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Global maximal inequality to a class of oscillatory integrals

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Abstract

In the present paper, we give the global L^q estimates for maximal operators generated by multiparameter oscillatory integral $S_{t,\Phi}$, which is defined by

$$S_{t,\Phi}f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{i(t_1\phi_1(|\xi_1|)+t_2\phi_2(|\xi_2|)+\cdots+t_n\phi_n(|\xi_n|))}\hat{f}(\xi)\,d\xi, \quad x\in\mathbb{R}^n,$$

where $n\geq 2$ and f is a Schwartz function in $\mathcal{S}(\mathbb{R}^n)$, $t=(t_1,t_2,\ldots,t_n)$, $\Phi=(\phi_1,\phi_2,\ldots,\phi_n)$, ϕ_i ($i=1,2,3,\ldots,n$) is a function on $\mathbb{R}^+\to\mathbb{R}$, which has a suitable growth condition. These estimates are apparently good extensions to the results of Sjölin and Soria (J. Math. Anal. Appl 411:129–143, 2014) for the multiparameter fractional Schrödinger equation.

MSC: 42B15; 42B25

Keywords: Maximal operator; Local estimate; Global estimate; Multiparameter oscillatory integrals

1 Introduction and main results

Let f be a Schwartz function in $\mathcal{S}(\mathbb{R}^n)$ and

$$S_tf(x) = u(x,t) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi+it|\xi|^a}\hat{f}(\xi)\,d\xi, \quad (x,t)\in\mathbb{R}^n\times\mathbb{R}.$$

It is well known that $S_tf(x)$ is the solution of the fractional Schrödinger equation

$$\begin{cases} i\partial_t u + (-\Delta)^{a/2}u = 0, & (x,t)\in\mathbb{R}^n\times\mathbb{R}, \\ u(x,0) = f(x). \end{cases} \tag{1.1}$$

Here $\hat{f}$ denotes the Fourier transform of f defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}f(x)\,dx$.

We recall the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ ($s\in\mathbb{R}$), which is defined by

$$\dot{H}^s(\mathbb{R}^n) = \left\{f\in\mathcal{S}' : \|f\|_{H^s} = \left(\int_{\mathbb{R}^n}|\xi|^{2s}\left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} < \infty\right\},$$

and the non-homogeneous Sobolev space $H^s(\mathbb{R}^n)$ ($s \in \mathbb{R}$), which is defined by

$$H^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' : \|f\|_{H^s} = \left(\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi \right)^{1/2} < \infty \right\}.$$

Maximal operator S^*f associated with the family of operators $\{S_t\}_{0<t<1}$ is defined by

$$S^*f(x) = \sup_{0<t<1} \left|S_t f(x)\right|, \quad x \in \mathbb{R}^n.$$

It is well known that if $a = 2$, u is the solution of the Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x). \end{cases} \tag{1.2}$$

In 1979, Carleson [4] proposed a problem: if $f \in H^s(\mathbb{R}^n)$ for which s does

$$\lim_{t \to 0} u(x,t) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n. \tag{1.3}$$

Carleson first considered this problem for dimension $n = 1$ in [4] and showed that the convergence (1.3) holds for $f \in H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$, which is sharp was shown by Dahlberg and Kenig [8]. The higher dimensional case of convergence (1.3) has been studied by several authors, see [1, 2, 9, 11–13, 24, 25, 30, 34, 35] for example. In fact, by a standard argument, for $f \in H^s(\mathbb{R}^n)$, the pointwise convergence (1.3) follows from the local estimate

$$\left\|S^*f\right\|_{L^q(\mathbb{B}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}, \quad f \in H^s\left(\mathbb{R}^n\right), \tag{1.4}$$

for some $q \geq 1$ and $s \in \mathbb{R}$. Here $\mathbb{B}^n$ is the unit ball centered at the origin in $\mathbb{R}^n$. On the other hand, the global estimates are of independent interest since they reveal global regularity properties of the corresponding oscillatory integrals. Next, we recall the global estimate

$$\left\|S^*f\right\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}. \tag{1.5}$$

Estimate (1.5) and related questions have been well studied in literature, see, e.g., Carbery [3], Cowling [7], Kenig and Ruiz [21], Kenig, Ponce, and Vega [20], Rogers and Villarroya [29], Rogers [28], Sjölin [30–32], and so on.

For $n \geq 2$ and a multiindex $a = (a_1, a_2, \ldots, a_n)$, with $a_j > 1$ and f being a Schwartz function in $\mathcal{S}(\mathbb{R}^n)$, we set

$$S_t f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{i(t_1|\xi_1|^{a_1} + t_2|\xi_2|^{a_2} + \cdots + t_n|\xi_n|^{a_n})} \hat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}^n,$$

where $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$. For $n \geq 2$, the local maximal operator M^* is defined by

$$M^*f(x) = \sup_{0<t_i<1} \left|S_t f(x)\right|, \quad x \in \mathbb{R}^n,$$

and the global maximal operator M^{**} is defined by

$$M^{**}f(x) = \sup_{t_i \in \mathbb{R}} \left|S_t f(x)\right|, \quad x \in \mathbb{R}^n.$$

The global estimate

$$\|M^{**}f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{\dot{H}^s(\mathbb{R}^n)} \tag{1.6}$$

and

$$\|M^{*}f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}. \tag{1.7}$$

In 2014, Sjolin and Soria [32] obtained the following results.

Theorem A ([32]) *Assume $n \geq 2$. Then, for every a, inequality (1.6) holds if and only if $4 \leq q < \infty$ and $s = n(\frac{1}{2} - \frac{1}{q})$.*

Theorem B ([32]) *Assume $n \geq 2$. Then, for every a and for $2 < q < 4$, inequality (1.7) holds if and only if $s \geq \frac{n}{2} - \frac{|a|}{4} + \frac{|a|}{q} - \frac{n}{q}$.*

Multiparameter singular integrals and related operators have been well studied and raised considerable attention in harmonic analysis, which can been seen in the work of Stein and Fefferman in [14–17], and so on. In the present paper, we consider the maximal estimates associated with multiparameter oscillatory integral $S_{t,\Phi}$ defined by

$$S_{t,\Phi}f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{i(t_1\phi_1(|\xi_1|)+t_2\phi_2(|\xi_2|)+\cdots+t_n\phi_n(|\xi_n|))}\hat{f}(\xi)\,d\xi, \quad x \in \mathbb{R}^n.$$

