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# A class of stochastic Gronwall's inequality and its application

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## Abstract

This paper puts forward the basic form of stochastic Gronwall's inequality and uses, respectively, the iterative method, the integral method and the martingale representation method to prove it. Then it presents an application to prove a comparison theorem of  $L^p$  solutions for one-dimensional backward stochastic differential equations under the stochastic Lipschitz condition.

**Keywords:** Gronwall's inequality; Stochastic; Backward stochastic differential equation; Comparison theorem

## 1 Introduction

Gronwall's inequality was first proposed and proved as its differential form by the Swedish mathematician called Thomas Hacon Gronwall [1] in 1911. The integral form was proved by the American mathematician Bellman [2] in 1943; see the following Proposition 1. Gronwall's inequality is an important tool to obtain various estimates in the theory of ordinary and stochastic differential equation. In particular, it provides a comparison theorem that can be used to prove uniqueness of a solution to the initial value problem.

**Proposition 1** Assume that  $a \geq 0$  and  $0 < T \leq +\infty$ . Let  $\beta(t)$  and  $\mu(t)$  be two non-negative continuous functions defined on  $[0, T]$ , which satisfy

$$\mu(t) \leq a + \int_0^t \beta(s)\mu(s) ds, \quad t \in [0, T].$$

Then

$$\mu(t) \leq ae^{\int_0^t \beta(s) ds}, \quad t \in [0, T].$$

With the development of differential equations, it was found that the original Gronwall inequality can no longer meet the needs of application, so people began to study the generalized Gronwall inequality. LaSalle [3] and Bihari [4] have put forward a nonlinear generalization of the Gronwall–Bellman inequality called the Bihari–LaSalle inequality. Pachpatte [5] has presented some other variants and generalizations of Gronwall's inequality. In recent years, Chen et al. [6] have established some new nonlinear Gronwall–Bellman–Ou–Iang type integral inequalities with two variables, which can be used as a simple tool to

study the qualitative and quantitative properties of solutions to differential equations. Lin [7] has provided several generalizations of Gronwall's inequality and presented their applications to proving the uniqueness of solutions for fractional differential equations with various derivatives. Zhang et al. [8] have studied a class of more general discrete fractional order Gronwall's inequality by using the definition of a new fractional order integral. Xu et al. [9] have developed some generalized discrete fractional Gronwall inequalities based on an iteration method, which can be used in the qualitative analysis of the solutions to fractional difference equation and summation equation. In particular, Fan et al. [10] have proposed the following backward Gronwall inequality, and used it to study the existence and uniqueness of solutions to backward stochastic differential equations.

**Proposition 2** *Assume that  $a \geq 0$ ,  $0 < T \leq +\infty$  and  $\beta(t) : [0, T] \mapsto [0, +\infty)$  satisfies  $\int_0^T \beta(s) ds < +\infty$ . Let  $\mu(t) : [0, T] \mapsto [0, +\infty)$  be a continuous function with  $\sup_{t \in [0, T]} \mu(t) < +\infty$  and satisfy*

$$\mu(t) \leq a + \int_t^T \beta(s)\mu(s) ds, \quad t \in [0, T].$$

Then

$$\mu(t) \leq ae^{\int_t^T \beta(s) ds}, \quad t \in [0, T].$$

This paper will put forward a class of stochastic Gronwall's inequalities, provide its three proof methods based on Propositions 1 and 2, and it will introduce an application in the field of backward stochastic differential equations.

## 2 Notation

In this paper, let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a complete probability space carrying a standard  $d$ -dimensional Brownian motion  $(B_t)_{t \geq 0}$  and  $(\mathcal{F}_t)_{t \geq 0}$  be the completed natural  $\sigma$ -algebra generated by  $(B_t)_{t \geq 0}$ . We always assume that  $0 < T \leq +\infty$ ,  $\mathcal{F}_T := \mathcal{F}$  and  $(\mathcal{F}_t)_{t \geq 0}$  is right-continuous and complete.  $\mathbf{E}[\xi]$  is used to express the mathematical expectation of the random variable  $\xi$ ,  $\mathbf{E}[\xi | \mathcal{F}_t]$  represents the conditional mathematical expectation of the random variable  $\xi$  with respect to  $\mathcal{F}_t$ , and  $\|\xi\|_\infty$  represents the infinity norm of the essentially bounded random variable  $\xi$ , i.e.,

