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Small operator ideals formed by s numbers on generalized Cesàro and Orlicz sequence spaces

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Abstract

In this article, we establish sufficient conditions on the generalized Cesàro and Orlicz sequence spaces \mathbb{E} such that the class $S_{\mathbb{E}}$ of all bounded linear operators between arbitrary Banach spaces with its sequence of s -numbers belonging to \mathbb{E} generates an operator ideal. The components of $S_{\mathbb{E}}$ as a pre-quasi Banach operator ideal containing finite dimensional operators as a dense subset and its completeness are proved. Some inclusion relations between the operator ideals as well as the inclusion relations for their duals are obtained. Finally, we show that the operator ideal formed by \mathbb{E} and approximation numbers is small under certain conditions.

Keywords: s -numbers; Small operator ideal; Orlicz sequence space; Generalized Cesàro sequence space

1 Introduction

The operator ideals theory is gaining importance in functional analysis, since it has many applications in spectral theory, geometry of Banach spaces, eigenvalue distributions theorem, fixed point theorem, etc. Throughout this paper, by w we denote the space of all real sequences, \mathbb{R} denotes the real numbers, $\mathbb{N} = \{0, 1, 2, \dots\}$, and $\mathcal{L}(U, V)$ is the space of all bounded linear operators from a normed space U into a normed space V . Some of operator ideals in the class of Banach spaces or Hilbert spaces are defined by different scalar sequence spaces. For example the ideal of compact operators is defined by the space c_0 of convergent to zero sequences and Kolmogorov numbers. Pietsch [1] examined the operator ideals formed by the classical sequence space ℓ^p ($0 < p < \infty$) and the approximation numbers. He showed that the ideal of nuclear operators and the ideal of Hilbert–Schmidt operators between Hilbert spaces are defined by ℓ^1 and ℓ^2 , respectively, and the sequence of approximation numbers. In [2], the authors studied the operator ideals constructed by generalized Cesàro and Orlicz sequence spaces ℓ_M and approximation numbers. With continuity in generalization, the idea of this paper is to study a generalized class $S_{\mathbb{E}}$ by using some sequences of s -numbers and \mathbb{E} . We give sufficient conditions on Orlicz and generalized Cesàro sequence spaces \mathbb{E} such that the class $S_{\mathbb{E}}$ forms an operator ideal. The completeness and denseness of its ideal components are specified. We also prove that the class $S_{\mathbb{E}}$, for any pre-modular special space of sequences (sss), is a pre-quasi Banach operator ideal which is more general than the usual classes of operator ideals. Moreover, we have

obtained various inclusion relations between the operator ideals as well as the inclusion relations for their duals. Finally, we give sufficient conditions on Orlicz and generalized Cesáro sequence spaces such that the operator ideal formed by approximation numbers is small. These results are considered as a generalization for the case of ℓ^p , ($0 < p < \infty$).

2 Definitions and preliminaries

Definition 2.1 ([3]) An s -number function is a map defined on $\mathfrak{L}(U, V)$ which associates to each operator $P \in \mathfrak{L}(U, V)$ a sequence of nonnegative numbers $(s_n(P))_{n=0}^\infty$ with some properties:

- (a) monotonicity: $\|P\| = s_0(P) \geq s_1(P) \geq s_2(P) \geq \cdots \geq 0$ for all $P \in \mathfrak{L}(U, V)$.
- (b) additivity: $s_{m+n-1}(P_1 + P_2) \leq s_m(P_1) + s_n(P_2)$ for all $P_1, P_2 \in \mathfrak{L}(U, V)$, $m, n \in \mathbb{N}$.
- (c) property of ideal: $s_n(TPR) \leq \|T\|s_n(P)\|R\|$ for all $R \in \mathfrak{L}(U_0, U)$, $P \in \mathfrak{L}(U, V)$, and $T \in \mathfrak{L}(V, V_0)$, where U_0 and V_0 are normed spaces.
- (d) $s_n(\lambda P) = |\lambda|s_n(P)$ for every $T \in \mathfrak{L}(U, V)$, $\lambda \in \mathbb{R}$.
- (e) rank property: If $\text{rank}(P) \leq n$, then $s_n(P) = 0$ for every $P \in \mathfrak{L}(U, V)$.
- (f) property of norming:

$$s_i(I_j) = \begin{cases} 1, & \text{if } i < j; \\ 0, & \text{if } i \geq j, \end{cases}$$

where I_j is the identity operator on the Euclidean space \mathbb{R}^j .

There are several examples of s -numbers, we mention the following:

- (1) The n th approximation number, denoted by $\alpha_n(P)$, is defined by $\alpha_n(P) = \inf\{\|P - A\| : A \in \mathfrak{L}(U, V) \text{ and } \text{rank}(A) \leq n\}$.
- (2) The n th Gel'fand number, denoted by $c_n(P)$, is defined by $c_n(P) = \alpha_n(J_V P)$, where J_V is a metric injection from the normed space V to a higher space $l_\infty(A)$ for an adequate index set A . This number is independent of the choice of the higher space $l_\infty(A)$.
- (3) The n th Kolmogorov number, denoted by $d_n(P)$, is defined by

$$d_n(P) = \inf_{\dim V \leq n} \sup_{\|x\| \leq 1} \inf_{y \in V} \|Px - y\|.$$

- (4) The n th Weyl number, denoted by $x_n(P)$, is defined by

$$x_n(P) = \inf\{\alpha_n(PA) : \|A : \ell_2 \rightarrow U\| \leq 1\}.$$

- (5) The n th Chang number, denoted by $y_n(P)$, is defined by

$$y_n(P) = \inf\{\alpha_n(BP) : \|B : V \rightarrow \ell_2\| \leq 1\}.$$

- (6) The n th Hilbert number, denoted by $h_n(P)$, is defined by

$$h_n(P) = \sup\{\alpha_n(BPA) : \|B : V \rightarrow \ell_2\| \leq 1 \text{ and } \|A : \ell_2 \rightarrow U\| \leq 1\}.$$

Remark 2.2 ([3]) Among all the s -number sequences defined above, it is easy to verify that the approximation number $\alpha_n(P)$ is the largest and the Hilbert number $h_n(P)$ is the smallest s -number sequence, i.e., $h_n(P) \leq s_n(P) \leq \alpha_n(P)$ for any bounded linear operator P . If P is compact and defined on a Hilbert space, then all the s -numbers coincide with the eigenvalues of $|P|$, where $|P| = (P^*P)^{\frac{1}{2}}$.