Here, $n \geq 2$ and f is a Schwartz function in $\mathcal{S}(\mathbb{R}^n)$, $\Phi = (\phi_1, \phi_2, \dots, \phi_n)$, ϕ_i $(i = 1, 2, 3, \dots, n)$ is a function on $\mathbb{R}^+ \to \mathbb{R}$. For $n \geq 2$, the local maximal operator M^*_Φ is defined by

$$M^*_\Phi f(x) = \sup_{0<t_i<1} |S_{t,\Phi}f(x)|, \quad x \in \mathbb{R}^n,$$

and the global maximal operator M^{**}_Φ is defined by

$$M^{**}_\Phi f(x) = \sup_{t_i\in\mathbb{R}} |S_{t,\Phi}f(x)|, \quad x \in \mathbb{R}^n.$$

The global estimates of maximal operators M^*_Φ and M^{**}_Φ are defined by

$$\|M^{**}_\Phi f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{\dot{H}^s(\mathbb{R}^n)} \tag{1.8}$$

and

$$\|M^{*}_\Phi f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}. \tag{1.9}$$

Assume that $\phi : \mathbb{R}^+ \to \mathbb{R}$ satisfies:

(H1) There exists $m_1 > 1$ such that $|\phi'(r)| \sim r^{m_1-1}$ and $|\phi''(r)| \gtrsim r^{m_1-2}$ for all $0 < r < 1$;

(H2) There exists $m_2 > 1$ such that $|\phi'(r)| \sim r^{m_2-1}$ and $|\phi''(r)| \gtrsim r^{m_2-2}$ for all $r \geq 1$;

(H3) Either $\phi''(r) > 0$ or $\phi''(r) < 0$ for all $r > 0$.

Now we state our main results as follows.

Theorem 1.1 *Assume that $n \geq 2$ and ϕ_i $(i = 1, 2, 3, \dots, n)$ satisfies* (H1)–(H3). *If $4 \leq q < \infty$ and $s = n(\frac{1}{2} - \frac{1}{q})$, then the global estimate* (1.8) *holds.*

Theorem 1.2 *Let $m = (m_{1,2}, m_{2,2}, \dots, m_{n,2})$ and set $|m| = m_{1,2} + m_{2,2} + \cdots + m_{n,2}$. Assume that $n \geq 2$ and ϕ_i $(i = 1, 2, 3, \dots, n)$ satisfies* (H1)–(H3) *with $m_{i,1} > 1$, $m_{i,2} > 1$. Then, for every m, inequality* (1.9) *holds if $2 < q < 4$ and $s \geq \frac{n}{2} - \frac{|m|}{4} + \frac{|m|}{q} - \frac{n}{q}$.*

Remark 1.1 There are many elements ϕ satisfying conditions (H1)–(H3), for instance, the fractional Schrödinger equation ($\phi(r) = r^a$), or ($\phi(r) = (1 + r^2)^{\frac{a}{2}}$), ($a \geq 1$), the Beam equation ($\phi(r) = \sqrt{1 + r^4}$), the fourth-order Schrödinger equation ($\phi(r) = r^2 + r^4$), iBq ($\phi(r) = r\sqrt{1 + r^2}$), and so on (see [5, 6, 18, 19, 22, 23, 27], and the references therein). Hence, Theorem 1.1 and Theorem 1.2 imply the sufficiency part of Theorem A and Theorem B, respectively. However, due to the complexity of the symbol ϕ, we cannot obtain the necessities of the range of q in Theorem 1.1 and Theorem 1.2.

This paper is organized as follows. The proofs of Theorem 1.1 and Theorem 1.2 are given in Sect. 2 and Sect. 3, respectively. To prove Theorem 1.1 and Theorem 1.2, we next need the following important lemmas, which play a key role in proving Theorem 1.1 and Theorem 1.2, respectively. The proof of Lemma 1.4 is given in Sect. 4.

Lemma 1.3 ([26]) *Assume that ϕ satisfies* (H1)–(H3) *with $m_1 > 1$, $m_2 > 1$. $\frac{1}{2} \leq s < 1$ and $\mu \in C_0^\infty(\mathbb{R})$. Then*

$$\left| \int_{\mathbb{R}} e^{ix\xi + it\phi(|\xi|)} |\xi|^{-s} \mu\left(\frac{\xi}{N}\right) d\xi \right| \leq C \frac{1}{|x|^{1-s}}$$

for $x \in \mathbb{R} \setminus \{0\}$, $t \in \mathbb{R}$, and $N = 1, 2, 3, \dots$. Here the constant C may depend on s and m_1, m_2, and μ but not on x, t, or N.

Remark 1.2 The proof of Lemma 1.3 is similar to that of Lemma 2.1 in [10].

Lemma 1.4 *Assume that ϕ satisfies* (H1)–(H3) *with $m_1 > 1$, $m_2 > 1$. $\frac{1}{2} \leq \alpha \leq \frac{m_2}{2}$, $-1 < d < 1$, and $\mu \in C_0^\infty(\mathbb{R})$. Then*

$$\left| \int_{\mathbb{R}} \frac{e^{i(d\phi(|\xi|) - x\xi)}}{(1 + \xi^2)^{\frac{\alpha}{2}}} \mu\left(\frac{\xi}{N}\right) d\xi \right| \leq C \frac{1}{|x|^{\beta}} \tag{1.10}$$

for $x \in \mathbb{R} \setminus \{0\}$ and $N = 1, 2, 3, \dots$, where $\beta = \frac{\alpha + \frac{m_2}{2} - 1}{m_2 - 1}$. Here the constant C may depend on α and m_1, m_2, and μ but not on x, d, and N.

Remark 1.3 Applying the result of Lemma 1.3, the proof of Lemma 1.4 is similar to that of Lemma 2.2 in [32]. The proof of Lemma 1.4 will be given in Sect. 4.