$$\|\xi\|_\infty := \sup\{x : \mathbf{P}(|\xi| > x) > 0\}.$$

Denote  $a^+ := \max\{a, 0\}$ ,  $\mathbf{R} := (-\infty, +\infty)$ ,  $\mathbf{R}_+ := [0, +\infty)$ . Let  $\mathbf{1}_A$  represent the indicator function of the set  $A$ , which means for each subset  $A \subset \Omega \times [0, T]$ ,  $\mathbf{1}_A(\omega, t) = 1$  in the case of  $(\omega, t) \in A$ , otherwise  $\mathbf{1}_A(\omega, t) = 0$ .

For each  $p > 0$ , let  $L^p(\Omega, \mathcal{F}_T, \mathbf{P})$  be the set of all  $\mathbf{R}$ -valued,  $\mathcal{F}_T$ -measurable random variables  $\xi$  such that  $\mathbf{E}[|\xi|^p] < +\infty$ . And let  $S^p(0, T; \mathbf{R})$  (or  $S^p$  simply) denote the set of  $\mathbf{R}$ -valued,  $(\mathcal{F}_t)$ -adapted and continuous processes  $(Y_t)_{t \in [0, T]}$  such that

$$\|Y\|_{S^p} := \left( \mathbf{E} \left[ \sup_{t \in [0, T]} |Y_t|^p \right] \right)^{\frac{1}{p} \wedge 1} < +\infty.$$

Finally, let  $M^p(0, T; \mathbf{R}^d)$  (or  $M^p$  simply) represent the set of  $(\mathcal{F}_t)$ -progressively measurable  $\mathbf{R}^d$ -valued processes  $(Z_t)_{t \in [0, T]}$  such that

$$\|Z\|_{M^p} := \left( \mathbf{E} \left[ \left( \int_0^T |Z_t|^2 dt \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p} \wedge 1} < +\infty.$$

### 3 Main result

We first give the basic form of a stochastic Gronwall’s inequality, which is the main result of this section.

**Theorem 1** *Let  $a > 0$  and the  $(\mathcal{F}_t)_{t \geq 0}$ -progressively measurable random process  $\beta(\omega, t) : \Omega \times [0, T] \mapsto \mathbf{R}_+$  satisfy*

$$\left\| \int_0^T \beta(\omega, s) ds \right\|_{\infty} < +\infty,$$

*i.e., there exists a constant  $M > 0$  such that  $\int_0^T \beta(\omega, s) ds \leq M$ ,  $d\mathbf{P}$ -a.s. If the  $(\mathcal{F}_t)_{t \geq 0}$ -progressively measurable random process  $\mu(\omega, t) : \Omega \times [0, T] \mapsto \mathbf{R}_+$  satisfies*

$$\mathbf{E} \left[ \sup_{t \in [0, T]} \mu(\omega, t) \right] < +\infty$$

and

$$\mu(t) \leq a + \mathbf{E} \left[ \int_t^T \beta(s) \mu(s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T], \tag{1}$$

then, for each  $t \in [0, T]$ , we have

$$\mu(t) \leq a \mathbf{E} \left[ e^{\int_t^T \beta(s) ds} \middle| \mathcal{F}_t \right], \quad d\mathbf{P}\text{-a.s.} \tag{2}$$

In particular,  $\mu(t) = 0$  when  $a = 0$ .

*Remark 1* In order to keep the simpleness of the notations, here and here after we usually omit “ $d\mathbf{P}$ -a.s.” and make random processes  $\beta(\omega, t)$  and  $\mu(\omega, t)$  abbreviated as  $\beta(t)$  and  $\mu(t)$  or  $\beta_t$  and  $\mu_t$ . The notations of the other processes are similar.

