mu_g + \alpha\mu - \alpha\alpha_a^2\frac{\Delta x^{aT}\Delta s^a}{n} + \alpha^2\frac{\Delta x^T\Delta s}{n} \\ &\leq (1-\alpha)\mu_g + \alpha\mu + \alpha\alpha_a^2\kappa\mu_g + \alpha^2 q\mu_g. \end{aligned} \tag{13}$$

To prove $x_i(\alpha)s_i(\alpha) \geq \gamma\mu_g(\alpha)$, one has to show that

$$\bigl(1-\alpha-\alpha t\alpha_a^2\bigr)\gamma\mu_g + \alpha\mu - \alpha^2 pn\mu_g \geq \gamma\bigl[(1-\alpha)\mu_g + \alpha\mu + \alpha\alpha_a^2\kappa\mu_g + \alpha^2 q\mu_g\bigr].$$

If $\mu = \frac{\gamma}{1-\gamma}\mu_g$, then the above inequality is equivalent to

$$\gamma - \gamma\kappa\alpha_a^2 - t\gamma\alpha_a^2 \geq \alpha(pn + q\gamma). \tag{14}$$

Since $\alpha_a = \sqrt{\frac{\gamma}{(4\kappa+1)n}}$, $\gamma < 1$ and $t \leq \frac{1}{4}$, we have

$$\gamma - \gamma\kappa\alpha_a^2 - t\gamma\alpha_a^2 \geq \gamma - \frac{4\kappa\gamma^2+\gamma^2}{4n(4\kappa+1)} \geq \gamma - \frac{4\kappa\gamma+\gamma}{4n(4\kappa+1)} \geq \frac{7}{8}\gamma.$$

Noting that $pn > q\gamma$, then inequality (14) is true if $0 \leq \alpha \leq \frac{7\gamma}{16pn}$. We can conclude that the corrector step size α satisfies

$$\alpha \geq \min\left\{\frac{1}{1+\frac{1}{4}\alpha_a^2}, \frac{7\gamma}{16pn}\right\} = \frac{7\gamma}{16pn}.$$

□

In Algorithm 1, the corrector step size α has an upper bound α_1, that is,

$$\alpha \le \alpha_1 = \frac{1-2\gamma-(1-\gamma)\kappa\alpha_a^2}{2q(1-\gamma)}. \tag{15}$$

The next theorem means that the upper bound is well defined.

Theorem 8 *Let α_a be the default predictor step size and $\gamma \in (0, \frac{1}{4\kappa+5})$. Then $\frac{7\gamma}{16pn} < \frac{1-2\gamma-(1-\gamma)\kappa\alpha_a^2}{2q(1-\gamma)} < 1$.*

Proof Since $q = \frac{14\kappa+11}{16} > \frac{1}{2}$, it is clear that $\frac{1-2\gamma-(1-\gamma)\kappa\alpha_a^2}{2q(1-\gamma)} < 1$. As $n \ge 2$ and $p > q > \frac{1}{2}$, we have

$$\frac{7\gamma}{16pn} \le \frac{7\gamma}{16}.$$

According to (12), one has $\kappa\alpha_a^2 = \frac{\kappa\gamma}{n(4\kappa+1)} \le \frac{\gamma}{8}$, thus

$$\frac{1-2\gamma-(1-\gamma)\kappa\alpha_a^2}{1-\gamma} \ge \frac{1-2\gamma-(1-\gamma)\frac{\gamma}{8}}{1-\gamma} = 2 + \frac{1}{\gamma-1} - \frac{\gamma}{8}.$$

From $\gamma < \frac{1}{4\kappa+5} \le \frac{1}{5}$, it follows that

$$2 + \frac{1}{\gamma-1} - \frac{\gamma}{8} > \frac{7\gamma}{16}.$$

Therefore

$$\frac{1-2\gamma-(1-\gamma)\kappa\alpha_a^2}{2q(1-\gamma)} > \frac{7\gamma}{16pn},$$

which completes the proof. □

In the following theorem, we obtain an upper bound of the iteration number.

Theorem 9 *Algorithm 1 stops after at most*

$$O\left((14\kappa+11)\sqrt{(1+4\kappa)(1+2\kappa)}\, n \log\frac{(x^0)^T s^0}{\varepsilon}\right)$$

iterations with a solution for which $x^T s \le \varepsilon$.