Theorem 2.3 ([3], p. 115) *If $P \in \mathfrak{L}(U, V)$, then*

$$h_n(P) \leq x_n(P) \leq c_n(P) \leq \alpha_n(P) \quad \text{and} \quad h_n(P) \leq y_n(P) \leq d_n(P) \leq \alpha_n(P).$$

Theorem 2.4 ([3], p. 90) *An s -number sequence is injective if, for any metric injection $J \in \mathfrak{L}(V, V_0)$, $s_n(P) = s_n(JP)$ for all $P \in \mathfrak{L}(U, V)$.*

Theorem 2.5 ([3], p. 95) *An s -number sequence is surjective if, for any metric surjection $Q \in \mathfrak{L}(U_0, U)$, $s_n(P) = s_n(PQ)$ for all $P \in \mathfrak{L}(U, V)$.*

Theorem 2.6 ([3], pp. 90–94) *The Gelfand numbers and the Weyl numbers are injective.*

Theorem 2.7 ([3], pp. 95) *The Kolmogorov numbers and the Chang numbers are surjective.*

Definition 2.8 A finite rank operator is a bounded linear operator whose dimension of the range space is finite.

Definition 2.9 ((Dual s -numbers) [4]) For each s -number sequence $s = (s_n)$, a dual s -number function $s^D = (s_n^D)$ is defined by

$$s_n^D(P) = s_n(P') \quad \text{for all } P \in \mathfrak{L}(U, V),$$

where P' is the dual of P .

Definition 2.10 ([5], p. 152) An s -number sequence is called symmetric if $s_n(P) \geq s_n(P')$ for all $P \in \mathfrak{L}(U, V)$. If $s_n(P) = s_n(P')$, then the s -number sequence is said to be completely symmetric.

Now we recall some known results related to the dual of an s -number sequence.

Theorem 2.11 ([5], p. 152) *The approximation numbers are symmetric, i.e., $\alpha_n(P') \leq \alpha_n(P)$ for $P \in \mathfrak{L}(U, V)$.*

Remark 2.12 ([6]) $\alpha_n(P') = \alpha_n(P)$ for every compact operator P .

Theorem 2.13 ([5], p. 153) *If $P \in \mathfrak{L}(U, V)$, then*

$$c_n(P) = d_n(P') \quad \text{and} \quad c_n(P') \leq d_n(P).$$

In addition, if P is a compact operator, then $c_n(P') = d_n(P)$.

Theorem 2.14 ([3], p. 96) *If $P \in \mathfrak{L}(U, V)$, then*

$$x_n(P) = y_n(P') \quad \text{and} \quad y_n(P') \leq x_n(P),$$

i.e., Weyl numbers and Chang numbers are dual to each other.

Theorem 2.15 ([5], p. 153) *The Hilbert numbers are completely symmetric, i.e., $h_n(P) = h_n(P')$ for all $P \in \mathfrak{L}(U, V)$.*

Definition 2.16 ([7, 8]) *The operator ideal $\mathbb{U} := \{\mathbb{U}(U, V); U \text{ and } V \text{ are Banach spaces}\}$ is a subclass of linear bounded operators such that its components $\mathbb{U}(U, V)$ which are subsets of $\mathfrak{L}(U, V)$ satisfy the following conditions:*

- (i) $I_K \in \mathbb{U}$ where K indicates a one-dimensional Banach space, where $\mathbb{U} \subset \mathfrak{L}$.
- (ii) For $P_1, P_2 \in \mathbb{U}(U, V)$, then $\lambda_1 P_1 + \lambda_2 P_2 \in \mathbb{U}(U, V)$ for any scalars λ_1, λ_2 .
- (iii) If $P \in \mathfrak{L}(U_0, U)$, $T \in \mathbb{U}(U, V)$, and $R \in \mathfrak{L}(V, V_0)$, then $RTP \in \mathbb{U}(U_0, V_0)$.

Definition 2.17 *An Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is convex, continuous, and nondecreasing with $M(0) = 0$, $M(u) > 0$ for $u > 0$ and $M(u) \rightarrow \infty$, as $u \rightarrow \infty$. See [9] and [10].*

Definition 2.18 *An Orlicz function M is said to satisfy Δ_2 -condition for every value of $u \geq 0$ if there is $k > 0$ such that $M(2u) \leq kM(u)$. The Δ_2 -condition is equivalent to $M(lu) \leq klM(u)$ for every value of $l > 1$ and u .*

Lindentrauss and Tzafriri [11] utilized the idea of an Orlicz function to define Orlicz sequence space:

$$\ell_M = \left\{ u \in \omega : \rho(\beta u) < \infty \text{ for some } \beta > 0 \right\} \quad \text{where } \rho(u) = \sum_{k=0}^{\infty} M(|u_k|),$$

$(\ell_M, \|\cdot\|)$ is a Banach space with the Luxemburg norm:

$$\|u\| = \inf \left\{ \beta > 0 : \rho\left(\frac{u}{\beta}\right) \leq 1 \right\}.$$

Every Orlicz sequence space contains a subspace that is isomorphic to c_0 or ℓ^q for some $1 \leq q < \infty$ ([12], Theorem 4.a.9).

Later, several classes of sequences have been introduced using Orlicz functions by Altin et al. [13], Et et al. ([14] and [15]), and Tripathy et al. [16–18].

For a sequence $q = (q_i)$ of positive real numbers with $q_i \geq 1$, for all $i \in \mathbb{N}$, the generalized Cesáro sequence space is defined by

$$\text{ces}((q_i)) = \left\{ u = (u_i) \in \omega : \rho(\beta u) < \infty \text{ for some } \beta > 0 \right\} \quad \text{and}$$

$$\rho(u) = \sum_{i=0}^{\infty} \left(\frac{\sum_{j=0}^i |u_j|}{i+1} \right)^{q_i}.$$

$(\text{ces}((q_i)), \|\cdot\|)$ is a Banach space with the Luxemburg norm. If (q_i) is bounded, one can simply write

$$\text{ces}((q_i)) = \left\{ u = (u_i) \in \omega : \sum_{i=0}^{\infty} \left(\frac{\sum_{j=0}^i |u_j|}{i+1} \right)^{q_i} < \infty \right\}.$$

Sanhan and Suantai [19] studied some geometric properties of $\text{ces}((q_i))$.