*Remark 2* In the case of  $\beta$ . being a deterministic process, i.e.,  $\beta$ . is independent of  $\omega$ , Theorem 1 can be regarded as a direct corollary of Proposition 2. In fact, in this case, due to the compatibility of the conditional mathematical expectation (see Theorem 5.1(vii), page 81 in [11] for details) and Fubini’s theorem, for each fixed  $t \in [0, T]$ , it follows from (1) that

$$\mathbf{E}[\mu(r)|\mathcal{F}_t] \leq a + \int_r^T \beta(s) \mathbf{E}[\mu(s)|\mathcal{F}_t] ds, \quad r \in [t, T].$$

Furthermore, let  $v(r) := \mathbf{E}[\mu(r)|\mathcal{F}_t]$ . Then

$$v(r) \leq a + \int_r^T \beta(s)v(s) ds, \quad r \in [t, T].$$

In this way, using  $[t, T]$  instead of  $[0, T]$  in Proposition 2 yields

$$\mathbf{E}[\mu(r)|\mathcal{F}_t] = v(r) \leq ae^{\int_t^T \beta(s) ds}, \quad r \in [t, T].$$

Finally, letting  $r = t, t \in [0, T]$  in the above equation, we obtain

$$\mu(t) \leq ae^{\int_t^T \beta(s) ds},$$

which means that the conclusion holds true.

It is necessary to point out that this proof method is no longer applicable when the stochastic process  $\beta$  in Theorem 1 is not independent of  $\omega$ , so we need to explore the new proof method.

### 3.1 Iterative method

*Proof of Theorem 1* Let  $v(t) := a\mathbf{E}[e^{\int_t^T \beta_s ds} | \mathcal{F}_t], t \in [0, T]$ . Then  $v(t)$  satisfies

$$v(t) = a + \mathbf{E}\left[\int_t^T \beta(s)v(s) ds \middle| \mathcal{F}_t\right], \quad t \in [0, T]. \tag{3}$$

In fact, due to the compatibility of the conditional mathematical expectation, it follows that, for each  $t \in [0, T]$ ,

$$\begin{aligned} a + \mathbf{E}\left[\int_t^T \beta_s v_s ds \middle| \mathcal{F}_t\right] &= a + \mathbf{E}\left[\int_t^T \beta_s a \mathbf{E}\left[e^{\int_s^T \beta_r dr} \middle| \mathcal{F}_s\right] ds \middle| \mathcal{F}_t\right] \\ &= a + a \mathbf{E}\left[\mathbf{E}\left[\int_t^T \beta_s e^{\int_s^T \beta_r dr} ds \middle| \mathcal{F}_s\right] \middle| \mathcal{F}_t\right] \\ &= a + a \mathbf{E}\left[\int_t^T \beta_s e^{\int_s^T \beta_r dr} ds \middle| \mathcal{F}_t\right] \\ &= a + a \mathbf{E}\left[-e^{\int_s^T \beta_s ds} \middle|_t^T \middle| \mathcal{F}_t\right] \\ &= a \mathbf{E}\left[e^{\int_t^T \beta_t dt} \middle| \mathcal{F}_t\right] \\ &= v_t. \end{aligned}$$

Now, let  $\varphi_t = \mu_t - v_t$ . Then it follows from (1) and (3) that

$$\begin{aligned} \varphi_t &\leq \left\{ a + \mathbf{E}\left[\int_t^T \beta_s \mu_s ds \middle| \mathcal{F}_t\right] \right\} - \left\{ a + \mathbf{E}\left[\int_t^T \beta_s v_s ds \middle| \mathcal{F}_t\right] \right\} \\ &= \mathbf{E}\left[\int_t^T \beta_s \varphi_s ds \middle| \mathcal{F}_t\right], \quad t \in [0, T]. \end{aligned} \tag{4}$$

In the sequel, we will use the mathematical induction method to prove that, for each  $n \geq 1$ ,

$$\varphi_t \leq \mathbf{E}\left[\int_t^T \frac{(\int_t^u \beta_s ds)^n}{n!} \beta_u \varphi_u du \middle| \mathcal{F}_t\right], \quad t \in [0, T]. \tag{5}$$