2 The proof of Theorem 1.1

Assume that $n \geq 2$, ϕ_i $(i = 1, 2, 3, \dots, n)$ satisfies (H1)–(H3). For $i = 1, 2, 3, \dots, n$, let $t_i(x)$ be a measurable function on $\mathbb{R}^n$ with $t_i(x) \in \mathbb{R}$. Denote $t(x) = (t_1(x), t_2(x), \dots, t_n(x))$, we set

$$S_{t(x),\Phi} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{i(t_1(x)\phi_1(|\xi_1|) + t_2(x)\phi_2(|\xi_2|) + \cdots + t_n(x)\phi_n(|\xi_n|))} \hat{f}(\xi)\, d\xi,$$

$$x \in \mathbb{R}^n, f \in \mathcal{S}(\mathbb{R}^n).$$

For $4 \le q < \infty$ and $s = n(\frac{1}{2} - \frac{1}{q})$, that is, $\frac{n}{4} \le s < \frac{n}{2}$ and $q = \frac{2n}{n-2s}$. By linearizing the maximal operator (see [30]) to prove the global estimate (1.8) holds, it suffices to show that

$$\|S_{t(x),\Phi} f\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{\dot{H}^s} = C\left(\int_{\mathbb{R}^n} |\xi|^{2s} \big|\hat{f}(\xi)\big|^2 \, d\xi\right)^{1/2}. \tag{2.1}$$

To prove (2.1) it suffices to prove that

$$\|S_{t(x),\Phi} f\|_{L^q(\mathbb{R}^n)} \le C\left(\int_{\mathbb{R}^n} |\xi_1|^{\frac{2s}{n}} |\xi_2|^{\frac{2s}{n}} | \cdots |\xi_n|^{\frac{2s}{n}} \big|\hat{f}(\xi)\big|^2 \, d\xi\right)^{1/2}. \tag{2.2}$$

Let $g(\xi) = |\xi_1|^{\frac{s}{n}} |\xi_2|^{\frac{s}{n}} \cdots |\xi_n|^{\frac{s}{n}} \hat{f}(\xi)$, then we have

$$\begin{aligned} S_{t(x),\Phi} f(x) &= \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{i(t_1(x)\phi_1(|\xi_1|)+t_2(x)\phi_2(|\xi_2|)+\cdots+t_n(x)\phi_n(|\xi_n|))} |\xi_1|^{-\frac{s}{n}} |\xi_2|^{-\frac{s}{n}} \cdots |\xi_n|^{-\frac{s}{n}} g(\xi) \, d\xi \\ &= R_\Phi g(x), \end{aligned} \tag{2.3}$$

where

$$R_\Phi g(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{i(t_1(x)\phi_1(|\xi_1|)+t_2(x)\phi_2(|\xi_2|)+\cdots+t_n(x)\phi_n(|\xi_n|))} |\xi_1|^{-\frac{s}{n}} |\xi_2|^{-\frac{s}{n}} \cdots |\xi_n|^{-\frac{s}{n}} g(\xi) \, d\xi.$$

To prove (2.2) it suffices to prove that

$$\|R_\Phi g\|_{L^q(\mathbb{R}^n)} \le C\|g\|_{L^2(\mathbb{R}^n)} \tag{2.4}$$

for g continuous and rapidly decreasing at infinity. We take a real-valued function $\rho \in C_0^\infty(\mathbb{R}^n)$ such that $\rho(x) = 1$ if $|x| \le 1$ and $\rho(x) = 0$ if $|x| \ge 2$. And we choose a real-valued function $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(x) = 1$ if $|x| \le 1$ and $\psi(x) = 0$ if $|x| \ge 2$, and set $\sigma(\xi) = \psi(\xi_1)\psi(\xi_2) \cdots \psi(\xi_n)$. For $\xi \in \mathbb{R}^n$ and $N = 1, 2, 3, \ldots$, we set $\rho_N(x) = \rho(\frac{x}{N})$ and $\sigma_N(\xi) = \sigma(\frac{\xi}{N})$. For $x \in \mathbb{R}^n$, $g \in L^2(\mathbb{R}^n)$, and for $N = 1, 2, 3, \ldots$, we define

$$\begin{aligned} R_{N,\Phi} g(x) = \rho_N(x) &\int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{i(t_1(x)\phi_1(|\xi_1|)+t_2(x)\phi_2(|\xi_2|)+\cdots+t_n(x)\phi_n(|\xi_n|))} |\xi_1|^{-\frac{s}{n}} |\xi_2|^{-\frac{s}{n}} \cdots \\ &\times |\xi_n|^{-\frac{s}{n}} \sigma_N(\xi) g(\xi) \, d\xi. \end{aligned}$$

The adjoint of $R_{N,\Phi}$ is given by

$$\begin{aligned} R'_{N,\Phi} h(\xi) = {}& \sigma_N(\xi) |\xi_1|^{-\frac{s}{n}} |\xi_2|^{-\frac{s}{n}} \cdots \\ &\times |\xi_n|^{-\frac{s}{n}} \int_{\mathbb{R}^2} e^{-ix\cdot\xi} e^{-i(t_1(x)\phi_1(|\xi_1|)+t_2(x)\phi_2(|\xi_2|)+\cdots+t_n(x)\phi_n(|\xi_n|))} \rho_N(x) h(x) \, dx, \end{aligned}$$

where $\xi \in \mathbb{R}^n$ and $h \in L^2(\mathbb{R}^n)$. To prove (2.4) it is sufficient to prove that

$$\|R_{N,\Phi} g\|_{L^q(\mathbb{R}^n)} \le C\|g\|_{L^2(\mathbb{R}^n)}. \tag{2.5}$$

By duality, to prove (2.5) it suffices to show that

$$\big\|R'_{N,\Phi}h\big\|_{L^2(\mathbb{R}^n)} \le C\|h\|_{L^{q'}(\mathbb{R}^n)}, \tag{2.6}$$

where $\frac{1}{q}+\frac{1}{q'}=1$. Thus, we have

$$\big\|R'_{N,\Phi}h\big\|^2_{L^2(\mathbb{R}^n)} = \int \big|R'_{N,\Phi}h(\xi)\big|^2\,d\xi = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_N(x,y)\rho_N(x)\rho_N(y)h(x)\overline{h(y)}\,dx\,dy, \tag{2.7}$$

where

$$K_N(x,y) = K_N^1(x,y)K_N^2(x,y)\cdots K_N^n(x,y) \tag{2.8}$$

and

$$K_N^i(x,y) = \int_{\mathbb{R}} |\xi_i|^{-\frac{2s}{n}} e^{i(y_i-x_i)\xi_i} e^{i(t_i(y)-t_i(x))\phi(|\xi_i|)}\psi_N(\xi_i)^2\,d\xi_i, \tag{2.9}$$

where $i=1,2,\dots,n$ and $N=1,2,\dots$. Since $\frac{n}{4}\le s<\frac{n}{2}$, we have $\frac{1}{2}\le\frac{2s}{n}<1$. Therefore, by Lemma 1.3, (2.9), and (2.8), we obtain

$$\big|K_N(x,y)\big| \le C\frac{1}{|x_1-y_1|^{1-\frac{2s}{n}}}\frac{1}{|x_2-y_2|^{1-\frac{2s}{n}}}\cdots\frac{1}{|x_n-y_n|^{1-\frac{2s}{n}}}. \tag{2.10}$$

We define

$$P_if(x_1,x_2,\dots,x_n) = \int_{\mathbb{R}} \frac{1}{|x_i-y_i|^{1-\frac{2s}{n}}} f(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots,x_n)\,dy_i,$$