Definition 2.19 ([2]) A class of linear sequence spaces \mathbb{E} is called a special space of sequences (sss) if

- (1) $e_i \in \mathbb{E}$ for all $i \in \mathbb{N}$,
- (2) if $u = (u_i) \in w$, $v = (v_i) \in \mathbb{E}$ and $|u_i| \leq |v_i|$ for every $i \in \mathbb{N}$, then $u \in \mathbb{E}$ “i.e., \mathbb{E} is solid”,
- (3) if $(u_i)_{i=0}^{\infty} \in \mathbb{E}$, then $(u_{[\frac{i}{2}]})_{i=0}^{\infty} \in \mathbb{E}$, wherever $[\frac{i}{2}]$ means the integral part of $\frac{i}{2}$.

Theorem 2.20 ℓ_M is a (sss) if M is an Orlicz function satisfying Δ_2 -condition.

Theorem 2.21 $\text{ces}((q_i))$ is a (sss) if (q_i) is an increasing sequence, $1 < q_0$ and $\sup q_i < \infty$.

Definition 2.22 ([2]) A subclass of the special space of sequences called a pre-modular (sss) if there is a function $\varrho : \mathbb{E} \rightarrow [0, \infty[$ satisfying the following conditions:

- (i) $\varrho(u) \geq 0$ for each $u \in \mathbb{E}$ and $\varrho(u) = 0 \Leftrightarrow u = \theta$, where θ is the zero element of \mathbb{E} ,
- (ii) there exists $L \geq 1$ such that $\varrho(\beta u) \leq L|\beta|\varrho(u)$ for all $u \in \mathbb{E}$ and for any scalar β ,
- (iii) for some $K \geq 1$, $\varrho(u + v) \leq K(\varrho(u) + \varrho(v))$ for every $u, v \in \mathbb{E}$,
- (iv) if $|u_i| \leq |v_i|$ for all $i \in \mathbb{N}$, then $\varrho((u_i)) \leq \varrho((v_i))$,
- (v) for some $K_0 \geq 1$, $\varrho((u_i)) \leq \varrho((u_{[\frac{i}{2}]})) \leq K_0\varrho((u_i))$,
- (vi) the set of all finite sequences is ϱ -dense in \mathbb{E} . This means that, for each $u = (u_i)_{i=0}^{\infty} \in \mathbb{E}$ and for each $\varepsilon > 0$, there exists $s \in \mathbb{N}$ such that $\varrho((u_i)_{i=s}^{\infty}) < \varepsilon$,
- (vii) there exists a constant $\xi > 0$ such that $\varrho(\beta, 0, 0, 0, \dots) \geq \xi|\beta|\varrho(1, 0, 0, 0, \dots)$ for any $\beta \in \mathbb{R}$.

From condition (ii), it is clear that ϱ is continuous at θ . We denote by $(\mathbb{E}_{\varrho}, \varrho)$ the linear space \mathbb{E} equipped with the metrizable topology generated by ϱ .

Example 2.23 ℓ^q is a pre-modular (sss) if $0 < q < \infty$.

Example 2.24 ℓ_M is a pre-modular (sss) if M is an Orlicz function satisfying Δ_2 -condition.

Example 2.25 ces_q is a pre-modular (sss) if $1 < q < \infty$.

Example 2.26 $\text{ces}((q_i))$ is a pre-modular (sss) if (q_i) is an increasing sequence, $1 < q_0$ and $\sup q_i < \infty$.

Theorem 2.27 ([20]) If U, V are infinite dimensional Banach spaces and (μ_i) is a monotonic decreasing sequence to zero, then there exists a bounded linear operator P such that

$$\frac{1}{16}\mu_{3i} \leq \alpha_i(P) \leq 8\mu_{i+1}.$$

Throughout this paper, $e_i = \{0, 0, \dots, 1, 0, \dots\}$ where 1 appears at the i th place for all $i \in \mathbb{N}$, the sequence (q_i) is a bounded sequence of positive numbers, and the following well-known inequality [21]: $|a_i + b_i|^{q_i} \leq K(|a_i|^{q_i} + |b_i|^{q_i})$, where $K = 2^{h-1}$, $h = \sup_i q_i$, and $q_i \geq 1$ for every $i \in \mathbb{N}$ are used.

3 Main results

Notations 3.1

$S_{\mathbb{E}} := \{S_{\mathbb{E}}(U, V); U \text{ and } V \text{ are Banach spaces}\}$, where

$S_{\mathbb{E}}(U, V) := \{P \in \mathcal{L}(U, V) : (s_i(P))_{i=0}^{\infty} \in \mathbb{E}\}$. Also

$S_{\mathbb{E}}^{\text{app}} := \{S_{\mathbb{E}}^{\text{app}}(U, V); U \text{ and } V \text{ are Banach spaces}\}$, where

$S_{\mathbb{E}}^{\text{app}}(U, V) := \{P \in \mathcal{L}(U, V) : (\alpha_i(P))_{i=0}^{\infty} \in \mathbb{E}\}$.

Theorem 3.2 *If \mathbb{E} is a (sss), then $S_{\mathbb{E}}$ is an operator ideal.*

Proof To show $S_{\mathbb{E}}$ is an operator ideal

(i) let $B \in F(U, V)$ and $\text{rank}(B) = n$ for each $n \in \mathbb{N}$, since $e_i \in \mathbb{E}$ for each $i \in \mathbb{N}$ and \mathbb{E} is a linear space, hence $(s_i(B))_{i=0}^{\infty} = (s_0(B), s_1(B), \dots, s_{n-1}(B), 0, 0, 0, \dots) = \sum_{i=0}^{n-1} s_i(B)e_i \in \mathbb{E}$; for that $B \in S_{\mathbb{E}}(U, V)$, which implies $F(U, V) \subseteq S_{\mathbb{E}}(U, V)$.

(ii) Let $P_1, P_2 \in S_{\mathbb{E}}(U, V)$ and $\beta_1, \beta_2 \in \mathbb{R}$, then from Definition 2.19 condition (3) we get $(s_{[\frac{i}{2}]}(P_1))_{i=0}^{\infty} \in \mathbb{E}$ and $(s_{[\frac{i}{2}]}(P_2))_{i=0}^{\infty} \in \mathbb{E}$, since $i \geq 2[\frac{i}{2}]$, from the definition of s -numbers and $s_i(P)$ is a decreasing sequence, we have $s_i(\beta_1 P_1 + \beta_2 P_2) \leq s_{2[\frac{i}{2}]}(\beta_1 P_1 + \beta_2 P_2) \leq s_{[\frac{i}{2}]}(\beta_1 P_1) + s_{[\frac{i}{2}]}(\beta_2 P_2) = |\beta_1| s_{[\frac{i}{2}]}(P_1) + |\beta_2| s_{[\frac{i}{2}]}(P_2)$ for all $i \in \mathbb{N}$. Since from Definition 2.19 condition (2) and \mathbb{E} is a linear space, we have $(s_i(\beta_1 P_1 + \beta_2 P_2))_{i=0}^{\infty} \in \mathbb{E}$, hence $\beta_1 P_1 + \beta_2 P_2 \in S_{\mathbb{E}}(U, V)$.