First, in view of (4) and the progressively measurable property of the process  $\beta$ , exchanging the integral order of the double integral in (4), we obtain

$$\begin{aligned} \varphi_t &\leq \mathbf{E} \left[ \int_t^T \beta_s \varphi_s \, ds \middle| \mathcal{F}_t \right] \\ &\leq \mathbf{E} \left[ \int_t^T \beta_s \mathbf{E} \left[ \int_s^T \beta_u \varphi_u \, du \middle| \mathcal{F}_s \right] ds \middle| \mathcal{F}_t \right] \\ &= \mathbf{E} \left[ \int_t^T \beta_s \int_s^T \beta_u \varphi_u \, du \, ds \middle| \mathcal{F}_t \right] \\ &= \mathbf{E} \left[ \int_t^T \beta_u \varphi_u \int_t^u \beta_s \, ds \, du \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \end{aligned}$$

Hence, (5) is true when  $n = 1$ . Now, assume that, when  $n = k$ , we have

$$\varphi(t) \leq \mathbf{E} \left[ \int_t^T \frac{(\int_t^u \beta_s \, ds)^k}{k!} \beta_u \varphi_u \, du \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \tag{6}$$

We will prove that, when  $n = k + 1$ ,

$$\varphi_t \leq \mathbf{E} \left[ \int_t^T \frac{(\int_t^u \beta_s \, ds)^{k+1}}{(k + 1)!} \beta_u \varphi_u \, du \middle| \mathcal{F}_t \right], \quad t \in [0, T].$$

In fact, in view of (4), (6) and the progressively measurable property of the process  $\beta$ , we can take advantage of the formula of integration by parts to obtain

$$\begin{aligned} \varphi_t &\leq \mathbf{E} \left[ \int_t^T \frac{(\int_t^u \beta_s \, ds)^k}{k!} \beta_u \mathbf{E} \left[ \int_u^T \beta_r \varphi_r \, dr \middle| \mathcal{F}_u \right] du \middle| \mathcal{F}_t \right] \\ &= \mathbf{E} \left[ \int_t^T \left( \frac{(\int_t^u \beta_s \, ds)^k}{k!} \beta_u \int_u^T \beta_r \varphi_r \, dr \right) du \middle| \mathcal{F}_t \right] \\ &= \mathbf{E} \left[ \int_t^T \left( \int_u^T \beta_r \varphi_r \, dr \right) d \left[ \frac{(\int_t^u \beta_s \, ds)^{k+1}}{(k + 1)!} \right] \middle| \mathcal{F}_t \right] \\ &= \mathbf{E} \left[ \frac{(\int_t^u \beta_s \, ds)^{k+1}}{(k + 1)!} \int_u^T \beta_r \varphi_r \, dr \bigg|_t^T + \int_t^T \frac{(\int_t^u \beta_s \, ds)^{k+1}}{(k + 1)!} \beta_u \varphi_u \, du \middle| \mathcal{F}_t \right] \\ &= \mathbf{E} \left[ \int_t^T \frac{(\int_t^u \beta_s \, ds)^{k+1}}{(k + 1)!} \beta_u \varphi_u \, du \middle| \mathcal{F}_t \right]. \end{aligned}$$

Above all, (5) is established.

Furthermore, it is easy to verify that the right side of (5) tends to 0 when  $n \rightarrow +\infty$  by the Lebesgue dominated convergence theorem together with the condition of Theorem 1, which means  $\varphi(t) \leq 0$  for each  $t \in [0, T]$ , that is,

$$\mu(t) \leq v(t) = a \mathbf{E} \left[ e^{\int_t^T \beta(s) \, ds} \middle| \mathcal{F}_t \right], \quad t \in [0, T].$$

Finally, in the case of  $a = 0$ , it follows from the above proof process that, for each  $n \geq 1$ ,

$$\mu(t) \leq \frac{1}{n} + \mathbf{E} \left[ \int_t^T \beta(s) \mu(s) \, ds \middle| \mathcal{F}_t \right]$$

$$\leq \frac{1}{n} \mathbf{E} \left[ e^{\int_t^T \beta(s) ds} \middle| \mathcal{F}_t \right], \quad t \in [0, T].$$