Proof If $\alpha_a \ge 0.3$ and $\alpha \ge \frac{7\gamma}{16pn}$, then Algorithm 1 adopts Mehrotra's strategy in the corrector step, i.e., $\mu = (\frac{g_a}{g})^2\frac{g_a}{n}$. Based on (13), we have

$$\begin{aligned}
\mu_g(\alpha) &\le \left[1-\left(1-\kappa\alpha_a^2 - q\alpha - \frac{\mu}{\mu_g}\right)\alpha\right]\mu_g \\
&\le \left[1-\left(1-\kappa\alpha_a^2 - q\frac{1-2\gamma-(1-\gamma)\kappa\alpha_a^2}{2q(1-\gamma)} - \left(1-\frac{3}{4}\alpha_a\right)^3\right)\alpha\right]\mu_g \\
&\le \left[1-\left(1-0.0125 - \frac{1-2\gamma}{2(1-\gamma)} - 0.47\right)\alpha\right]\mu_g
\end{aligned}$$

$$\begin{aligned}
&\le \left[1 - \frac{\gamma}{2(1-\gamma)}\alpha\right]\mu_g \\
&\le \left[1 - \frac{7\gamma^2}{32pn(1-\gamma)}\right]\mu_g,
\end{aligned}$$

where the second inequality follows from (10) and (15), the third inequality is due to $\kappa\alpha_a^2 \le 0.025$ and $1 - \frac{3}{4}\alpha_a \le \frac{31}{40}$, and the fourth inequality comes from $1 - 0.0125 - 0.47 \ge \frac{1}{2}$.

If $\alpha_a < 0.3$ or $\alpha < \frac{7\gamma}{16pn}$, then Algorithm 1 adopts the safeguard strategy in the corrector step, that is, $\mu = \frac{\gamma}{1-\gamma}\mu_g$. In this case, from $\frac{7\gamma}{16pn} \le \alpha \le \frac{1-2\gamma-(1-\gamma)\kappa\alpha_a^2}{2q(1-\gamma)}$, we get

$$\begin{aligned}
\mu_g(\alpha) &\le \left[1 - \left(1 - \kappa\alpha_a^2 - q\alpha - \frac{\mu}{\mu_g}\right)\alpha\right]\mu_g \\
&\le \left[1 - \left(1 - \kappa\alpha_a^2 - q\frac{1-2\gamma-(1-\gamma)\kappa\alpha_a^2}{2q(1-\gamma)} - \frac{\gamma}{1-\gamma}\right)\alpha\right]\mu_g \\
&= \left[1 - \left(\frac{1-2\gamma}{2(1-\gamma)} - \frac{\kappa\alpha_a^2}{2}\right)\alpha\right]\mu_g \\
&\le \left[1 - \frac{(39-79\gamma)7\gamma}{1280pn(1-\gamma)}\right]\mu_g,
\end{aligned}$$

where the last inequality follows from $\kappa\alpha_a^2 \le 0.025$ and $\alpha \ge \frac{7\gamma}{16pn}$. This completes the proof by Theorem 3.2 of [1]. □

4 Numerical results

In this section, some numerical results are reported. The results are obtained by using MATLAB R2014a.

The algorithm presented in this paper is compared with a Mizuno–Todd–Ye (MTY) type predictor–corrector algorithm [6], Cho's algorithm [8] and an interior point algorithm based on the classical kernel function $\varphi(t) = \frac{t^2-1}{2} - \log t$ (IPMCKF) [20]. We consider the following two problems.

Problem 1 ([21])

$$M = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \qquad q = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad x_0 = \begin{pmatrix} 0.4 \\ 0.45 \end{pmatrix}, \qquad s_0 = \begin{pmatrix} 2.45 \\ 2.2 \end{pmatrix}.$$

Problem 2 ([22])

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \cdots & 10 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 6 & 10 & \cdots & 4n-3 \end{pmatrix}, \qquad q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}, \qquad x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Problem 1 is a $P_*(\frac{1}{4})$ LCP, and Problem 2 is a $P_*(0)$ LCP. We solve the two problems by using the above-mentioned algorithms. For all algorithms we set the accuracy parameter $\epsilon = 10^{-8}$. In Algorithm 1, we set $\gamma = 0.01$.

Table 1 The iteration numbers of Problem 1

Algorithm 1	MTY	Cho's	IPMCKF
4	55	19	24

Table 2 The iteration numbers of Problem 2

n	Algorithm 1	MTY	Cho's	IPMCKF
10	10	31	31	34
20	11	37	44	39
30	12	40	49	43
40	13	40	52	44
50	13	49	53	47
100	15	58	64	52
150	15	73	68	55
200	16	85	71	57

Table 1 shows the iteration numbers of Algorithm 1, MTY algorithm, Cho's algorithm and IPMCKF algorithm for Problem 1. From the results we conclude that Algorithm 1 reduced the iteration numbers.

Table 2 gives the iteration numbers of the four algorithms for Problem 2 with $n \in \{10, 20, 30, 40, 50, 100, 150, 200\}$. The numerical results illustrate that Algorithm 1 has the least iteration numbers. Since the safeguard step helps Algorithm 1 to avoid small steps, Algorithm 1 is efficient.

