

RESEARCH

Open Access



Bohr-type inequalities of analytic functions

Ming-Sheng Liu^{1*} , Yin-Miao Shang¹ and Jun-Feng Xu²

*Correspondence:

liumsh65@163.com

¹School of Mathematical Sciences, South China Normal University, Guangzhou, China

Full list of author information is available at the end of the article

Abstract

In this paper, we investigate the Bohr-type radii for several different forms of Bohr-type inequalities of analytic functions in the unit disk, we also investigate the Bohr-type radius of the alternating series associated with the Taylor series of analytic functions. We will prove that most of the results are sharp.

MSC: Primary 30C45; secondary 30A10; 30H05

Keywords: Bounded analytic function; Bohr radius; Bohr–Rogosinski sum; Bohr-type radius

1 Introduction and preliminaries

Bohr's inequality states that if

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad (1.1)$$

is analytic in the unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and $|f(z)| < 1$ for all $z \in D$, then

$$\sum_{k=0}^{\infty} |a_k| |z|^k \leq 1 \quad (1.2)$$

for all $|z| \leq \frac{1}{3}$. This inequality was discovered by Bohr in 1914 [6]. Bohr actually obtained the inequality for $|z| \leq \frac{1}{6}$, but subsequently later, Wiener, Riesz and Schur, independently established the inequality for $|z| \leq \frac{1}{3}$ and the constant $1/3$ cannot be improved [12, 16, 17]. Other proofs were also given in [13, 14]. The problem was considered by Bohr when he was working on the absolute convergence problem for Dirichlet series of the form $\sum a_n n^{-s}$, but now it has become a very interesting problem. Bohr's idea naturally extends to functions of several complex variables [1, 2, 5, 11] and a variety of results on Bohr's theorem in higher dimensions appeared recently.

The majorant series $M_f(z) = \sum_{k=0}^{\infty} |a_k| |z|^k$ belongs to a very important class of series of non-negative terms. In analogy to the Bohr radius, there is also the notion of the Rogosinski radius [10, 15], which is described as follows: If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is an analytic function in D such that $|f(z)| < 1$ in D , then, for every $N \geq 1$, we have $|s_N(z)| < 1$ in the disk $|z| < \frac{1}{2}$ and this radius is sharp, where $s_N(z) = \sum_{k=0}^{N-1} a_k z^k$ denotes the partial sums of f . There is

a relevant quantity, which we call the Bohr–Rogosinski sum $R_N^f(z)$ of f defined by

$$R_N^f(z) := |f(z)| + \sum_{k=N}^{\infty} |a_k| r^k, \quad |z| = r. \tag{1.3}$$

We remark that, for $N = 1$, this quantity is related to the classical Bohr sum in which $f(0)$ is replaced by $f(z)$. More recently, Kayumov and Ponnusamy [9] obtained the following result on the Bohr–Rogosinski radius for analytic functions.

Theorem A ([9]) *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in the unit disk D and $|f(z)| < 1$ in D . Then*

$$|f(z)| + \sum_{k=N}^{\infty} |a_k| r^k \leq 1 \quad \text{for } r \leq R_N,$$

where R_N is the positive root of the equation $2(1+r)r^N - (1-r)^2 = 0$. The radius R_N is the best possible. Moreover,

$$|f(z)|^2 + \sum_{k=N}^{\infty} |a_k| r^k \leq 1 \quad \text{for } r \leq R'_N,$$

where R'_N is the positive root of the equation $(1+r)r^N - (1-r)^2 = 0$. The radius R'_N is the best possible.

In 2017, Ali, Barnard and Solynin defined the associated alternating series of series (1.1) as $A_f(z) = \sum_{k=0}^{\infty} (-1)^k |a_k| |z|^k$, they obtained the following result in [4].

Theorem B ([4]) *If $|\sum_{k=0}^{\infty} a_k z^k| \leq 1$ in D , then*

$$\left| \sum_{k=0}^{\infty} (-1)^k |a_k| |z|^k \right| \leq 1$$

in the disk $D_{1/\sqrt{3}} = \{z \in \mathbb{C} \mid |z| < 1/\sqrt{3}\}$. The radius $r = 1/\sqrt{3}$ is the best possible.

Theorem C ([3]) *If $f(z) = \sum_{k=0}^{\infty} a_{nk} z^{nk}$ is analytic in D satisfying $\operatorname{Re} f(z) \leq 1$ in D and $f(0) = a_0$ is positive, then $M_f(r) \leq 1$ for $0 \leq r \leq 1/\sqrt[n]{3}$.*

Remark 1.1 By a simple calculation in Theorem A, we observe that $R_1 = \sqrt{5} - 2$ is unequal to $\frac{1}{3}$ when $|f(0)|$ is replaced by $|f(z)|$ in Bohr’s inequality. Therefore, it is interesting to note what will happen to the Bohr radius if we use higher order derivatives of $f(z)$ to replace some Taylor coefficients of analytic functions in Bohr’s inequality.