Therefore, let  $n \rightarrow +\infty$ , then  $\mu(t) \equiv 0$ . The proof is then complete. □

### 3.2 Integral method

*Proof of Theorem 1* By replacing  $t$  in (1) with  $r$ , multiplying both sides of (1) by  $b_r = \beta_r e^{\int_0^r \beta_s ds}$ , and then taking the integral on  $[t, T]$  and the conditional mathematical expectation with respect to  $\mathcal{F}_t$ , it follows that, for each  $t \in [0, T]$ ,

$$\begin{aligned} \mathbf{E} \left[ \int_t^T b_r \mu_r dr \middle| \mathcal{F}_t \right] &\leq a \mathbf{E} \left[ \int_t^T b_r dr \middle| \mathcal{F}_t \right] + \mathbf{E} \left[ \int_t^T b_r \mathbf{E} \left[ \int_r^T \beta_s \mu_s ds \middle| \mathcal{F}_r \right] dr \middle| \mathcal{F}_t \right] \\ &= a \mathbf{E} \left[ e^{\int_t^T \beta_s ds} \middle| \mathcal{F}_t \right] + \mathbf{E} \left[ \int_t^T \left( b_r \int_r^T \beta_s \mu_s ds \right) dr \middle| \mathcal{F}_t \right]. \end{aligned} \tag{7}$$

On the other hand, exchanging the integral order of the double integral yields

$$\begin{aligned} \int_t^T \left( b_r \int_r^T \beta_s \mu_s ds \right) dr &= \int_t^T \left( \beta_s \mu_s \int_t^s b_r dr \right) ds \\ &= \int_t^T \beta_s \mu_s \cdot e^{\int_0^r \beta_u du} \Big|_t^s ds \\ &= \int_t^T b_s \mu_s ds - e^{\int_0^t \beta_u du} \int_t^T \beta_s \mu_s ds, \quad t \in [0, T]. \end{aligned} \tag{8}$$

Now, substituting (8) into the right side of (7) and rewriting this formula, we have

$$e^{\int_0^t \beta_s ds} \mathbf{E} \left[ \int_t^T \beta_s \mu_s ds \middle| \mathcal{F}_t \right] \leq a \mathbf{E} \left[ e^{\int_0^T \beta_s ds} - e^{\int_0^t \beta_s ds} \middle| \mathcal{F}_t \right], \quad t \in [0, T].$$

Hence,

$$\mathbf{E} \left[ \int_t^T \beta_s \mu_s ds \middle| \mathcal{F}_t \right] \leq a \mathbf{E} \left[ e^{\int_t^T \beta_r dr} \middle| \mathcal{F}_t \right] - a, \quad t \in [0, T].$$

Furthermore, in view of (1), (2) follows immediately.

The proof of the case of  $a = 0$  is the same as in Sect. 3.1. The proof is complete. □

### 3.3 Martingale representation method

*Proof of Theorem 1* Denote  $\xi = \int_0^T \beta_s \mu_s ds$ . Then  $\mathbf{E}[|\xi|] < +\infty$  by the condition of Theorem 1. By the martingale representation theorem (see Theorem 2.46, page 120 in [12] for details), it follows that there exists a stochastic process  $(z_t)_{t \in [0, T]} \in \bigcap_{\beta \in (0, 1)} M^\beta$  such that, for each  $t \in [0, T]$ ,

$$\mathbf{E}[\xi | \mathcal{F}_t] = \mathbf{E}[\xi] + \int_0^t z_s \cdot dB_s.$$

Now, let

$$\bar{\mu}_t := a + \mathbf{E} \left[ \int_t^T \beta_s \mu_s ds \middle| \mathcal{F}_t \right], \quad t \in [0, T].$$

Then  $\bar{\mu}_t$  is  $(\mathcal{F}_t)$ -progressively measurable,  $\mu_t \leq \bar{\mu}_t$ , and

$$\begin{aligned} \bar{\mu}_t &= a + \mathbf{E}\left[\xi - \int_0^t \beta_s \mu_s \, ds \mid \mathcal{F}_t\right] \\ &= a + \mathbf{E}[\xi \mid \mathcal{F}_t] - \int_0^t \beta_s \mu_s \, ds \\ &= a + \mathbf{E}[\xi] + \int_0^t z_s \cdot dB_s - \int_0^t \beta_s \mu_s \, ds, \quad t \in [0, T]. \end{aligned}$$