(iii) If $P \in \mathcal{L}(U_0, U)$, $T \in S_{\mathbb{E}}(U, V)$, and $R \in \mathcal{L}(V, V_0)$, then we get $(s_i(P))_{i=0}^{\infty} \in \mathbb{E}$ and since $s_i(RTP) \leq \|R\| s_i(T) \|P\|$, from Definition 2.19 conditions (1) and (2) we get $(s_i(RTP))_{i=0}^{\infty} \in \mathbb{E}$, then $RTP \in S_{\mathbb{E}}(U_0, V_0)$. \square

Corollary 3.3 *If M is an Orlicz function satisfying Δ_2 -condition, then S_{ℓ_M} is an operator ideal.*

Corollary 3.4 *S_{ℓ^q} is an operator ideal if $0 < q < \infty$.*

Corollary 3.5 *If (q_i) is a bounded increasing sequence and $q_0 > 1$, then $S_{\text{ces}((q_i))}$ is an operator ideal.*

Corollary 3.6 *S_{ces_q} is an operator ideal if $1 < q < \infty$.*

The following question arises naturally: for which Orlicz and Cesàro sequence spaces \mathbb{E} , is the ideal of the finite rank operators dense in $S_{\mathbb{E}}(U, V)$?

Theorem 3.7 *$S_{\ell_M}(U, V) = \overline{F(U, V)}$ if M is an Orlicz function and U, V are Banach spaces.*

Proof Define $\varrho(u) = \sum_{i=0}^{\infty} M(|u_i|)$ on ℓ_M . First, we show, if $P \in F(U, V)$, then it belongs to $S_{\ell_M}(U, V)$. Since $e_i \in \ell_M$ for each $i \in \mathbb{N}$ and ℓ_M is a linear space, for each finite operator $P \in F(U, V)$, we obtain $(s_i(P))_{i=0}^{\infty}$ containing only finitely many terms different from zero.

Currently we prove that $S_{\ell_M}(U, V) \subseteq \overline{F(U, V)}$, let $P \in S_{\ell_M}(U, V)$ we have $(s_i(P))_{i=0}^\infty \in \ell_M$ and since $\sum_{i=0}^\infty M(s_i(P)) < \infty$, let $\varepsilon \in (0, 1]$, hence there exists $i_0 \in \mathbb{N} - \{0\}$ with $\sum_{i=i_0}^\infty M(s_i(P)) < \frac{\varepsilon}{4}$. Since $s_i(P)$ is decreasing for each $i \in \mathbb{N}$ and ϱ is nondecreasing, we obtain

$$i_0 M(s_{2i_0}(P)) \leq \sum_{i=i_0+1}^{2i_0} M(s_i(P)) \leq \sum_{i=i_0}^\infty M(s_i(P)) < \frac{\varepsilon}{4}.$$

Hence, there exists $B \in F_{2i_0}(U, V)$ such that $\text{rank } B \leq 2i_0$ with $M(\|P - B\|) < \frac{\varepsilon}{4i_0}$, and M is an Orlicz function, hence

$$\begin{aligned} d(P, B) &= \varrho(s_i(P - B))_{i=0}^\infty \\ &= \sum_{i=0}^\infty M(s_i(P - B)) \\ &\quad \times \sum_{i=0}^{3i_0-1} M(s_i(P - B)) + \sum_{i=3i_0}^\infty M(s_i(P - B)) \\ &\leq \sum_{i=0}^{3i_0-1} M(\|P - B\|) + \sum_{i=3i_0}^\infty M(s_i(P - B)) \\ &\leq 3i_0 M(\|P - B\|) + \sum_{i=i_0}^\infty M(s_{i+2i_0}(P - B)) \\ &\leq 3i_0 M(\|P - B\|) + \sum_{i=i_0}^\infty M(s_i(P)) < \varepsilon. \end{aligned}$$

□

Corollary 3.8 $S_{\ell^q}(U, V) = \overline{F(U, V)}$ if $0 < q < \infty$ and U, V are Banach spaces.

Theorem 3.9 $S_{\text{ces}((q_i))}(U, V) = \overline{F(U, V)}$ if (q_i) is an increasing sequence, $q_0 > 1$ and U, V are Banach spaces.

Proof We prove first that $\overline{F(U, V)} \subseteq S_{\text{ces}((q_i))}(U, V)$. Since $e_i \in \text{ces}((q_i))$ for each $i \in \mathbb{N}$ and $\text{ces}((q_i))$ is a linear space, for each finite operator $P \in F(U, V)$, i.e., one obtains that $(s_i(P))_{i=0}^\infty$ holds main finitely a significant unique number in relation to zero. Now we prove that $S_{\text{ces}((q_i))}(U, V) \subseteq \overline{F(U, V)}$. Since $q_0 > 1$ and (q_i) is an increasing sequence, we have $\sum_{i=0}^\infty (\frac{1}{i+1})^{q_i} < \infty$, let $P \in S_{\text{ces}((q_i))}(U, V)$, we get $(s_i(P))_{i=0}^\infty \in \text{ces}((q_i))$ and since $\varrho(s_i(P))_{i=0}^\infty < \infty$, let $\varepsilon \in (0, 1)$, hence there exists $i_0 \in \mathbb{N} - \{0\}$ such that $\varrho((s_i(P))_{i=i_0}^\infty) < \frac{\varepsilon}{2^{h+3}\delta C}$ for some $c \geq 1$, where $\delta = \max\{1, \sum_{i=i_0}^\infty (\frac{1}{i+1})^{q_i}\}$. As $s_i(P)$ is decreasing for every $i \in \mathbb{N}$, we have

$$\sum_{i=i_0+1}^{2i_0} \left(\frac{\sum_{j=0}^i s_{2i_0}(P)}{i+1} \right)^{q_i} \leq \sum_{i=i_0+1}^{2i_0} \left(\frac{\sum_{j=0}^i s_j(P)}{i+1} \right)^{q_i} \leq \sum_{i=i_0}^\infty \left(\frac{\sum_{j=0}^i s_j(P)}{i+1} \right)^{q_i} < \frac{\varepsilon}{2^{h+3}\delta C}. \quad (1)$$