$i=1,2,\dots,n$. Thus, by (2.7) and (2.10), we obtain

$$\begin{aligned}
&\int \big|R'_{N,\Phi}h(\xi)\big|^2\,d\xi \\
&\quad\le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{1}{|x_1-y_1|^{1-\frac{2s}{n}}}\frac{1}{|x_2-y_2|^{1-\frac{2s}{n}}}\cdots\frac{1}{|x_n-y_n|^{1-\frac{2s}{n}}}\big|h(x)\big|\big|h(y)\big|\,dx\,dy \\
&\quad= C\int_{\mathbb{R}^n}\biggl(\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{|x_n-y_n|^{1-\frac{2s}{n}}}\frac{1}{|x_{n-1}-y_{n-1}|^{1-\frac{2s}{n}}}\cdots\frac{1}{|x_3-y_3|^{1-\frac{2s}{n}}} \\
&\qquad\times\frac{1}{|x_2-y_2|^{1-\frac{2s}{n}}}\biggl(\int\frac{1}{|x_1-y_1|^{1-\frac{2s}{n}}}\big|h(y_1,y_2,\dots,y_n)\big|\,dy_1\biggr)\,dy_2\,dy_3\cdots dy_n\biggr)\big|h(x)\big|\,dx \\
&\quad= C\int_{\mathbb{R}^n}P_nP_{n-1}\cdots P_2P_1|h|(x)\big|h(x)\big|\,dx.
\end{aligned} \tag{2.11}$$

Invoking Hölder's inequality, we get

$$\int \big|R'_{N,\Phi}h(\xi)\big|^2\,d\xi \le C\big\|P_nP_{n-1}\cdots P_2P_1|h|\big\|_{L^q(\mathbb{R}^n)}\|h\|_{L^{q'}(\mathbb{R}^n)}. \tag{2.12}$$

Since $q = \frac{2n}{n-2s}$, it follows that $q' = \frac{2n}{n+2s}$ and the fact $\frac{1}{q} = \frac{1}{q'} - \frac{2s}{n}$. Denote by I_σ the Riesz potential of order σ, which is defined by

$$I_\sigma(f)(u) = \int_{\mathbb{R}} \frac{f(v)}{|u-v|^{1-\sigma}}\, dv.$$

Applying the fact I_s is bounded from $L^{q'}(\mathbb{R})$ to $L^q(\mathbb{R})$, we have

$$\left(\int_{\mathbb{R}} \left|P_j h(x)\right|^q dx_j\right)^{1/q} \le C\left(\int_{\mathbb{R}} \left|h(x)\right|^{q'} dx_j\right)^{1/q'}, \tag{2.13}$$

where $j = 1, 2, \ldots, n$. By (2.13) and Minkowski's inequality, we have

$$\left\|P_n P_{n-1} \cdots P_2 P_1 |h|\right\|_{L^q(\mathbb{R}^n)} \le C\|h\|_{L^{q'}(\mathbb{R}^n)}. \tag{2.14}$$

Therefore, (2.6) follows from (2.12) and (2.14). Now we complete the proof of Theorem 1.1.

3 The proof of Theorem 1.2

Assume that $n \ge 2$, ϕ_i $(i = 1, 2, 3, \ldots, n)$ satisfies (H1)–(H3) with $m_{i,1} > 1$, $m_{i,2} > 1$. For every $m = (m_{1,2}, m_{2,2}, \ldots, m_{n,2})$ and $2 < q < 4$, we will prove that inequality (1.9) holds if $s = \frac{n}{2} - \frac{|m|}{4} + \frac{|m|}{q} - \frac{n}{q}$, where $|m| = m_{1,2} + m_{2,2} + \cdots + m_{n,2}$. For $i = 1, 2, 3, \ldots, n$, let $t_i(x)$ be a measurable function on $\mathbb{R}^n$ with $0 < t_i(x) < 1$. Denote $t(x) = (t_1(x), t_2(x), \ldots, t_n(x))$, we set

$$S_{t(x),\Phi} f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{i(t_1(x)\phi_1(|\xi_1|)+t_2(x)\phi_2(|\xi_2|)+\cdots+t_n(x)\phi_n(|\xi_n|))} \hat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}^n, f \in \mathcal{S}(\mathbb{R}^n).$$

By linearizing the maximal operator, to prove the global estimate (1.9) it suffices to show that

$$\|S_{t(x),\Phi} f\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{H^s} = C\left(\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2}. \tag{3.1}$$

Since $s = \frac{n}{2} - \frac{|m|}{4} + \frac{|m|}{q} - \frac{n}{q} = n\frac{1}{2} - \frac{m_{1,2}+m_{2,2}+\cdots+m_{n,2}}{4} + \frac{m_{1,2}+m_{2,2}+\cdots+m_{n,2}}{q} - n\frac{1}{q} =: s_1 + s_2 + \cdots + s_n$, where $s_i = \frac{1}{2} - \frac{m_{i,2}}{4} + \frac{m_{i,2}}{q} - \frac{1}{q}$, $i = 1, 2, \ldots, n$. Therefore, to prove (3.1) it suffices to prove that

$$\|S_{t(x),\Phi} f\|_{L^q(\mathbb{R}^n)} \le C\left(\int_{\mathbb{R}^2} \left(1 + |\xi_1|^2\right)^{s_1} \left(1 + |\xi_2|^2\right)^{s_2} | \cdots | \left(1 + |\xi_n|^2\right)^{s_n} \left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2}. \tag{3.2}$$

Let $g(\xi) = (1 + |\xi_1|^2)^{\frac{s_1}{2}} (1 + |\xi_2|^2)^{\frac{s_2}{2}} \cdots (1 + |\xi_n|^2)^{\frac{s_n}{2}} \hat{f}(\xi)$, then we have

$$S_{t(x),\Phi} f(x) = R_\Phi g(x), \tag{3.3}$$

where

$$R_\Phi g(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{i(t_1(x)\phi_1(|\xi_1|)+t_2(x)\phi_2(|\xi_2|)+\cdots+t_n(x)\phi_n(|\xi_n|))} \left(1 + |\xi_1|^2\right)^{-\frac{s_1}{2}} \left(1 + |\xi_2|^2\right)^{-\frac{s_2}{2}} \cdots$$
$$\times \left(1 + |\xi_n|^2\right)^{-\frac{s_n}{2}} g(\xi)\, d\xi.$$