Next, using Itô's formula and the fact that  $\mu_t \leq \bar{\mu}_t$  yields

$$\begin{aligned} d(\bar{\mu}_t e^{\int_0^t \beta_s \, ds}) &= e^{\int_0^t \beta_s \, ds} [-\beta_t \mu_t \, dt + z_t \cdot dB_t + \beta_t \bar{\mu}_t \, dt] \\ &\geq e^{\int_0^t \beta_s \, ds} z_t \cdot dB_t. \end{aligned} \tag{9}$$

By the fact that the process  $(\int_0^t e^{\int_0^r \beta_s \, ds} z_r \cdot dB_r)_{t \in [0, T]}$  is  $(\mathcal{F}_t)$ -martingale, note that

$$\begin{aligned} \mathbf{E}\left[\int_t^T e^{\int_0^r \beta_s \, ds} z_r \cdot dB_r \mid \mathcal{F}_t\right] &= \mathbf{E}\left[\int_0^T e^{\int_0^r \beta_s \, ds} z_r \cdot dB_r \mid \mathcal{F}_t\right] - \int_0^t e^{\int_0^r \beta_s \, ds} z_r \cdot dB_r \\ &= 0, \quad t \in [0, T]. \end{aligned}$$

Integrating on  $[t, T]$  and taking the conditional mathematical expectation with respect to  $\mathcal{F}_t$  on both sides of (9) leads to, in view of  $\bar{\mu}_T = a$ ,

$$\mathbf{E}\left[ae^{\int_0^T \beta_s \, ds} - \bar{\mu}_t e^{\int_0^t \beta_s \, ds} \mid \mathcal{F}_t\right] \geq 0, \quad t \in [0, T].$$

Furthermore, since the process  $\bar{\mu}_t e^{\int_0^t \beta_s \, ds}$  is  $\mathcal{F}_t$ -measurable, we have  $\mathbf{E}[\bar{\mu}_t e^{\int_0^t \beta_s \, ds} \mid \mathcal{F}_t] = \bar{\mu}_t e^{\int_0^t \beta_s \, ds}$ , and then

$$\bar{\mu}_t e^{\int_0^t \beta_s \, ds} \leq \mathbf{E}\left[ae^{\int_0^T \beta_s \, ds} \mid \mathcal{F}_t\right], \quad t \in [0, T].$$

So

$$\mu_t \leq \bar{\mu}_t \leq a \mathbf{E}\left[e^{\int_0^T \beta_s \, ds} \mid \mathcal{F}_t\right], \quad t \in [0, T].$$

Thus, (2) is established.

The proof of the case of  $a = 0$  is the same as in Sect. 3.1. The proof is completed. □

#### 4 Application

This section will introduce an application of the stochastic Gronwall's inequality.

We are concerned with the following one-dimensional backward stochastic differential equation (BSDE for short in the remaining):

$$y_t = \xi + \int_t^T g(s, y_s, z_s) \, ds - \int_t^T z_s \cdot dB_s, \quad t \in [0, T], \tag{10}$$

where  $0 < T \leq +\infty$  is called the terminal time,  $\xi$  is  $\mathcal{F}_T$ -measurable random variable called the terminal condition, the random function  $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R}^d \rightarrow \mathbf{R}$  is  $(\mathcal{F}_t)$ -progressively measurable for each  $(y, z)$  called the generator of BSDE (9). The solution

$(y_t, z_t)_{t \in [0, T]}$  is a pair of  $(\mathcal{F}_t)$ -progressively measurable processes and the triple  $(\xi, T, g)$  is called the parameters of BSDE (9). BSDE with the parameters  $(\xi, T, g)$  is usually denoted by BSDE  $(\xi, T, g)$ .