Hence, there exists $B \in F_{2i_0}(U, V)$ such that $\text{rank } B \leq 2i_0$ and

$$\sum_{i=2i_0+1}^{3i_0} \left(\frac{\sum_{j=0}^i \|P - B\|}{i+1} \right)^{q_i} \leq \sum_{i=i_0+1}^{2i_0} \left(\frac{\sum_{j=0}^i \|P - B\|}{i+1} \right)^{q_i} < \frac{\varepsilon}{2^{h+3}\delta C} \quad (2)$$

for the bounded sequence (q_i) . Then consider

$$\sup_{i=i_0}^{\infty} \left(\sum_{j=0}^{i_0} \|P-B\| \right)^{q_i} < \frac{\varepsilon}{2^{2h+2}\delta}, \quad (3)$$

hence by setting

$$\sum_{i=0}^{i_0} \left(\frac{\sum_{j=0}^i \|P-B\|}{i+1} \right)^{q_i} < \frac{\varepsilon}{2^{h+3}\delta C}. \quad (4)$$

Since (q_i) is increasing and by using (1), (2), (3), and (4), we have

$$\begin{aligned} d(P, B) &= \varrho(s_i(P-B))_{i=0}^{\infty} = \sum_{i=0}^{3i_0-1} \left(\frac{\sum_{j=0}^i s_j(P-B)}{i+1} \right)^{q_i} + \sum_{i=3i_0}^{\infty} \left(\frac{\sum_{j=0}^i s_j(P-B)}{i+1} \right)^{q_i} \\ &\leq \sum_{i=0}^{3i_0} \left(\frac{\sum_{j=0}^i \|P-B\|}{i+1} \right)^{q_i} + \sum_{i=i_0}^{\infty} \left(\frac{\sum_{j=0}^{i+2i_0} s_j(P-B)}{i+1} \right)^{q_{i+2i_0}} \\ &\leq \sum_{i=0}^{3i_0} \left(\frac{\sum_{j=0}^i \|P-B\|}{i+1} \right)^{q_i} + \sum_{i=i_0}^{\infty} \left(\frac{\sum_{j=0}^{i+2i_0} s_j(P-B)}{i+1} \right)^{q_i} \\ &\leq \sum_{i=0}^{3i_0} \left(\frac{\sum_{j=0}^i \|P-B\|}{i+1} \right)^{q_i} + \sum_{i=i_0}^{\infty} \left(\frac{\sum_{j=0}^{i+2i_0} s_j(P-B)}{i+1} \right)^{q_i} \\ &\leq 3 \sum_{i=0}^{i_0} \left(\frac{\sum_{j=0}^i \|P-B\|}{i+1} \right)^{q_i} + \sum_{i=i_0}^{\infty} \left(\frac{\sum_{j=0}^{2i_0-1} s_j(P-B) + \sum_{j=2i_0}^{i+2i_0} s_j(P-B)}{i+1} \right)^{q_i} \\ &\leq 3 \sum_{i=0}^{i_0} \left(\frac{\sum_{j=0}^i \|P-B\|}{i+1} \right)^{q_i} \\ &\quad + 2^{h-1} \left[\sum_{i=i_0}^{\infty} \left(\frac{\sum_{j=0}^{2i_0-1} s_j(P-B)}{i+1} \right)^{q_i} + \sum_{i=i_0}^{\infty} \left(\frac{\sum_{j=2i_0}^{i+2i_0} s_j(P-B)}{i+1} \right)^{q_i} \right] \\ &\leq 3 \sum_{i=0}^{i_0} \left(\frac{\sum_{j=0}^i \|P-B\|}{i+1} \right)^{q_i} \\ &\quad + 2^{h-1} \left[\sum_{i=i_0}^{\infty} \left(\frac{\sum_{j=0}^{2i_0-1} \|P-B\|}{i+1} \right)^{q_i} + \sum_{i=i_0}^{\infty} \left(\frac{\sum_{j=0}^i s_{j+2i_0}(P-B)}{i+1} \right)^{q_i} \right] \\ &\leq 3 \sum_{i=0}^{i_0} \left(\frac{\sum_{j=0}^i \|P-B\|}{i+1} \right)^{q_i} + 2^{h-1} \sup_{i=i_0}^{\infty} \left(\sum_{j=0}^{2i_0-1} \|P-B\| \right)^{q_i} \sum_{i=i_0}^{\infty} (i+1)^{-q_i} \\ &\quad + 2^{h-1} \sum_{i=i_0}^{\infty} \left(\frac{\sum_{j=0}^i s_j(P)}{i+1} \right)^{q_i} \end{aligned}$$

$< \varepsilon$.

□

Corollary 3.10 $S_{ces_q}(U, V) = \overline{F(U, V)}$ if $1 < q < \infty$ and U, V are Banach spaces.

The following question arises naturally: for which Orlicz and generalized Cesáro sequence spaces \mathbb{E} , are the components of the ideal $S_{\mathbb{E}}^{\text{app}}$ complete?

Definition 3.11 A function $g : \Omega \rightarrow [0, \infty)$ is said to be a pre-quasi norm on the ideal Ω if the following conditions hold:

- (1) for all $P \in \Omega(U, V)$, $g(P) \geq 0$ and $g(P) = 0$ if and only if $P = 0$,
- (2) there exists a constant $L \geq 1$ such that $g(\lambda P) \leq L|\lambda|g(P)$ for all $P \in \Omega(U, V)$ and $\lambda \in \mathbb{R}$,
- (3) there exists a constant $K \geq 1$ such that $g(P_1 + P_2) \leq K[g(P_1) + g(P_2)]$ for all $P_1, P_2 \in \Omega(U, V)$,
- (4) there exists a constant $C \geq 1$ such that if $T \in \mathfrak{L}(U_0, U)$, $P \in \Omega(U, V)$, and $R \in \mathfrak{L}(V, V_0)$, then $g(RPT) \leq C\|R\|g(P)\|T\|$, where U_0 and V_0 are normed spaces.

We state the following two theorems without proof, those can be established using standard techniques.

Theorem 3.12 Every quasi norm on the ideal Ω is a pre-quasi norm on the ideal Ω .

Theorem 3.13 The function $g(P) = \varrho(s_i(P))_{i=0}^\infty$ is a pre-quasi norm on $S_{\mathbb{E}_\varrho}$, where \mathbb{E}_ϱ is a pre-modular (sss).

Theorem 3.14 $(S_{\mathbb{E}_\varrho}, g)$ is a pre-quasi Banach operator ideal if \mathbb{E}_ϱ is a pre-modular (sss).