By (3.3), to prove (3.2) it is sufficient to show that

$$\|R_{\Phi}g\|_{L^q(\mathbb{R}^n)} \le C\|g\|_{L^2(\mathbb{R}^n)} \tag{3.4}$$

for g continuous and rapidly decreasing at infinity. We take a real-valued function $\rho \in C_0^\infty(\mathbb{R}^n)$ such that $\rho(x) = 1$ if $|x| \le 1$ and $\rho(x) = 0$ if $|x| \ge 2$. And we choose a real-valued function $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(x) = 1$ if $|x| \le 1$ and $\psi(x) = 0$ if $|x| \ge 2$, and set $\sigma(\xi) = \psi(\xi_1)\psi(\xi_2)\cdots\psi(\xi_n)$ for $\xi \in \mathbb{R}^n$. For $N = 1, 2, 3, \ldots$, we set $\rho_N(x) = \rho(\frac{x}{N})$ and $\sigma_N(\xi) = \sigma(\frac{\xi}{N})$. For $x \in \mathbb{R}^n$, $g \in L^2(\mathbb{R}^n)$, and $N = 1, 2, 3, \ldots$, we define

$$R_{N,\Phi}g(x) = \rho_N(x)\int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{i(t_1(x)\phi_1(|\xi_1|)+t_2(x)\phi_2(|\xi_2|)+\cdots+t_n(x)\phi_n(|\xi_n|))}\left(1+|\xi_1|^2\right)^{-\frac{s_1}{2}}\left(1+|\xi_2|^2\right)^{-\frac{s_2}{2}}$$
$$\times\cdots\times\left(1+|\xi_n|^2\right)^{-\frac{s_n}{2}}\sigma_N(\xi)g(\xi)\,d\xi.$$

The adjoint of $R_{N,\Phi}$ is given by

$$R'_{N,\Phi}h(\xi) = \sigma_N(\xi)\left(1+|\xi_1|^2\right)^{-\frac{s_1}{2}}\left(1+|\xi_2|^2\right)^{-\frac{s_2}{2}}\cdots\left(1+|\xi_n|^2\right)^{-\frac{s_n}{2}}\int_{\mathbb{R}^2} e^{-ix\cdot\xi}e^{-it_1(x)\phi_1(|\xi_1|)}$$
$$\times e^{-i(t_2(x)\phi_2(|\xi_2|)+\cdots+t_n(x)\phi_n(|\xi_n|))}\rho_N(x)h(x)\,dx,$$

where $\xi \in \mathbb{R}^n$ and $h \in L^2(\mathbb{R}^n)$. To prove (3.4) it suffices to prove that

$$\|R_{N,\Phi}g\|_{L^q(\mathbb{R}^n)} \le C\|g\|_{L^2(\mathbb{R}^n)}. \tag{3.5}$$

By duality, to prove (3.5) it is sufficient to show that

$$\left\|R'_{N,\Omega}h\right\|_{L^2(\mathbb{R}^n)} \le C\|h\|_{L^{q'}(\mathbb{R}^n)}, \tag{3.6}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Thus, we have

$$\left\|R'_{N,\Phi}h\right\|^2_{L^2(\mathbb{R}^n)} = \int\left|R'_{N,\Phi}h(\xi)\right|^2 d\xi = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_N(x,y)\rho_N(x)\rho_N(y)h(x)\overline{h(y)}\,dx\,dy, \tag{3.7}$$

where

$$K_N(x,y) = K_N^1(x,y)K_N^2(x,y)\cdots K_N^n(x,y), \tag{3.8}$$

and

$$K_N^i(x,y) = \int_{\mathbb{R}}\left(1+\xi_i^2\right)^{-s_i} e^{i(y_i-x_i)\xi_i}e^{i(t_i(y)-t_i(x))\phi(|\xi_i|)}\psi_N(\xi_i)^2\,d\xi_i, \tag{3.9}$$

where $i = 1, 2, \ldots, n$ and $N = 1, 2, \ldots$. Denote $\alpha_i = 2s_i$, since $s_i = \frac{1}{2} - \frac{m_{i,2}}{4} + \frac{m_{i,2}}{q} - \frac{1}{q}$, $i = 1, 2, \ldots, n$, and $2 < q < 4$, it follows that $\frac{1}{2} < \alpha_i < \frac{m_{i,2}}{2}$, $i = 1, 2, \ldots, n$. Therefore, by (3.9) and Lemma 1.4, we obtain

$$\left|K_N^i(x,y)\right| \le C\frac{1}{|x_i - y_i|^{\beta_i}}, \tag{3.10}$$

where $\beta_i = \frac{\alpha_i + \frac{m_{i,2}}{2} - 1}{m_{i,2} - 1}$. Denote $\sigma_i = 1 - \beta_i$, we define

$$P_i f(x_1, x_2, \ldots, x_n) = \int_{\mathbb{R}} \frac{1}{|x_i - y_i|^{1-\sigma_i}} f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)\, dy_i,$$

$i = 1, 2, \ldots, n$. Thus, by (3.7), (3.8), and (3.10), we obtain

$$\begin{aligned}
&\int \left|R'_{N,\Phi} h(\xi)\right|^2 d\xi \\
&\quad \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x_1 - y_1|^{1-\sigma_1}} \frac{1}{|x_2 - y_2|^{1-\sigma_2}} \cdots \frac{1}{|x_n - y_n|^{1-\sigma_n}} \left|h(x)\right| \left|h(y)\right| dx\, dy \\
&\quad = C \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{|x_n - y_n|^{1-\sigma_n}} \frac{1}{|x_{n-1} - y_{n-1}|^{1-\sigma_{n-1}}} \cdots \frac{1}{|x_3 - y_3|^{1-\sigma_3}} \right. \\
&\qquad \left. \times \frac{1}{|x_2 - y_2|^{1-\sigma_2}} \left(\int \frac{1}{|x_1 - y_1|^{1-\sigma_1}} \left|h(y_1, y_2, \ldots, y_n)\right| dy_1 \right) dy_2\, dy_3 \cdots dy_n \right) \left|h(x)\right| dx \\
&\quad = C \int_{\mathbb{R}^n} P_n P_{n-1} \cdots P_2 P_1 |h|(x) \left|h(x)\right| dx.
\end{aligned} \tag{3.11}$$

Invoking Hölder's inequality, we get

$$\int \left|R'_{N,\Phi} h(\xi)\right|^2 d\xi \leq C \left\| P_n P_{n-1} \cdots P_2 P_1 |h| \right\|_{L^q(\mathbb{R}^n)} \|h\|_{L^{q'}(\mathbb{R}^n)}. \tag{3.12}$$

Since $\beta_i = \frac{\alpha_i + \frac{m_{i,2}}{2} - 1}{m_{i,2} - 1}$ and $\alpha_i = 2s_i$, $s_i = \frac{1}{2} - \frac{m_{i,2}}{4} + \frac{m_{i,2}}{q} - \frac{1}{q}$, $i = 1, 2, \ldots, n$. It follows that $\beta_i = \frac{2}{q}$ and $\sigma_i = 1 - \beta_i = 1 - \frac{2}{q}$, $\frac{1}{q} = \frac{1}{q'} - \sigma_i$. Thus, estimate (3.6) follows from (3.12) and estimate (2.14) in the proof of Theorem 1.1. Now we complete the proof of Theorem 1.2.