**Definition 1** Assume  $p > 1$ . If  $(y_t, z_t)_{t \in [0, T]} \in \mathcal{S}^p(0, T; \mathbf{R}) \times M^p(0, T; \mathbf{R}^d)$  and  $\mathbf{dP}$ -a.s., BSDE (9) holds true, then the processes  $(y_t, z_t)_{t \in [0, T]}$  is called an  $L^p$  solution of BSDE (9).

Assume that the generator  $g$  satisfies the following assumption:

(H) There exist two stochastic processes  $(u_t)_{t \in [0, T]}$ ,  $(v_t)_{t \in [0, T]}$  satisfying

$$\int_0^T [u_t(\omega) + v_t^2(\omega)] dt \leq M, \quad \mathbf{dP}\text{-a.s.},$$

for some constant  $M > 0$  and such that  $\mathbf{dP} \times dt$ -a.e., for each  $(y_i, z_i) \in \mathbf{R} \times \mathbf{R}^d$ ,  $i = 1, 2$ ,

$$|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq u_t(\omega)|y_1 - y_2| + v_t(\omega)|z_1 - z_2|.$$

**Theorem 2** Let  $p > 1$ ,  $0 < T \leq +\infty$ ,  $\xi, \xi' \in L^p(\Omega, \mathcal{F}_T, \mathbf{P})$ ,  $g$  and  $g'$  be two generators of BSDEs, and let  $(y, z)$  and  $(y', z')$  be an  $L^p$  solution to BSDE  $(\xi, T, g)$  and BSDE  $(\xi', T, g')$ , respectively. If  $\mathbf{dP}$ -a.s.,  $\xi \leq \xi'$  and one of the following two statements is satisfied:

- (i)  $g$  satisfies (H) and  $\mathbf{1}_{y_t > y'_t}(g(t, y'_t, z'_t) - g(t, y_t, z_t)) \leq 0$ ,  $\mathbf{dP} \times dt$ -a.e.;
  - (ii)  $g'$  satisfies (H) and  $\mathbf{1}_{y_t > y'_t}(g(t, y_t, z_t) - g(t, y'_t, z'_t)) \leq 0$ ,  $\mathbf{dP} \times dt$ -a.e.,
- then, for each  $t \in [0, T]$ ,  $y_t \leq y'_t$ ,  $\mathbf{dP}$ -a.s.

*Proof* We only prove the case (i). The other case can be proved in the same way. Set  $\hat{y} := y - y'$ ,  $\hat{z} := z - z'$  and  $\hat{\xi} := \xi - \xi'$ . Then

$$\hat{y}_t = \hat{\xi} + \int_t^T [g(s, y_s, z_s) - g'(s, y'_s, z'_s)] ds - \int_t^T \hat{z}_s \cdot dB_s, \quad t \in [0, T].$$

Tanaka’s formula (see page 220 in [13] for details) yields, in view of  $\mathbf{dP}$ -a.s.,  $\hat{\xi}^+ = 0$ ,

$$\begin{aligned} \hat{y}_t^+ &= \hat{\xi}^+ + \int_t^T \mathbf{1}_{\hat{y}_s > 0}(g(s, y_s, z_s) - g'(s, y'_s, z'_s)) ds - \int_t^T \mathbf{1}_{\hat{y}_s > 0} \hat{z}_s \cdot dB_s - \frac{1}{2} L_t^0 \\ &\leq \int_t^T \mathbf{1}_{\hat{y}_s > 0}(g(s, y_s, z_s) - g'(s, y'_s, z'_s)) ds - \int_t^T \mathbf{1}_{\hat{y}_s > 0} \hat{z}_s \cdot dB_s, \end{aligned}$$

where  $L_t^0$  is the semimartingale local time of the process  $\hat{y}_t$  at 0, it is an increasing process and  $L_0^0 = 0$ . Since  $\mathbf{1}_{y_t > y'_t}(g(t, y'_t, z'_t) - g(t, y_t, z_t))$  is non-positive, we have

$$\begin{aligned} &\mathbf{1}_{\hat{y}_s > 0}(g(s, y_s, z_s) - g'(s, y'_s, z'_s)) \\ &= \mathbf{1}_{\hat{y}_s > 0}(g(s, y_s, z_s) - g(s, y'_s, z'_s)) + \mathbf{1}_{\hat{y}_s > 0}(g(s, y'_s, z'_s) - g'(s, y'_s, z'_s)) \\ &\leq \mathbf{1}_{\hat{y}_s > 0}(g(s, y_s, z_s) - g(s, y'_s, z'_s)), \quad \mathbf{dP} \times ds\text{-a.e.}, \end{aligned}$$