In this paper, we mainly study the Bohr-type radii for several forms of Bohr-type inequalities of analytic functions when the Taylor coefficients of classical Bohr inequality are partly replaced and when the Taylor coefficients of the classical Bohr inequality are completely replaced by the higher order derivatives of $f(z)$, respectively. We obtain the Bohr-type radii under certain conditions. Moreover, we also discuss the Bohr-type radius of the alternating series associated with the Taylor series of analytic functions.

In order to establish our main results, we need the following lemmas, which will play the key role in proving the main results of this paper.

Lemma 1.2 ([8]) *If $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic and $|\varphi(z)| \leq 1$ in the unit disk D . Then $|a_n| \leq 1 - |a_0|^2$ for all $n = 1, 2, \dots$.*

Lemma 1.3 (Schwarz–Pick lemma) *If $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic and $|\varphi(z)| < 1$ in the unit disk D . Then:*

- (1) $|\varphi(z_1) - \varphi(z_2)| / |1 - \overline{\varphi(z_1)}\varphi(z_2)| \leq |z_1 - z_2| / |1 - \overline{z_1}z_2|$ holds for $z_1, z_2 \in D$, and the equality holds for distinct $z_1, z_2 \in D$ if and only if φ is a Möbius transformation;
- (2) $|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2}$ holds for $z \in D$, and the equality holds for some $z \in U$ if and only if φ is a Möbius transformation.

Lemma 1.4 ([7]) *If $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic and $|\varphi(z)| < 1$ in D . Then, for all $k = 1, 2, \dots$, we have*

$$|\varphi^{(k)}(z)| \leq \frac{k!(1 - |\varphi(z)|^2)}{(1 - |z|^2)^k} (1 + |z|)^{k-1}, \quad |z| < 1.$$

Lemma 1.5 ([3]) *If $p(z) = \sum_{k=0}^{\infty} p_k z^k$ is analytic in D such that $\operatorname{Re} p(z) > 0$ in D , then $|p_k| \leq 2 \operatorname{Re} p_0$ for all $k \geq 1$.*

2 Main results

We first provide a result involves computing Bohr-type radius for the analytic functions $f(z)$ for which $|a_0|$ and $|a_1|$ are replaced by $|f(z)|$ and $|f'(z)|$, respectively.

Theorem 2.1 *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in D and $|f(z)| < 1$ in D . Then*

$$|f(z)| + |f'(z)||z| + \sum_{k=2}^{\infty} |a_k||z|^k \leq 1 \quad \text{for } |z| = r \leq \frac{\sqrt{17} - 3}{4}.$$

The radius $r = \frac{\sqrt{17} - 3}{4}$ is the best possible.

Proof By assumption, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in D and $|f(z)| < 1$ in D . Since $f(0) = a_0$, by the Schwarz–Pick lemma, we obtain, for $z \in D$,

$$\frac{|f(z) - a_0|}{|1 - \overline{a_0}f(z)|} \leq |z|, \quad |f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

Thus it follows from the above inequality and Lemma 1.2 that, for $z = re^{i\theta} \in D$,

$$|f(z)| \leq \frac{r + |a_0|}{1 + r|a_0|}, \quad |a_k| \leq 1 - |a_0|^2$$

for $k = 1, 2, \dots$

Using these inequalities, we have

$$|f(z)| + |f'(z)|r + \sum_{k=2}^{\infty} |a_k|r^k$$

$$\begin{aligned}
 &\leq \frac{r}{1-r^2}(1-|f(z)|^2) + |f(z)| + (1-|a_0|^2)\frac{r^2}{1-r} \\
 &\leq \frac{r}{1-r^2}\left[1 - \left(\frac{r+|a_0|}{1+|a_0|r}\right)^2\right] + \frac{r+|a_0|}{1+|a_0|r} + (1-|a_0|^2)\frac{r^2}{1-r} \\
 &= \frac{|a_0| + 2r + |a_0|r^2}{(1+|a_0|r)^2} + (1-|a_0|^2)\frac{r^2}{1-r}, \tag{2.1}
 \end{aligned}$$

where the second inequality holds for any $r \in [0, \sqrt{2} - 1]$, since $\frac{1-r^2}{2r} \geq 1$ if $r \in [0, \sqrt{2} - 1]$.

Notice $|a_0| < 1$, we know (2.1) is smaller than or equal to 1 provided $\varphi(r) \leq 0$, where

$$\begin{aligned}
 \varphi(r) &= (|a_0| + 2r + |a_0|r^2)(1-r) + (1+|a_0|r)^2(1-|a_0|^2)r^2 - (1+|a_0|r)^2(1-r) \\
 &= (1-|a_0|)[-1 + 3r + (2|a_0| - 1)r^2 + |a_0|(2|a_0| + 1)r^3 + |a_0|^2(1+|a_0|)r^4] \\
 &\leq (1-|a_0|)(-1 + 3r + r^2 + 3r^3 + 2r^4) \\
 &= (1-|a_0|)2(1+r^2)\left(r + \frac{\sqrt{17}+3}{4}\right)\left(r - \frac{\sqrt{17}-3}{4}\right).
 \end{aligned}$$

Now, $\varphi(r) \leq 0$ if $\eta(r) := (1+r^2)(r + \frac{\sqrt{17}+3}{4})(r - \frac{\sqrt{17}-3}{4}) \leq 0$, which holds for $r \leq \frac{\sqrt{17}-3}{4}$. The first part of the theorem follows.