4 The proof of Lemma 1.4

To prove Lemma 1.4, we need to present the following lemma.

Lemma 4.1 (see [33], pp. 309–312) *Assume that $a < b$ and set $I = [a, b]$. Let $F \in C^\infty(I)$ be real-valued and assume that $\psi \in C^\infty(I)$.*

(i) *Assume that $|F'(x)| \geq \lambda > 0$ for $x \in I$ and that F' is monotonic on I. Then*

$$\left| \int_a^b e^{iF(x)} \psi(x)\, dx \right| \leq C \frac{1}{\lambda} \left\{ \left|\psi(b)\right| + \int_a^b \left|\psi'(x)\right| dx \right\},$$

where C does not depend on F, ψ, or I.

(ii) *Assume that $|F''(x)| \geq \lambda > 0$ for $x \in I$. Then*

$$\left| \int_a^b e^{iF(x)} \psi(x)\, dx \right| \leq C \frac{1}{\lambda^{1/2}} \left\{ \left|\psi(b)\right| + \int_a^b \left|\psi'(x)\right| dx \right\},$$

where C does not depend on F, ψ, or I.

Proof of Lemma 1.4 By conditions (H1) and (H2), there exist positive constants C_i ($i = 1, 2, \ldots, 6$) so that for $r \geq 1$ and $m_2 > 1$ such that

$$C_1 r^{m_2 - 1} \leq \left|\phi'(r)\right| \leq C_2 r^{m_2 - 1} \quad \text{and} \quad \left|\phi''(r)\right| \geq C_3 r^{m_2 - 2}, \tag{4.1}$$

and for $0 < r < 1$ and $m_1 > 1$ such that

$$C_4 r^{m_1-1} \leq \left|\phi'(r)\right| \leq C_5 r^{m_1-1} \quad \text{and} \quad \left|\phi''(r)\right| \geq C_6 r^{m_1-2}. \tag{4.2}$$

Set

$$J = \int_{\mathbb{R}} \frac{e^{i(d\phi(|\xi|)-x\xi)}}{(1+\xi^2)^{\frac{\alpha}{2}}} \mu\left(\frac{\xi}{N}\right) d\xi.$$

To prove Lemma 1.4, it suffices to show that there exists a constant C such that for $x \in \mathbb{R} \setminus \{0\}$, $\beta = \frac{\alpha+\frac{m_2}{2}-1}{m_2-1}$ and $N \in \mathbb{N}$,

$$|J| \leq C \frac{1}{|x|^\beta}, \tag{4.3}$$

where C depends only on α, m_1, m_2, C_i $(i = 1, 2, \dots, 6)$, and μ.

Without loss of generality, we may assume $\xi, d > 0$. Denote $\psi(\xi) = (1+\xi^2)^{-\frac{\alpha}{2}} \mu(\frac{\xi}{N})$, then we have

$$\max_{\xi \geq 0} \left|\psi(\xi)\right| + \int_0^\infty \left|\psi'(\xi)\right| d\xi \leq C. \tag{4.4}$$

In fact, since $\mu \in C_0^\infty(\mathbb{R})$ and $\frac{1}{2} \leq \alpha \leq \frac{m_2}{2}$, we get

$$\max_{\xi \geq 0} \left|\psi(\xi)\right| \leq C. \tag{4.5}$$

Noting that

$$\psi'(\xi) = -\alpha\xi\left(1+\xi^2\right)^{-\frac{\alpha}{2}-1} \mu\left(\frac{\xi}{N}\right) + \left(1+\xi^2\right)^{-\frac{\alpha}{2}} \frac{1}{N} \mu'\left(\frac{\xi}{N}\right), \tag{4.6}$$

we have

$$\begin{aligned} \int_0^\infty \left|\psi'(\xi)\right| d\xi &\leq \alpha \int_0^\infty \xi\left(1+\xi^2\right)^{-\frac{\alpha}{2}-1} \left|\mu\left(\frac{\xi}{N}\right)\right| d\xi \\ &\quad + \int_0^\infty \left(1+\xi^2\right)^{-\frac{\alpha}{2}} \frac{1}{N} \left|\mu'\left(\frac{\xi}{N}\right)\right| d\xi \\ &=: G_1 + G_2. \end{aligned} \tag{4.7}$$

Since $\mu \in C_0^\infty(\mathbb{R})$ and $\frac{1}{2} \leq \alpha \leq \frac{m_2}{2}$, we obtain

$$G_1 \leq C \int_0^\infty \xi\left(1+\xi^2\right)^{-\frac{\alpha}{2}-1} d\xi = C \int_0^\infty \left(1+\xi^2\right)^{-\frac{\alpha}{2}-1} d\left(1+\xi^2\right) = C \tag{4.8}$$

and

$$G_2 \leq C \int_0^\infty \frac{1}{N} \left|\mu'\left(\frac{\xi}{N}\right)\right| d\xi \leq C. \tag{4.9}$$

By (4.7), (4.8), and (4.9), we get

$$\int_0^\infty \left|\psi'(\xi)\right| d\xi \le C. \tag{4.10}$$

Therefore, (4.4) follows from (4.5) and (4.10).

To estimate (4.3), we choose a positive constant M such that $M = \max\{(\frac{1}{\delta})^{m_2-1}, 2C_5, 2\}$, where δ is a small positive constant such that $\delta^{m_2-1}C_2 \le \frac{1}{2}$. Below, we show (4.3) by dividing two cases $|x| \ge M$ and $|x| < M$.