Proof Since \mathbb{E}_ϱ is a pre-modular (sss), then the function $g(P) = \varrho(s_i(P))_{i=0}^\infty$ is a pre-quasi norm on $S_{\mathbb{E}_\varrho}$. Let (P_m) be a Cauchy sequence in $S_{\mathbb{E}_\varrho}(U, V)$. Hence, by using Part (vii) of Definition 2.22 and since $\mathfrak{L}(U, V) \supseteq S_{\mathbb{E}_\varrho}(U, V)$, one gets

$$\begin{aligned} g(P_i - P_j) &= \varrho((s_n(P_i - P_j))_{n=0}^\infty) \geq \varrho(s_0(P_i - P_j), 0, 0, 0, \dots) = \varrho(\|P_i - P_j\|, 0, 0, 0, \dots) \\ &\geq \xi \|P_i - P_j\| \varrho(1, 0, 0, 0, \dots), \end{aligned}$$

then $(P_m)_{m \in \mathbb{N}}$ is a Cauchy sequence in $\mathfrak{L}(U, V)$. While the space $\mathfrak{L}(U, V)$ is a Banach space, so there exists $P \in \mathfrak{L}(U, V)$ such that $\lim_{m \rightarrow \infty} \|P_m - P\| = 0$, and while $(s_n(P_m))_{n=0}^\infty \in \mathbb{E}$ for every $m \in \mathbb{N}$, so, by using parts (iii) and (iv) of Definition 2.22 and that ϱ is continuous at θ , we obtain

$$\begin{aligned} g(P) &= \varrho((s_n(P))_{n=0}^\infty) = \varrho((s_n(P - P_m + P_m))_{n=0}^\infty) \\ &\leq K\varrho((s_{[\frac{n}{2}]}(P - P_m))_{n=0}^\infty) + K\varrho((s_{[\frac{n}{2}]}(P_m))_{n=0}^\infty) \\ &\leq K\varrho(\|P_m - P\|_{n=0}^\infty) + K\varrho((s_n(P_m))_{n=0}^\infty) < \infty, \end{aligned}$$

we have $(s_n(P))_{n=0}^\infty \in \mathbb{E}$, then $P \in S_{\mathbb{E}_\varrho}(U, V)$. □

Corollary 3.15 (S_{ℓ_M}, g) is a pre-quasi Banach operator ideal if M is an Orlicz function satisfying Δ_2 -condition.

Corollary 3.16 (S_{ℓ^q}, g) is a quasi Banach operator ideal if $0 < q < \infty$.

Corollary 3.17 $(S_{\text{ces}((q_i))}, g)$ is a pre-quasi Banach operator ideal if (q_i) is an increasing sequence and $q_0 > 1$.

Corollary 3.18 $(S_{\text{ces}_q}^{\text{app}}, g)$ is a quasi Banach operator ideal, $1 < q < \infty$.

Theorem 3.19 ([20]) For any infinite dimensional Banach spaces U, V and for any $q > p > 0$, it is true that $S_{\ell^p}^{\text{app}}(U, V) \subsetneq S_{\ell^q}^{\text{app}}(U, V) \subsetneq \mathcal{L}(U, V)$.

Theorem 3.20 For any infinite dimensional Banach spaces U, V and for any $1 < p_n < q_n$ for all $n \in \mathbb{N}$, it is true that $S_{\text{ces}(p_n)}^{\text{app}}(U, V) \subsetneq S_{\text{ces}(q_n)}^{\text{app}}(U, V) \subsetneq \mathcal{L}(U, V)$, where p_n and q_n are monotonic decreasing sequences.

Proof Let U and V be infinite dimensional Banach spaces and for any $1 < p_n < q_n$ for every $n \in \mathbb{N}$, if $P \in S_{\text{ces}(p_n)}^{\text{app}}(U, V)$, then $(\alpha_n(P)) \in \text{ces}(p_n)$. Since $\text{ces}(p_n) \subset \text{ces}(q_n)$, hence $P \in S_{\text{ces}(q_n)}^{\text{app}}(U, V)$. Next, if $q_n > p_n > 1$ for every $n \in \mathbb{N}$ and $\mu_n = \frac{1}{p_n^{1/n+1}}$. So, by using Theorem 2.27, one can find $P \in \mathcal{L}(U, V)$ with $\frac{1}{16^{1/(3n+1)}} \leq \alpha_n(P) \leq \frac{8}{p_n^{1/(n+2)}}$ such that P does not belong to $S_{\text{ces}(p_n)}^{\text{app}}(U, V)$ and $P \in S_{\text{ces}(q_n)}^{\text{app}}(U, V)$. It is easy to see that $S_{\text{ces}(q_n)}^{\text{app}}(U, V) \subset \mathcal{L}(U, V)$. Next, if we take $\mu_n = \frac{1}{q_n^{1/n+1}}$. So, by using Theorem 2.27, one can find $P \in \mathcal{L}(U, V)$ with $\frac{1}{16^{1/(3n+1)}} \leq \alpha_n(P) \leq \frac{8}{q_n^{1/(n+2)}}$ such that P does not belong to $S_{\text{ces}(q_n)}^{\text{app}}(U, V)$. \square

Corollary 3.21 For any infinite dimensional Banach spaces U, V and $1 < p < q < \infty$, then $S_{\text{ces}_p}^{\text{app}}(U, V) \subsetneq S_{\text{ces}_q}^{\text{app}}(U, V) \subsetneq \mathcal{L}(U, V)$.

We now study some properties of the pre-quasi Banach operator ideal $S_{\mathbb{E}}$.

Theorem 3.22 If the s -number sequence is injective, then the pre-quasi Banach operator ideal $(S_{\mathbb{E}_0}, g)$ is injective.

Proof Let $T \in \mathcal{L}(U, V)$ and $J \in \mathcal{L}(V, V_0)$ be any metric injection. Suppose that $JT \in S_{\mathbb{E}_0}(U, V_0)$, then $\varrho(s_n(JT)) < \infty$. Since the s -number sequence is injective, we have $s_n(JT) = s_n(T)$ for all $T \in \mathcal{L}(U, V)$, $n \in \mathbb{N}$. So $\varrho(s_n(T)) = \varrho(s_n(JT)) < \infty$. Hence $T \in S_{\mathbb{E}_0}(U, V)$ and clearly $g(T) = g(JT)$ holds. \square

Remark 3.23 The pre-quasi Banach operator ideal $(S_{\mathbb{E}_0}^{\text{Gel}}, g)$ and the pre-quasi Banach operator ideal $(S_{\mathbb{E}_0}^{\text{Weyl}}, g)$ are injective pre-quasi Banach operator ideals.