To show the sharpness of the number $r = \frac{\sqrt{17}-3}{4}$, we let $a \in [0, 1)$ and consider the function

$$f(z) = \frac{a-z}{1-az} = a - (1-a^2) \sum_{k=1}^{\infty} a^{k-1}z^k, \quad z \in D.$$

For this function, we find that

$$|f(-r)| + |f'(-r)|r + \sum_{k=2}^{\infty} |a_k|r^k = \frac{a+r}{1+ar} + \frac{1-a^2}{(1+ar)^2}r + (1-a^2)\frac{ar^2}{1-ar}. \tag{2.2}$$

The last expression is larger than 1 if and only if

$$(1-a)(-1 + (2+a)r + a^2r^2 + a^2(2a+1)r^3 + a^3(1+a)r^4) > 0. \tag{2.3}$$

Let $P_3(a, r) = -1 + (2+a)r + a^2r^2 + a^2(2a+1)r^3 + a^3(1+a)r^4$. After elementary calculation, we find that $\frac{\partial P_3}{\partial a} = r + 2ar^2 + 6a^2r^3 + 2ar^3 + 3a^2r^4 + 4a^3r^4$ is equal to or greater than 0 for any $r \in [0, 1)$. The latter equation implies that

$$P_3(a, r) \leq P_3(1, r) = -1 + 3r + r^2 + 3r^3 + 2r^4 = 2(1+r^2)\left(r + \frac{\sqrt{17}+3}{4}\right)\left(r - \frac{\sqrt{17}-3}{4}\right).$$

Therefore, Eq. (2.2) is smaller than or equal to 1 for all $a \in [0, 1)$, only in the case when $r \leq \frac{\sqrt{17}-3}{4}$. Finally, it also suggests that $a \rightarrow 1$ in (2.3) shows that Eq. (2.2) is larger than 1 if $r > \frac{\sqrt{17}-3}{4}$. This proves the sharpness. \square

Next, we discuss the Bohr-type radius when the coefficients of the series of missing series are completely replaced by the higher order derivatives.

Theorem 2.2 *Suppose that $N(\geq 2)$ is an integer, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in D and $|f(z)| < 1$ in D . Then*

$$|f(z)| + \sum_{k=N}^{\infty} \left| \frac{f^{(k)}(z)}{k!} \right| |z|^k \leq 1 \quad \text{for } |z| = r \leq R_N,$$

where R_N is the minimum positive root of the equation $\psi_N(r) = (1+r)(1-2r)(1-r)^{N-1} - 2r^N = 0$. The radius R_N is the best possible.

Proof By simple calculations we can know that

$$r \leq R_N < 1/2 \quad \text{if and only if} \quad \frac{2r^N}{(1+r)(1-2r)(1-r)^{N-1}} \leq 1.$$

By assumption, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in D and $|f(z)| < 1$ in D . Since $f(0) = a_0$, it follows from the Schwarz–Pick lemma and Lemma 1.4 that, for $z = re^{i\theta} \in D$,

$$|f(z)| \leq \frac{r + |a_0|}{1 + |a_0|r} \quad \text{and} \quad |f^{(k)}(z)| \leq \frac{k!(1 - |f(z)|^2)}{(1 - |z|^2)^k} (1 + |z|)^{k-1} \quad \text{for } k = 1, 2, \dots$$

Using these inequalities, we have

$$\begin{aligned} &|f(z)| + \sum_{k=N}^{\infty} \left| \frac{f^{(k)}(z)}{k!} \right| r^k \\ &\leq |f(z)| + \sum_{k=N}^{\infty} \frac{(1 - |f(z)|^2)}{(1 - |z|^2)^k} (1 + |z|)^{k-1} r^k \\ &= |f(z)| + (1 - |f(z)|^2) \sum_{k=N}^{\infty} \frac{(1+r)^{k-1} r^k}{(1-r^2)^k} \\ &\leq |f(z)| + (1 - |f(z)|^2) \frac{r^N}{(1+r)(1-2r)(1-r)^{N-1}} \\ &= |f(z)| - \frac{r^N}{(1+r)(1-2r)(1-r)^{N-1}} |f(z)|^2 + \frac{r^N}{(1+r)(1-2r)(1-r)^{N-1}} \\ &\leq \frac{(|a_0| + r)(1 + |a_0|r)(1-r)^N(1-2r) + (1 - |a_0|^2)(1-r)^2 r^N}{(1-r)^N(1-2r)(1 + |a_0|r)^2} := \omega_N(r) \end{aligned} \tag{2.4}$$

for $0 \leq r \leq R_N < 1/2$.