Case (I): $|x| \ge M$. Let $F(\xi) = d\phi(\xi) - x\xi$, we have

$$F'(\xi) = d\phi'(\xi) - x, \qquad F''(\xi) = d\phi''(\xi).$$

Denote $\rho = (\frac{|x|}{d})^{\frac{1}{m_2-1}}$, then we have $\delta\rho \ge 1$. In fact, noting that $|x| \ge (\frac{1}{\delta})^{m_2-1}$, $0 < d < 1$, $m_2 > 1$, and $\frac{|x|}{d} > |x|$, it follows that $\delta\rho > \delta|x|^{\frac{1}{m_2-1}} \ge 1$. We choose a large positive constant λ such that $\lambda \ge \max\{(\frac{2}{C_1})^{\frac{1}{m_2-1}}, \delta\}$. Denote

$$I_1 = [0, \delta\rho], \qquad I_2 = [\delta\rho, \lambda\rho], \qquad I_3 = [\lambda\rho, \infty).$$

Thus, we obtain

$$|J| = \left|\int_0^\infty e^{iF(\xi)}\psi(\xi)\, d\xi\right| \le \sum_{j=1}^3 \left|\int_{I_j} e^{iF(\xi)}\psi(\xi)\, d\xi\right| =: \sum_{j=1}^3 J_j. \tag{4.11}$$

Firstly, we estimate J_1. We will show that the following estimate holds:

$$\left|F'(\xi)\right| \ge \frac{|x|}{2}, \quad \xi \in [0, \delta\rho]. \tag{4.12}$$

Now we divide the verification of (4.12) into two cases according to the value of ξ.

Case (I-a): $\xi \in [0, 1)$. Since $m_1 > 1$ and $0 < d < 1$, we have

$$d\left|\phi'(\xi)\right| \le C_5\, d\xi^{m_1-1} \le C_5 \le \frac{M}{2} \le \frac{|x|}{2}. \tag{4.13}$$

By (4.13), if $\xi \in [0, 1)$, we get

$$\left|F'(\xi)\right| \ge |x| - d\left|\phi'(\xi)\right| \ge \frac{|x|}{2}. \tag{4.14}$$

Case (I-b): $\xi \in [1, \delta\rho]$. Since $m_2 > 1$, we have

$$d\left|\phi'(\xi)\right| \le C_2\, d\xi^{m_2-1} \le C_2\, d\delta^{m_2-1}\frac{|x|}{d} \le C_2\delta^{m_2-1}|x| \le \frac{|x|}{2}. \tag{4.15}$$

By (4.15), we get

$$\left|F'(\xi)\right| \ge |x| - d\left|\phi'(\xi)\right| \ge \frac{|x|}{2}. \tag{4.16}$$

Therefore (4.12) follows from (4.14) and (4.16). Since ϕ' is monotonic on $\mathbb{R}^+$ by condition (H3) and $d > 0$, it follows that F' is monotonic on $\xi \in I_1$. Thus, by (i) of Lemma 4.1 and estimate (4.12), (4.4), we have

$$|J_1| \le C\frac{1}{|x|} \le C\frac{1}{|x|^\beta}, \tag{4.17}$$

where we use $|x| \ge 2$ and the fact $\frac{1}{2} \le \beta \le 1$. Next we prove estimate J_3. Since $\xi \ge \lambda(\frac{|x|}{d})^{\frac{1}{m_2-1}} > 1$ and $\lambda \ge (\frac{2}{C_1})^{\frac{1}{m_2-1}}$,

$$d\left|\phi'(\xi)\right| \ge C_1\, d\xi^{m_2-1} \ge C_1\, d\lambda^{m_2-1}\frac{|x|}{d} \ge 2|x|,$$

it follows that

$$\left|F'(\xi)\right| \ge 2|x| - |x| = |x|, \quad \xi \in [\lambda\rho, \infty). \tag{4.18}$$

Thus, by (i) of Lemma 4.1 and estimate (4.18), (4.4), we have

$$|J_3| \le C\frac{1}{|x|} \le C\frac{1}{|x|^\beta}, \tag{4.19}$$

where we use $|x| \ge 2$ and the fact $\frac{1}{2} \le \beta \le 1$. Now, we give estimate J_2. Since $\xi \in I_2$, we have $|\xi| \ge 1$. By (4.1), we obtain

$$\left|F''(\xi)\right| \ge d\left|\phi''(\xi)\right| \ge C_3\, d\xi^{m_2-2} \ge C_3\, d\left(\frac{|x|}{d}\right)^{\frac{m_2-2}{m_2-1}}. \tag{4.20}$$

We first prove that the following estimate holds:

$$\max_{I_2}|\psi| + \int_{I_2}\left|\psi'\right| d\xi \le C\left(\frac{|x|}{d}\right)^{-\frac{\alpha}{m_2-1}}. \tag{4.21}$$

In fact, since $\mu \in C_0^\infty(\mathbb{R})$ and $\frac{1}{2} \le \alpha \le \frac{m_2}{2}$, we get

$$\max_{\xi\in A_2}\left|\psi(\xi)\right| \le C(\delta\rho)^{-\alpha} = C\delta^{-\alpha}(\rho)^{-\alpha} = C\delta^{-\alpha}\left(\frac{|x|}{d}\right)^{-\frac{\alpha}{m_2-1}}. \tag{4.22}$$

By (4.6), we have

$$\begin{aligned}\int_{A_2}\left|\psi'(\xi)\right| d\xi &\le \alpha\int_{A_2}\xi\left(1+\xi^2\right)^{-\frac{\alpha}{2}-1}\left|\mu\left(\frac{\xi}{N}\right)\right| d\xi + \int_{A_2}\left(1+\xi^2\right)^{-\frac{\alpha}{2}}\frac{1}{N}\left|\mu'\left(\frac{\xi}{N}\right)\right| d\xi \\ &=: L_1 + L_2. \end{aligned}\tag{4.23}$$

Since $\mu \in C_0^\infty(\mathbb{R})$ and $\frac{1}{2} \le \alpha \le \frac{m_2}{2}$, we obtain

$$L_1 \le C\int_{A_2}\xi\left(1+\xi^2\right)^{-\frac{\alpha}{2}-1} d\xi \le C\int_{\delta\rho}^{\lambda\rho}\xi^{-\alpha-1}\, d\xi = C\left(\frac{|x|}{d}\right)^{-\frac{\alpha}{m_2-1}} \tag{4.24}$$

and

$$L_2 \leq C(\delta\rho)^{-\alpha} \int_{A_2} \frac{1}{N} \left| \mu' \left(\frac{\xi}{N} \right) \right| d\xi \leq C \left(\frac{|x|}{d} \right)^{-\frac{\alpha}{m_2-1}}. \tag{4.25}$$

By (4.23), (4.24), and (4.25), we get

$$\int_0^\infty \left| \psi'(\xi) \right| d\xi \leq C \left(\frac{|x|}{d} \right)^{-\frac{\alpha}{m_2-1}}. \tag{4.26}$$

Therefore, (4.21) follows from (4.22) and (4.26). Thus, by (ii) of Lemma 4.1 and estimate (4.20), (4.21), we have

$$|J_2| \leq d^{-\frac{1}{2}} \left(\frac{|x|}{d} \right)^{-\frac{m_2-2}{2(m_2-1)}} \left(\frac{|x|}{d} \right)^{-\frac{\alpha}{m_2-1}} = C \frac{d^{\frac{\alpha-\frac{1}{2}}{m_2-1}}}{|x|^{\frac{\alpha+\frac{m_2}{2}-1}{m_2-1}}} \leq C \frac{1}{|x|^\beta}. \tag{4.27}$$

Here in the last inequality we use the fact $\frac{\alpha-\frac{1}{2}}{m_2-1} \geq 0$ and $0 < d < 1$. Therefore, for $|x| \geq M$, by estimates (4.11), (4.17), (4.19), and (4.27), it follows (4.3).