Theorem 3.24 If the s -number sequence is surjective, then the pre-quasi Banach operator ideal $(S_{\mathbb{E}_0}, g)$ is surjective.

Proof Let $T \in \mathcal{L}(U, V)$ and $Q \in \mathcal{L}(U_0, U)$ be any metric surjection. Suppose that $TQ \in S_{\mathbb{E}_0}(U_0, V)$. Then $\varrho(s_n(TQ)) < \infty$. Since the s -number sequence is surjective, we have $s_n(TQ) = s_n(T)$ for all $T \in \mathcal{L}(U, V)$, $n \in \mathbb{N}$. So $\varrho(s_n(T)) = \varrho(s_n(TQ)) < \infty$. Hence $T \in S_{\mathbb{E}_0}(U, V)$ and clearly $g(T) = g(TQ)$ holds. \square

Remark 3.25 The pre-quasi Banach operator ideal $(S_{\mathbb{E}_0}^{\text{Kol}}, g)$ and the pre-quasi Banach operator ideal $(S_{\mathbb{E}_0}^{\text{Chang}}, g)$ are surjective pre-quasi Banach operator ideals.

Also, we have the following inclusion relations between the pre-quasi Banach operator ideals.

Theorem 3.26

- (1) $S_{\mathbb{E}_\varrho}^{\text{app}} \subseteq S_{\mathbb{E}_\varrho}^{\text{Gel}} \subseteq S_{\mathbb{E}_\varrho}^{\text{Weyl}} \subseteq S_{\mathbb{E}_\varrho}^{\text{Hilb}}.$
- (2) $S_{\mathbb{E}_\varrho}^{\text{app}} \subseteq S_{\mathbb{E}_\varrho}^{\text{Kol}} \subseteq S_{\mathbb{E}_\varrho}^{\text{Chang}} \subseteq S_{\mathbb{E}_\varrho}^{\text{Hilb}}.$

Proof Since $h_n(P) \leq x_n(P) \leq c_n(P) \leq \alpha_n(P)$ and $h_n(P) \leq y_n(P) \leq d_n(P) \leq \alpha_n(P)$ for every $n \in \mathbb{N}$ and ϱ is nondecreasing, we obtain

$$\begin{aligned}\varrho(h_n(P)) &\leq \varrho(x_n(P)) \leq \varrho(c_n(P)) \leq \varrho(\alpha_n(P)), \\ \varrho(h_n(P)) &\leq \varrho(y_n(P)) \leq \varrho(d_n(P)) \leq \varrho(\alpha_n(P)).\end{aligned}$$

Hence the result. \square

We now state the dual of the operator ideal formed by different s -number sequences.

Theorem 3.27 *The operator ideal $S_{\mathbb{E}_\varrho}^{\text{app}}$ is symmetric and the operator ideal $S_{\mathbb{E}_\varrho}^{\text{Hilb}}$ is completely symmetric.*

Proof Since $\alpha_n(P') \leq \alpha_n(P)$ and $h_n(P') = h_n(P)$ for all $P \in \mathfrak{L}(U, V)$, we have $S_{\mathbb{E}_\varrho}^{\text{app}} \subseteq (S_{\mathbb{E}_\varrho}^{\text{app}})'$ and $S_{\mathbb{E}_\varrho}^{\text{Hilb}} = (S_{\mathbb{E}_\varrho}^{\text{Hilb}})'$. \square

In view of Theorem 2.13, we state the following result without proof.

Theorem 3.28 *The operator ideal $S_{\mathbb{E}_\varrho}^{\text{Gel}} = (S_{\mathbb{E}_\varrho}^{\text{Kol}})'$ and $S_{\mathbb{E}_\varrho}^{\text{Kol}} \subseteq (S_{\mathbb{E}_\varrho}^{\text{Gel}})'$. In addition if T is a compact operator from U to V , then $S_{\mathbb{E}_\varrho}^{\text{Kol}} = (S_{\mathbb{E}_\varrho}^{\text{Gel}})'$.*

In view of Theorem 2.14, we state the following result without proof.

Theorem 3.29 *The operator ideal $S_{\mathbb{E}_\varrho}^{\text{Weyl}} = (S_{\mathbb{E}_\varrho}^{\text{Chang}})'$ and $S_{\mathbb{E}_\varrho}^{\text{Chang}} = (S_{\mathbb{E}_\varrho}^{\text{Weyl}})'$.*

Theorem 3.30 *If (q_i) is an increasing sequence and $q_0 > 1$, then the pre-quasi Banach operator ideal $S_{\text{ces}(q_i)}^{\text{app}}$ is small.*

Proof Since (q_i) is an increasing sequence and $q_0 > 1$, take $\lambda = (\sum_{i=0}^{\infty} \frac{1}{(i+1)^{q_i}})^{\frac{1}{h}}$. Then $(S_{\text{ces}(q_i)}^{\text{app}}, g)$, where $g(P) = \frac{1}{\lambda} (\sum_{i=0}^{\infty} (\frac{\sum_{j=0}^i \alpha_j(P)}{(i+1)})^{q_i})^{\frac{1}{h}}$ is a pre-quasi Banach operator ideal. Let U and V be any two Banach spaces. Suppose that $S_{\text{ces}(q_i)}^{\text{app}}(U, V) = \mathfrak{L}(U, V)$, then there exists a constant $C > 0$ such that $g(P) \leq C\|P\|$ for all $P \in \mathfrak{L}(U, V)$. Assume that U and V are infinite dimensional Banach spaces. Hence by Dvoretzky's theorem [5] for $m \in \mathbb{N}$, we have quotient spaces U/N_m and subspaces M_m of V which can be mapped onto ℓ_2^m by isomorphisms H_m and A_m such that $\|H_m\| \|H_m^{-1}\| \leq 2$ and $\|A_m\| \|A_m^{-1}\| \leq 2$. Let I_m be the identity map on ℓ_2^m , Q_m be the quotient map from U onto U/N_m , and J_m be the natural embedding