Now, $\omega_N(r) \leq 1$ if $v_N(r) \leq 0$, where

$$\begin{aligned} v_N(r) &= (|a_0| + r)(1 + |a_0|r)(1-r)^N(1-2r) + (1 - |a_0|^2)(1-r)^2 r^N \\ &\quad - (1-r)^N(1-2r)(1 + |a_0|r)^2 \\ &= (1 - |a_0|) [(-1 + (3 - |a_0|)r + (3|a_0| - 2)r^2 - 2|a_0|r^3)(1-r)^N \\ &\quad + (1 + |a_0|)r^N(1-r)^2] \\ &= (1 - |a_0|) [(-1 + 3r - 2r^2)(1-r)^N + r^N(1-r)^2] \\ &\quad + (1 - |a_0|)|a_0|r(1-r)^2 [r^{N-1} - (1-2r)(1-r)^{N-1}]. \end{aligned}$$

Now we split all this into two cases to prove that $v_N(r) \leq 0$ for $r \leq R_N$.

Case 1. $r \leq R_{N,1}$, where $R_{N,1}$ is the minimum positive root of the equation $\varphi_N(r) = (1 - 2r)(1 - r)^{N-1} - r^{N-1} = 0$. Since $r^{N-1} - (1 - 2r)(1 - r)^{N-1} \leq 0$ and $|a_0| < 1$, we have

$$\begin{aligned} v_N(r) &\leq (1 - |a_0|)(1 - r)^2[r \cdot r^{N-1} - (1 - 2r)(1 - r)^{N-1}] \\ &\leq (1 - |a_0|)(1 - r)^2[r^{N-1} - (1 - 2r)(1 - r)^{N-1}] \leq 0. \end{aligned}$$

Case 2. $R_{N,1} < r \leq R_N$. Notice that $R_{N,1} < R_N$ and $r^{N-1} - (1 - 2r)(1 - r)^{N-1} > 0$ for $r > R_{N,1}$, we have

$$\begin{aligned} v_N(r) &\leq (1 - |a_0|)[(-1 + 3r - 2r^2)(1 - r)^N + r^N(1 - r)^2] \\ &\quad + (1 - |a_0|)r(1 - r)^2[r^{N-1} - (1 - 2r)(1 - r)^{N-1}] \\ &\leq (1 - |a_0|)(1 - r)^2[2r^N - (1 + r)(1 - 2r)(1 - r)^{N-1}] \leq 0. \end{aligned}$$

The first part of the theorem follows.

To show the sharpness of the number R_N , we let $a \in [0, 1)$ and consider the function

$$f(z) = \frac{a - z}{1 - az} = a - (1 - a^2) \sum_{k=1}^{\infty} a^{k-1} z^k, \quad z \in D.$$

For this function, we find that

$$\begin{aligned} |f(r)| &+ \sum_{k=N}^{\infty} \left| \frac{f^{(k)}(r)}{k!} \right| r^k \\ &= |f(r)| + \sum_{k=N}^{\infty} \frac{a^{k-1}(1 - a^2)}{(1 - ar)^{k+1}} r^k \\ &= \frac{a - r}{1 - ar} + (1 - a^2) \frac{a^{N-1} r^N}{(1 - ar)^N (1 - 2ar)} \quad \left(\text{when } r < \frac{1}{2a} \right). \end{aligned} \tag{2.5}$$

The last expression is larger than 1 if and only if

$$(1 - a)[(-1 + (2a - 1)r + 2ar^2)(1 - ar)^{N-1} + (1 + a)a^{N-1}r^N] > 0. \tag{2.6}$$

Let $P_4(a, r) = (-1 + (2a - 1)r + 2ar^2)(1 - ar)^{N-1} + (1 + a)a^{N-1}r^N$. After elementary calculation, we find that $\frac{\partial P_4}{\partial a} = (2r + 2r^2)(1 - ar)^{N-1} + r(N - 1)(1 + r)(1 - 2ar)(1 - ar)^{N-2} + a^{N-1}r^N + (1 + a)(N - 1)a^{N-2}r^N$ is equal to or greater than 0 for any $r < \frac{1}{2}$. The latter equation implies that

$$P_4(a, r) \leq P_4(1, r) = (-1 + r + 2r^2)(1 - r)^{N-1} + 2r^N = 2r^N - (r + 1)(1 - 2r)(1 - r)^{N-1}$$

holds for $r < \frac{1}{2}$. Therefore, Eq. (2.5) is smaller than or equal to 1 for all $a \in [0, 1)$, only in the case when $r \leq R_N$.

Finally, allowing $a \rightarrow 1$ in (2.6) shows that Eq. (2.5) is larger than 1 if $r > R_N$. This proves the sharpness. \square

Corollary 2.3 *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in D and $|f(z)| < 1$ in D . Then*

$$|f(z)|^2 + \sum_{k=N}^{\infty} \left| \frac{f^{(k)}(z)}{k!} \right| |z|^k \leq 1 \quad \text{for } |z| = r \leq R'_N,$$

where R'_N is the positive root of the equation $(1+r)(1-2r)(1-r)^{N-1} - r^N = 0$. The radius R'_N is the best possible.