5 Concluding remarks

In this paper, a Mehrotra-type predictor–corrector algorithm for $P_*(\kappa)$ LCPs is studied. Since $P_*(\kappa)$ LCPs are the generalization of linear programming, the search directions Δx and Δs are not orthogonal, therefore the analysis is different from that in linear programming. The iteration bound of our algorithm is $O((14\kappa + 11)\sqrt{(1+4\kappa)(1+2\kappa)}\, n \log \frac{(x^0)^T s^0}{\varepsilon})$. Numerical results show that this algorithm is efficient.

Funding

This work was supported by the National Natural Science Foundation of China (71471102, 71561022 and 71771025).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

Author details

[1]College of Science, China Three Gorges University, Yichang, P.R. China. [2]Three Gorges Mathematical Research Center, China Three Gorges University, Yichang, P.R. China. [3]College of Economics and Management, China Three Gorges University, Yichang, P.R. China.

Publisher's Note

Received: 7 August 2018 Accepted: 1 January 2019 Published online: 11 January 2019

References

1. Wright, S.J.: Primal–Dual Interior-Point Methods. Society for Industrial and Applied Mathematics, Philadelphia (1997)
2. Billups, S.C., Murty, K.G.: Complementarity problems. J. Comput. Appl. Math. **124**(1), 303–318 (2000)
3. Cottle, R.W., Pang, J.S., Stone, R.E.: The Linear Complementarity Problem. Academic Press, New York (1992)
4. Roos, C., Terlaky, T., Vial, J.P.: Interior Point Methods for Linear Optimization. Springer, Boston (2006)
5. Kojima, M., Megiddo, N., Noma, T., Yoshise, A.: A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems. Springer, Berlin (1991)

6. Illes, T., Nagy, M.: A Mizuno–Todd–Ye type predictor–corrector algorithm for sufficient linear complementarity problems. Eur. J. Oper. Res. **181**(3), 1097–1111 (2007)
7. Miao, J.: A quadratically convergent $O((1+\kappa)\sqrt{n}L)$-iteration algorithm for the $P_*(\kappa)$-matrix linear complementarity problem. Math. Program. **69**(1), 355–368 (1995)
8. Cho, G.M.: A new large-update interior point algorithm for $P_*(\kappa)$ linear complementarity problems. J. Comput. Appl. Math. **216**(1), 265–278 (2008)
9. Cho, G.M., Kim, M.K.: A new large-update interior point algorithm for $P_*(\kappa)$ LCPs based on kernel functions. Appl. Math. Comput. **182**(2), 1169–1183 (2006)
10. Liu, H., Liu, X., Liu, C.: Mehrotra-type predictor–corrector algorithms for sufficient linear complementarity problem. Appl. Numer. Math. **62**(12), 1685–1700 (2012)
11. Czyzyk, J., Mehrotra, S., Wagner, M., Wright, S.: Pcx: an interior-point code for linear programming. Optim. Methods Softw. **11**(1–4), 397–430 (1999)
12. Zhang, Y.: Solving large-scale linear programs by interior-point methods under the Matlab environment. Optim. Methods Softw. **10**(1), 1–31 (1998)
13. Mehrotra, S.: On finding a vertex solution using interior point methods. Linear Algebra Appl. **152**(91), 233–253 (1991)
14. Jarre, F., Wechs, M.: Extending Mehrotra's corrector for linear programs. Adv. Model. Optim. **1**(2), 38–60 (1999)
15. Zhang, Y., Zhang, D.: On polynomiality of the Mehrotra-type predictor–corrector interior-point algorithms. Math. Program. **68**(1), 303–318 (1995)
16. Salahi, M., Peng, J., Terlaky, T.: On Mehrotra-type predictor–corrector algorithms. SIAM J. Optim. **18**(4), 1377–1397 (2007)
17. Salahi, M., Terlaky, T.: Mehrotra-type predictor–corrector algorithm revisited. Optim. Methods Softw. **23**(2), 259–273 (2008)
18. Liu, X., Liu, H., Wang, W.: Polynomial convergence of Mehrotra-type predictor–corrector algorithm for the Cartesian $P_*(\kappa)$-LCP over symmetric cones. Optimization **64**(4), 815–837 (2015)
19. Yang, X., Liu, H., Dong, X.: Polynomial convergence of Mehrotra-type prediction-corrector infeasible-IPM for symmetric optimization based on the commutative class directions. Appl. Math. Comput. **230**(2), 616–628 (2014)
20. Lesaja, C., Roos, C.: Unified analysis of kernel-based interior-point methods for $P_*(\kappa)$-linear complementarity problems. SIAM J. Optim. **20**(6), 3014–3039 (2010)
21. Lee, Y.-H., Cho, Y.-Y., Cho, G.-M.: Interior-point algorithms for $P_*(\kappa)$ LCP based on a new class of kernel functions. J. Glob. Optim. **58**, 137–149 (2014)
22. Harker, P.T., Pang, J.-S.: A damped-Newton method for the linear complementarity problem. Lect. Appl. Math. **26**, 265–284 (1990)