Case (II): $|x| < M$. Now we divide the verification of (4.3) into three cases according to the value of α for $|x| < M$.

Case (II-a): $\alpha > 1$. Since $\mu \in C_0^\infty(\mathbb{R})$ and $\alpha > 1$, we get

$$|J| = \left| \int_0^\infty \frac{e^{i(d\phi(|\xi|)-x\xi)}}{(1+\xi^2)^{\frac{\alpha}{2}}} \mu \left(\frac{\xi}{N} \right) d\xi \right| \leq C \int_0^\infty \frac{1}{(1+\xi^2)^{\frac{\alpha}{2}}} d\xi \leq C. \tag{4.28}$$

Noting that $|x| < M$ and $\frac{1}{2} \leq \beta \leq 1$, by (4.28), we have

$$|J| \leq C = C|x|^\beta \frac{1}{|x|^\beta} \leq CM^\beta \frac{1}{|x|^\beta} = C \frac{1}{|x|^\beta},$$

which follows (4.3).

Case (II-b): $\frac{1}{2} \leq \alpha < 1$. By the mean value theorem, when $\frac{1}{2} \leq \alpha < 1$, we have

$$0 < \left(1+\xi^2\right)^{\frac{\alpha}{2}} - \xi^\alpha = \left(1+\xi^2\right)^{\frac{\alpha}{2}} - \left(\xi^2\right)^{\frac{\alpha}{2}} \leq \frac{\alpha}{2} \left(\xi^2\right)^{\frac{\alpha}{2}-1} \leq \xi^{\alpha-2}. \tag{4.29}$$

By (4.29), we obtain

$$\frac{1}{\xi^\alpha} - \frac{1}{(1+\xi^2)^{\frac{\alpha}{2}}} = O\left(\frac{1}{\xi^{\alpha+2}} \right), \quad \xi \to \infty. \tag{4.30}$$

Noting that $\frac{1}{2} \leq \alpha < 1$, by (4.30), we have

$$\int_0^\infty \left| \frac{1}{\xi^\alpha} - \frac{1}{(1+\xi^2)^{\frac{\alpha}{2}}} \right| d\xi \leq C \tag{4.31}$$

and

$$|J| = \left| \int_0^\infty \frac{e^{i(d\phi(|\xi|)-x\xi)}}{(1+\xi^2)^{\frac{\alpha}{2}}} \mu \left(\frac{\xi}{N} \right) d\xi \right| \leq \left| \int_0^\infty e^{i(d\phi(|\xi|)-x\xi)} \left(\frac{1}{(1+\xi^2)^{\frac{\alpha}{2}}} - \frac{1}{\xi^\alpha} \right) \mu \left(\frac{\xi}{N} \right) d\xi \right|$$

$$
\begin{aligned}
&+\left|\int_0^\infty e^{i(d\phi(|\xi|)-x\xi)}\frac{1}{\xi^\alpha}\mu\left(\frac{\xi}{N}\right)d\xi\right| \\
&=:K_1+K_2.
\end{aligned}
$$

By (4.31), we have

$$|K_1|\le C=C|x|^\beta\frac{1}{|x|^\beta}\le CM^\beta\frac{1}{|x|^\beta}=C\frac{1}{|x|^\beta}. \tag{4.32}$$

By Lemma 1.3, we obtain

$$|K_2|\le C\frac{1}{|x|^{1-\alpha}}. \tag{4.33}$$

Noting that $\frac{1}{|x|}>\frac{1}{M}$, and the fact $\beta\ge 1-\alpha$, it follows from $\frac{1}{2}\le\alpha<1$, $\frac{1}{2}\le\beta\le 1$ that

$$|x|^{1-\alpha}=|x|^\beta|x|^{1-\alpha-\beta}=|x|^\beta\left(\frac{1}{|x|}\right)^{\beta-(1-\alpha)}\ge C|x|^\beta. \tag{4.34}$$

Therefore, by (4.33) and (4.34), we have

$$|K_2|\le C\frac{1}{|x|^\beta}. \tag{4.35}$$

Hence, (4.3) holds from (4.32) and (4.35).

Case (II-c): $\alpha=1$. From the proof of Lemma 1.3, noting that $M\ge 2$, we may get

$$|J|\le C\log\left(\frac{1}{|x|}\right)\quad\text{if } 0<|x|\le\frac{1}{2} \tag{4.36}$$

and

$$|J|\le C\quad\text{if } \frac{1}{2}<|x|<M. \tag{4.37}$$

By (4.36) and $\frac{1}{2}\le\beta\le 1$, for $0<|x|\le\frac{1}{2}$, we have

$$|J|\le C\log\left(\frac{1}{|x|}\right)\le C\frac{1}{|x|^\beta}. \tag{4.38}$$

By (4.37), for $\frac{1}{2}<|x|<M$, we have

$$|J|\le C=C|x|^\beta\frac{1}{|x|^\beta}\le CM^\beta\frac{1}{|x|^\beta}=C\frac{1}{|x|^\beta}. \tag{4.39}$$

Thus, for $\alpha=1$, $|x|<M$, by (4.38) and (4.39), we get

$$|J|\le C\frac{1}{|x|^\beta},$$

which is just estimate (4.3).

Summing up all the above estimates, we complete the proof of estimate (4.3) and finish the proof of Lemma 1.4. □

5 Conclusion

In this paper, by linearizing the maximal operator and duality methods, and applying the results of Lemma 1.3 and Lemma 1.4, we obtain the maximal global L^q inequalities (1.8) and (1.9) for multiparameter oscillatory integral $S_{t,\Phi}$. These estimates are apparently good extensions to maximal global L^q inequalities (1.6) and (1.7) for the multiparameter fractional Schrödinger equation in [32].

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Authors' contributions
YN participated in the design of the study and in the discussions of all results. YX participated in the discussions of all results. All authors read and approved the final manuscript.

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