Proof By simple calculations we can know that

$$r \leq R'_N \quad \text{if and only if} \quad \frac{(1+r)(1-r)^N(1-2r) - r^N(1-r)}{(1+r)(1-2r)(1-r)^N} \geq 0.$$

In analogy to the calculation of Theorem 2.2, we have

$$\begin{aligned} &|f(z)|^2 + \sum_{k=N}^{\infty} \left| \frac{f^{(k)}(z)}{k!} \right| r^k \\ &\leq |f(z)|^2 + \sum_{k=N}^{\infty} \frac{(1-|f(z)|^2)}{(1-|z|^2)^k} (1+|z|)^{k-1} r^k \\ &\leq \left(1 - \frac{r^N(1-r)}{(1+r)(1-2r)(1-r)^N} \right) |f(z)|^2 + \frac{r^N(1-r)}{(1+r)(1-2r)(1-r)^N} \\ &\leq \frac{(|a_0|+r)^2(1-r)^N(1-2r) + (1-|a_0|^2)r^N(1-r)^2}{(1-r)^N(1+|a_0|r)^2(1-2r)}. \end{aligned} \tag{2.7}$$

So (2.7) is smaller than or equal to 1 provided $\omega_N(r) \leq 1$, where

$$\omega_N(r) := \frac{(|a_0|+r)^2(1-r)^N(1-2r) + (1-|a_0|^2)r^N(1-r)^2}{(1-r)^N(1+|a_0|r)^2(1-2r)}.$$

Now, $\omega_N(r) \leq 1$ if $v_N(r) \leq 0$, where

$$\begin{aligned} v_N(r) &= (|a_0|+r)^2(1-r)^N(1-2r) + (1-|a_0|^2)r^N(1-r)^2 - (1-r)^N(1+|a_0|r)^2(1-2r) \\ &= (1-|a_0|^2)[(1-r)^N(-1+r^2+2r-2r^3) + r^N(1-r)^2] \\ &= (1-|a_0|^2)[(1-r)^N(1-r)(1+r)(2r-1) + r^N(1-r)^2]. \end{aligned}$$

Now, $v_N(r) \leq 0$ if $(1+r)(1-2r)(1-r)^{N-1} - r^N \geq 0$, which holds for $r \leq R'_N$, where R'_N is as in the statement of the theorem.

To show the sharpness of the number R'_N , we let $a \in [0, 1)$ and consider the function

$$f(z) = \frac{a-z}{1-az} = a - (1-a^2) \sum_{k=1}^{\infty} a^{k-1} z^k, \quad z \in D.$$

For this function, we find that

$$|f(r)|^2 + \sum_{k=N}^{\infty} \left| \frac{f^{(k)}(r)}{k!} \right| r^k = |f(r)|^2 + \sum_{k=N}^{\infty} \frac{a^{k-1}(1-a^2)}{(1-ar)^{k+1}} r^k$$

$$= \frac{(a - r)^2(1 - ar)^{N-2}(1 - 2ar) + (1 - a^2)a^{N-1}r^N}{(1 - ar)^N(1 - 2ar)}, \tag{2.8}$$

(2.8) is larger than 1 if and only if

$$(1 - a^2)[(-1 + 2ar + r^2 - 2ar^3)(1 - ar)^{N-2} + a^{N-1}r^N] > 0. \tag{2.9}$$

In analogy to the processing methods of Theorem 2.2. After elementary calculation, we find that allowing $a \rightarrow 1$ in (2.9), it follows that Eq. (2.8) is larger than 1 if $r > R'_N$. This proves the sharpness and we complete the proof of Corollary 2.3. \square

Applying a method similar to Theorem 2.2, we may obtain the following corollary.

Corollary 2.4 *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in D and $|f(z)| < 1$ in D . Then*

$$\sum_{k=0}^{\infty} \left| \frac{f^{(k)}(z)}{k!} \right| |z|^k \leq 1 \quad \text{for } |z| = r \leq \frac{\sqrt{17} - 3}{4}.$$

The radius $r = \frac{\sqrt{17}-3}{4}$ is the best possible.

In analogy to Theorem C, we now consider the Bohr-type radius when conditions of $|f(z)| < 1$ are replaced by $\text{Re}f(z) \leq 1$ and $f(0) = a_0$ is positive.

Theorem 2.5 *If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in D satisfying $\text{Re}f(z) \leq 1$ in D and $f(0) = a_0$ is positive, then*

$$|f(z)| + \sum_{k=1}^{\infty} |a_{nk}| |z|^{nk} \leq 1 \quad \text{for } |z| = r \leq R_n, \tag{2.10}$$

where R_n is the positive root of the equation $\varphi_n(r) = 0$, $\varphi_n(r) = r^{n+1} + 3r^n + r - 1$. The radius R_n is the best possible.