map from M_m into V . Let u_n be the Bernstein numbers [4], then

$$\begin{aligned} 1 &= u_n(I_m) = u_n(A_m A_m^{-1} I_m H_m H_m^{-1}) \\ &\leq \|A_m\| u_n(A_m^{-1} I_m H_m) \|H_m^{-1}\| \\ &= \|A_m\| u_n(J_m A_m^{-1} I_m H_m) \|H_m^{-1}\| \\ &\leq \|A_m\| d_n(J_m A_m^{-1} I_m H_m) \|H_m^{-1}\| \\ &= \|A_m\| d_n(J_m A_m^{-1} I_m H_m Q_m) \|H_m^{-1}\| \\ &\leq \|A_m\| \alpha_n(J_m A_m^{-1} I_m H_m Q_m) \|H_m^{-1}\| \end{aligned} \quad (5)$$

for $1 \leq i \leq m$. Now

$$\begin{aligned} \sum_{j=0}^i (1) &\leq \sum_{j=0}^i \|A_m\| \alpha_j(J_m A_m^{-1} I_m H_m Q_m) \|H_m^{-1}\| \\ \Rightarrow \frac{1}{i+1} (i+1) &\leq \|A_m\| \left(\frac{1}{i+1} \sum_{j=0}^i \alpha_j(J_m A_m^{-1} I_m H_m Q_m) \right) \|H_m^{-1}\| \\ \Rightarrow 1 &\leq (\|A_m\| \|H_m^{-1}\|)^{q_i} \left(\frac{1}{i+1} \sum_{j=0}^i \alpha_j(J_m A_m^{-1} I_m H_m Q_m) \right)^{q_i}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left(\sum_{i=0}^m (1) \right)^{\frac{1}{h}} &\leq L \|A_m\| \|H_m^{-1}\| \left[\sum_{i=0}^m \left(\frac{1}{i+1} \sum_{j=0}^i \alpha_j(J_m A_m^{-1} I_m H_m Q_m) \right)^{q_i} \right]^{\frac{1}{h}} \\ \Rightarrow \frac{1}{\lambda} (m+1)^{\frac{1}{h}} &\leq L \|A_m\| \|H_m^{-1}\| \frac{1}{\lambda} \left[\sum_{i=0}^m \left(\frac{1}{i+1} \sum_{j=0}^i \alpha_j(J_m A_m^{-1} I_m H_m Q_m) \right)^{q_i} \right]^{\frac{1}{h}} \\ \Rightarrow \frac{1}{\lambda} (m+1)^{\frac{1}{h}} &\leq L \|A_m\| \|H_m^{-1}\| g(J_m A_m^{-1} I_m H_m Q_m), \\ \frac{1}{\lambda} (m+1)^{\frac{1}{h}} &\leq LC \|A_m\| \|H_m^{-1}\| \|J_m A_m^{-1} I_m H_m Q_m\|, \\ \frac{1}{\lambda} (m+1)^{\frac{1}{h}} &\leq LC \|A_m\| \|H_m^{-1}\| \|J_m A_m^{-1}\| \|I_m\| \|H_m Q_m\| \\ &= LC \|A_m\| \|H_m^{-1}\| \|A_m^{-1}\| \|I_m\| \|H_m\|, \\ \frac{1}{\lambda} (m+1)^{\frac{1}{h}} &\leq 4LC \end{aligned}$$

for some $L \geq 1$. Thus we arrive at a contradiction since m is arbitrary. Thus U and V both cannot be infinite dimensional when $S_{\text{ces}(q_i)}^{\text{app}}(U, V) = \mathfrak{L}(U, V)$. Hence the result. \square

Theorem 3.31 *If (q_i) is increasing and $q_0 > 1$, then the pre-quasi Banach operator ideal $S_{\text{ces}(q_i)}^{\text{Kol}}$ is small.*

Corollary 3.32 *If $1 < q < \infty$, then the quasi Banach operator ideal $S_{\text{ces}_q}^{\text{app}}$ is small.*

Corollary 3.33 *If $1 < q < \infty$, then the quasi Banach operator ideal $S_{\text{ces}_q}^{\text{Kol}}$ is small.*

Theorem 3.34 *If M is an Orlicz function satisfying Δ_2 -condition, then the pre-quasi Banach operator ideal $S_{\ell_M}^{\text{app}}$ is small.*

Proof Since M is an Orlicz function satisfying Δ_2 -condition, then $(S_{\ell_M}^{\text{app}}, g)$, where $g(P) = \sum_{i=0}^{\infty} M(\alpha_i(P))$ is a pre-quasi Banach operator ideal. Let U and V be any two Banach spaces. Suppose that $S_{\ell_M}^{\text{app}}(U, V) = \mathfrak{L}(U, V)$, then there exists a constant $C > 0$ such that $g(P) \leq C\|P\|$ for all $P \in \mathfrak{L}(U, V)$. Assume that U and V are infinite dimensional Banach spaces. By using inequality (5) and since M is an Orlicz function satisfying Δ_2 -condition, one obtains

$$\begin{aligned} M(1) &\leq L\|A_m\|M(\alpha_i(J_m A_m^{-1} I_m H_m Q_m))\|H_m^{-1}\| \\ &\Rightarrow \sum_{i=0}^m 1 \leq L\|A_m\|\|H_m^{-1}\| \sum_{i=0}^m M(\alpha_i(J_m A_m^{-1} I_m H_m Q_m)) \\ &\Rightarrow (m+1) \leq L\|A_m\|\|H_m^{-1}\|g(J_m A_m^{-1} I_m H_m Q_m), \\ &\quad (m+1) \leq LC\|A_m\|\|H_m^{-1}\|\|J_m A_m^{-1} I_m H_m Q_m\|, \\ &\quad (m+1) \leq LC\|A_m\|\|H_m^{-1}\|\|J_m A_m^{-1}\|\|I_m\|\|H_m Q_m\| \\ &\quad \quad = LC\|A_m\|\|H_m^{-1}\|\|A_m^{-1}\|\|I_m\|\|H_m\|, \\ &\quad (m+1) \leq 4LC \end{aligned}$$

for some $L \geq 1$. Thus we arrive at a contradiction since m is arbitrary. Thus U and V both cannot be infinite dimensional when $S_{\ell_M}^{\text{app}}(U, V) = \mathfrak{L}(U, V)$. Hence the result. \square

Corollary 3.35 ([22]) *If $0 < p < \infty$, then the quasi Banach operator ideal $S_{\ell_p}^{\text{app}}$ is small.*

Corollary 3.36 *If $0 < p < \infty$, then the quasi Banach operator ideal $S_{\ell_p}^{\text{Kol}}$ is small.*

Acknowledgements

The authors thank the anonymous referees for their constructive suggestions and helpful comments which led to significant improvement of the original manuscript of this paper.

Funding

Not applicable.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Received: 26 March 2018 Accepted: 10 December 2018 Published online: 29 December 2018

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