Furthermore, using the assumption (H) for  $g$ , we deduce that

$$\mathbf{1}_{\hat{y}_s > 0}(g(s, y_s, z_s) - g(s, y'_s, z'_s)) \leq u_s \hat{y}_s^+ + \mathbf{1}_{\hat{y}_s > 0} v_s |\hat{z}_s|, \quad \mathbf{dP} \times ds\text{-a.e.}$$

Therefore, we obtain

$$\begin{aligned} \hat{y}_t^+ &\leq \int_t^T (u_s \hat{y}_s^+ + \mathbf{1}_{\hat{y}_s > 0} v_s |\hat{z}_s|) \, ds - \int_t^T \mathbf{1}_{\hat{y}_s > 0} \hat{z}_s \cdot dB_s \\ &= \int_t^T u_s \hat{y}_s^+ \, ds - \int_t^T \mathbf{1}_{\hat{y}_s > 0} \hat{z}_s \cdot \left[ dB_s - \frac{v_s \hat{z}_s}{|\hat{z}_s|} \mathbf{1}_{|\hat{z}_s| \neq 0} \, ds \right], \quad t \in [0, T]. \end{aligned} \tag{11}$$

In the sequel, let  $\mathbf{Q}$  be the probability measure on  $(\Omega, \mathcal{F}_T)$  which is equivalent to  $\mathbf{P}$  and defined by

$$\frac{d\mathbf{Q}}{d\mathbf{P}} := \exp \left\{ \int_0^T \frac{v_s \hat{z}_s}{|\hat{z}_s|} \mathbf{1}_{|\hat{z}_s| \neq 0} \cdot dB_s - \frac{1}{2} \int_0^T \mathbf{1}_{|\hat{z}_s| \neq 0} |v_s|^2 \, ds \right\}.$$

It is worth noting that  $d\mathbf{Q}/d\mathbf{P}$  has moments of all orders since  $\int_0^T v^2(s) \, ds \leq M$ ,  $d\mathbf{P}$ -a.s. By Girsanov’s theorem, under  $\mathbf{Q}$  the process

$$\tilde{B}(t) = B_t - \int_0^t \frac{v_s \hat{z}_s}{|\hat{z}_s|} \mathbf{1}_{|\hat{z}_s| \neq 0} \, ds, \quad t \in [0, T],$$

is a Brownian motion. Moreover, the process

$$\left( \int_0^t \mathbf{1}_{\hat{y}_s > 0} \hat{z}_s \cdot d\tilde{B}_s \right)_{t \in [0, T]}$$

is an  $(\mathcal{F}_t, \mathbf{Q})$ -martingale. Thus, taking the conditional expectation with respect to  $\mathcal{F}_t$  under  $\mathbf{Q}$  in (10), we obtain, for each  $t \in [0, T]$ ,

$$\hat{y}_t^+ \leq \mathbf{E}_{\mathbf{Q}} \left[ \int_t^T u_s \hat{y}_s^+ \, ds \mid \mathcal{F}_t \right], \quad d\mathbf{Q}\text{-a.s.}$$

Thus, it follows from Theorem 1 that  $\hat{y}_t^+ = 0$ ,  $d\mathbf{Q}$ -a.s. and then  $y_t \leq y'_t$ ,  $d\mathbf{P}$ -a.s. for each  $t \in [0, T]$  since  $\mathbf{Q}$  is equivalent to  $\mathbf{P}$ . Theorem 2 is then proved.  $\square$

*Remark 3* It follows from Theorem 2 that if  $g$  satisfies the assumption (H), then the  $L^p$  solution of BSDE  $(\xi, T, g)$  must be unique.

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**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

XW completed the manuscript. Both authors have on equal level discussed, posed the research questions and proved the results in this paper. Both authors read and approved the final manuscript.

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