Proof By assumption, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic and $\text{Re}f(z) \leq 1$ in D .

Since $f(0) = a_0$ is positive. Applying the result of Lemma 1.5 to $p(z) = 1 - f(z)$ and the Schwarz–Pick lemma that, for $z = re^{i\theta} \in D$, we have

$$|a_{nk}| \leq 2(1 - a_0) \quad \text{for } k = 1, 2, \dots$$

and

$$|f(z)| \leq \frac{r + a_0}{1 + ra_0}.$$

Using the last two inequalities, we have

$$|f(z)| + \sum_{k=1}^{\infty} |a_{nk}| r^{nk} \leq \frac{r + a_0}{1 + a_0 r} + 2(1 - a_0) \frac{r^n}{1 - r^n}, \tag{2.11}$$

for which (2.11) is smaller than or equal to 1 provided $\phi(r) \leq 0$, where

$$\begin{aligned} \phi(r) &= (r + a_0)(1 - r^n) + 2(1 - a_0)r^n(1 + a_0r) - (1 + a_0r)(1 - r^n) \\ &= (1 - a_0)[(2a_0 - 1)r^{n+1} + 3r^n - 1 + r] \\ &\leq (1 - a_0)(r^{n+1} + 3r^n + r - 1) \quad \text{since } |a_0| < 1. \end{aligned}$$

Now, $\phi(r) \leq 0$ if $\psi(r) := r^{n+1} + 3r^n + r - 1 \leq 0$, which holds for $r \leq R_n$. This completes the proof of inequality (2.10).

To show that the radius $r = R_n$ is the best possible, we let $a \in [0, 1)$ and consider the function

$$f(z) = \frac{a - z}{1 - az} = a - (1 - a^2) \sum_{k=1}^{\infty} a^{k-1} z^k, \quad z \in D.$$

For this function, we find that

$$|f(-r)| + \sum_{k=1}^{\infty} |a_{nk}| r^{nk} = \frac{a + r}{1 + ar} + (1 - a^2) \frac{a^{n-1} r^n}{1 - a^n r^n}, \quad \text{where } r = |z|. \tag{2.12}$$

We claim that, for every r such that $R_n < r < 1$, there is a such that $0 < a < 1$, and

$$\frac{(a + r)(1 - a^n r^n) + (1 - a^2)(1 + ar)a^{n-1} r^n}{(1 + ar)(1 - a^n r^n)} > 1. \tag{2.13}$$

Indeed, inequality (2.13) is equivalent to the inequality

$$(1 - a)[(a^{n-1} + 2a^n)r^n + a^{n+1}r^{n+1} + r - 1] > 0. \tag{2.14}$$

Let $P_1(a, r) = (a^{n-1} + 2a^n)r^n + a^{n+1}r^{n+1} + r - 1$ denote a part of the left-hand side of (2.14). After elementary calculation, we find that $\frac{\partial P_1}{\partial a} \geq 0$ apparently. The latter inequality implies that $P_1(a, r) \leq P_1(1, r) = r^{n+1} + 3r^n + r - 1$ holds for all $r \in [0, 1)$. Therefore, Eq. (2.12) is smaller than or equal to 1 for all $a \in [0, 1)$, only in the case when $r \leq R_n$.

Finally, allowing $a \rightarrow 1$ in (2.14) shows that Eq. (2.13) is larger than 1 if $r > R_n$. This proves the sharpness. \square

Setting $n = 1$ in Theorem 2.5, we have the following corollary.

Corollary 2.6 *If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in D satisfying $\operatorname{Re} f(z) \leq 1$ in D and $f(0) = a_0$ is positive, then*

$$|f(z)| + \sum_{k=1}^{\infty} |a_k| |z|^k \leq 1 \quad \text{for } |z| = r \leq \sqrt{5} - 2,$$

where the radius $\sqrt{5} - 2$ is the best possible.

Remark 2.7 By simple calculation, we can know the Bohr-type radius in Theorem 2.5 with the condition of $\operatorname{Re} f(z) \leq 1$ and $f(0) = a_0 > 0$ is the same as the condition of $|f(z)| < 1$.

Finally, we consider a new Bohr-type radius of the alternating series associated the Taylor series of analytic functions where $|a_0|$ is replaced by $|f(z)|$. We have

$$R_f(z) = |f(z)| + \sum_{k=1}^{\infty} (-1)^k |a_k| |z|^k.$$

Lemma 2.8 *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in the unit disk D and $|f(z)| < 1$ in D . Then*

$$|f(z)| + \sum_{k=1}^{\infty} |a_{2k}| |z|^{2k} \leq 1 \quad \text{for } |z| = r \leq \sqrt{2} - 1. \tag{2.15}$$

The radius $r = \sqrt{2} - 1$ is the best possible.

Proof By assumption, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic and $|f(z)| < 1$ in D . Since $f(0) = a_0$, it follows from Lemma 1.2 and the Schwarz–Pick lemma that, for $z = re^{i\theta} \in D$,

$$|a_k| \leq 1 - |a_0|^2 \quad \text{for } k = 1, 2, \dots$$

and

$$|f(z)| \leq \frac{r + |a_0|}{1 + r|a_0|}.$$

Using the last two inequalities, we have

$$|f(z)| + \sum_{k=1}^{\infty} |a_{2k}| r^{2k} \leq \frac{r + |a_0|}{1 + |a_0|r} + (1 - |a_0|^2) \frac{r^2}{1 - r^2}, \tag{2.16}$$

and (2.16) is smaller than or equal to 1 provided $\phi(r) \leq 0$, where

$$\begin{aligned} \phi(r) &= (r + |a_0|)(1 - r^2) + (1 - |a_0|^2)r^2(1 + |a_0|r) - (1 + |a_0|r)(1 - r^2) \\ &= (1 - |a_0|)[-1 + r + (2 + |a_0|)r^2 + (|a_0|^2 + |a_0| - 1)r^3] \\ &\leq (1 - |a_0|)(3r^2 + r - 1 + r^3) \\ &= (1 - |a_0|)(r + 1 + \sqrt{2})(r + 1 - \sqrt{2})(r + 1), \quad \text{since } |a_0| < 1. \end{aligned}$$

Now, $\phi(r) \leq 0$ if $\psi(r) := (r + 1 + \sqrt{2})(r + 1 - \sqrt{2})(r + 1) \leq 0$, which holds for $r \leq \sqrt{2} - 1$. This completes the proof of inequality (2.15).

To shows that the radius $r = \sqrt{2} - 1$ is the best possible, we let $a \in [0, 1)$ and consider the function

$$f(z) = \frac{a - z}{1 - az} = a - (1 - a^2) \sum_{k=1}^{\infty} a^{k-1} z^k, \quad z \in D.$$

For this function, we find that

$$|f(-r)| + \sum_{k=1}^{\infty} |a_{2k}| r^{2k} = \frac{a + r}{1 + ar} + (1 - a^2) \frac{ar^2}{1 - a^2 r^2}, \quad \text{where } r = |z|. \tag{2.17}$$

We claim that, for every r such that $\sqrt{2} - 1 < r < 1$, there is a such that $0 < a < 1$, and

$$\frac{(a+r)(1-ar) + (1-a^2)ar^2}{(1+ar)(1-ar)} > 1. \tag{2.18}$$

Indeed, inequality (2.18) is equivalent to the inequality

$$(1-a)[-1 + (1+a)r + a^2r^2] > 0. \tag{2.19}$$

Let $P_1(a, r) = -1 + (1+a)r + a^2r^2$ denote a part of the left-hand side of (2.19). After elementary calculation, we find that $\frac{\partial P_1}{\partial a} = r + 2ar^2 \geq 0$. The latter inequality implies that $P_1(a, r) \leq P_1(1, r) = -1 + 2r + r^2$ holds for all $r \in [0, 1)$. Therefore, Eq. (2.17) is smaller than or equal to 1 for all $a \in [0, 1)$, only in the case when $r \leq \sqrt{2} - 1$.

Finally, allowing $a \rightarrow 1$ in (2.19) shows that Eq. (2.18) is larger than 1 if $r > \sqrt{2} - 1$. This proves the sharpness. \square

Theorem 2.9 *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in D and $|f(z)| < 1$ in D . Then*

$$\left| |f(z)| + \sum_{k=1}^{\infty} (-1)^k |a_k| |z|^k \right| \leq 1 \quad \text{for } |z| = r \leq \sqrt{2} - 1. \tag{2.20}$$

Proof By the proof of Lemma 2.8, we have

$$\begin{aligned} |f(z)| + \sum_{k=1}^{\infty} (-1)^k |a_k| r^k &\leq \frac{r + |a_0|}{1 + |a_0|r} + \sum_{k=1}^{\infty} |a_{2k}| r^{2k} - \sum_{k=1}^{\infty} |a_{2k-1}| r^{2k-1} \\ &\leq \frac{r + |a_0|}{1 + |a_0|r} + \sum_{k=1}^{\infty} |a_{2k}| r^{2k} \\ &\leq \frac{r + |a_0|}{1 + |a_0|r} + (1 - |a_0|^2) \frac{r^2}{1 - r^2}. \end{aligned} \tag{2.21}$$

We know that Eq. (2.21) is smaller than or equal to 1, which holds for $r \leq \sqrt{2} - 1$ and for all $a \in [0, 1)$.

To find a lower bound for $R_f(z)$, we consider the following chain of relations:

$$\begin{aligned} R_f(z) &= |f(z)| + \sum_{k=1}^{\infty} |a_{2k}| r^{2k} - \sum_{k=1}^{\infty} |a_{2k-1}| r^{2k-1} \\ &\geq - \sum_{k=1}^{\infty} |a_{2k-1}| r^{2k-1} = - \left(|a_1|r + \sum_{k=1}^{\infty} |a_{2k+1}| r^{2k+1} \right) \\ &\geq - \left((1 - |a_0|^2)r + \sum_{k=1}^{\infty} |a_{2k+1}| r^{2k} \right) \\ &\geq - \left(\frac{r + |a_0|}{1 + r|a_0|} + \sum_{k=1}^{\infty} |a_{2k+1}| r^{2k} \right), \end{aligned}$$

where the last inequality is obtained by a simple calculation.

Combining this with (2.21), we conclude that $R_f(z) \geq -1$ for all $r \leq \sqrt{2} - 1$. This completes the proof of inequality (2.20). \square

Notice that we have not proved that the number $r = \sqrt{2} - 1$ is the best possible in Theorem 2.9, therefore the following problem remains open.

Problem 2.10 Find the largest radius r_0 for the class of analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in D with $|f(z)| < 1$ in D such that

$$\left| |f(z)| + \sum_{k=1}^{\infty} (-1)^k |a_k| |z|^k \right| \leq 1 \quad \text{for } |z| = r \leq r_0.$$

3 Conclusion

From the results that we have given in this paper, we can get the exact Bohr-type radius when we replace the coefficient of Bohr's inequality with $f(z)$ or its higher order derivatives, and we conclude that the Bohr-type radius obtained after the change of coefficients is smaller than the Bohr radius.

Acknowledgements

The authors would like to express their deep gratitude to the referee for his/her careful reading and invaluable comments.

Funding

This research is partly supported by Guangdong Natural Science Foundations (Grant No. 2018A030313508, 2016A030313002) and Funds of Education Department of Guangdong (2016KTSCX145).

Availability of data and materials

The data set supporting the conclusions of this article is included within the article.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All the authors conceived of the study, participated in its design and read and approved the final manuscript.

Author details

¹School of Mathematical Sciences, South China Normal University, Guangzhou, China. ²Department of Mathematics, Wuyi University, Jiangmen, China.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 13 July 2018 Accepted: 5 December 2018 Published online: 13 December 2018

References

1. Aizenberg, L.: Multidimensional analogues of Bohr's theorem on power series. *Proc. Am. Math. Soc.* **128**, 1147–1155 (2000)
2. Aizenberg, L.: Generalization of Caratheodory's inequality and the Bohr radius for multidimensional power series. *Oper. Theory, Adv. Appl.* **158**, 87–94 (2005)
3. Ali, R.M., Abu-Muhanna, Y., Ponnusamy, S.: On the Bohr inequality. In: Govil, N.K., et al. (eds.) *Progress in Approximation Theory and Applicable Complex Analysis*. Springer Optimization and Its Applications, vol. 117, pp. 265–295 (2016)
4. Ali, R.M., Barnard, R.W., Solynin, A.Y.: A note on Bohr's phenomenon for power series. *J. Math. Anal. Appl.* **44**, 154–167 (2017)
5. Boas, H.P., Khavinson, D.: Bohr's power series theorem in several variables. *Proc. Am. Math. Soc.* **125**, 2975–2979 (1997)
6. Bohr, H.: A theorem concerning power series. *Proc. Lond. Math. Soc.* **2**(13), 1–5 (1914)
7. Dai, S.Y., Pan, Y.F.: Note on Schwarz–Pick estimates for bounded and positive real part analytic functions. *Proc. Am. Math. Soc.* **136**, 635–640 (2008)
8. Graham, I., Kohr, G.: *Geometric Function Theory in One and Higher Dimensions*. Marcel Dekker Inc, New York (2003)
9. Kayumov, I.R., Ponnusamy, S.: Bohr–Rogosinski radius for analytic functions (2017). [arXiv:1708.05585](https://arxiv.org/abs/1708.05585)
10. Landau, E., Gaier, D.: *Darstellung und Begründung Einiger Neuerer Ergebnisse der Funktionen-Theorie*. Springer, Berlin (1986)

11. Li, PR: Some classes of singular integral equations of convolution type in the class of exponentially increasing functions. *Arch. Inequal. Appl.* **2017**, 307 (2017)
12. Paulsen, V.I., Popescu, G., Singh, D.: On Bohr's inequality. *Proc. Lond. Math. Soc.* **3**(85), 493–512 (2002)
13. Paulsen, V.I., Singh, D.: Bohr's inequality for uniform algebras. *Proc. Am. Math. Soc.* **132**, 3577–3579 (2004)
14. Paulsen, V.I., Singh, D.: Extensions of Bohr's inequality. *Bull. Lond. Math. Soc.* **38**(6), 991–999 (2006)
15. Rogosinski, W.: Über Bildschranken bei Potenzreihen und ihren Abschnitten. *Math. Z.* **17**, 260–276 (1923)
16. Sidon, S.: Über einen satz von Herrn Bohr. *Math. Z.* **26**, 731–732 (1927)
17. Tomic, M.: Sur un theoreme de H. Bohr. *Math. Scand.* **11**, 103–106 (1962)

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com
