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Improved McClelland and Koolen–Moulton bounds for distance energy of a graph

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Abstract

Let G be a graph with n vertices and m edges. The term energy of a graph G was introduced by I. Gutman in chemistry due to its relevance to the total π -electron energy of a carbon compound. An analogous energy $\mathcal{E}_D(G)$, called the distance energy, was defined by Indulal et al. (MATCH Commun. Math. Comput. Chem. 60:461–472, 2008) in 2008. McClelland and Koolen–Moulton bounds for distance energy were established subsequently by Ramane et al. (Kragujev. J. Math. 31:59–68, 2008). The lower and upper bounds for $\mathcal{E}_D(G)$ obtained in this paper are better than the McClelland and Koolen–Moulton bounds.

MSC: 05C50; 05C69

Keywords: Distance matrix; Distance spectrum; Bounds for distance energy of graph

1 Introduction

Let G be a simple undirected graph with n vertices and m edges. The distance between two vertices v_i and v_j is denoted by d_{ij} and is defined as the length of the shortest path from v_i to v_j . The distance sum $W(G) = \sum_{i < j} d_{ij}$ is called the Wiener index of the graph G . For simplicity, we write the Wiener index as W . The distance matrix of the graph G is defined as $A(G) = A_D(G) = [d_{ij}]$. Clearly, $A_D(G)$ is a symmetric matrix. Its eigenvalues are called D -eigenvalues and are ordered in the form $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$. The largest eigenvalue μ_1 is called the distance spectral radius of the graph G . We also write its absolute eigenvalues in decreasing order as $\rho_1 \geq \rho_2 \geq \dots \geq \rho_n$. Given a graph G , the distance energy of G is defined by $\mathcal{E}_D(G) = \sum_{i=1}^n |\mu_i| = \sum_{i=1}^n \rho_i$. For any vertex v_i in the connected graph G , the eccentricity $e(v_i)$ is the distance between v_i and a vertex that is farthest from v_i in G . The minimum eccentricity among the vertices of G is called the radius of graph G , and the maximum eccentricity among the vertices is called the diameter of the graph G , which are respectively denoted by $\text{rad}(G)$ and $\text{diam}(G)$.

The distance energy is analogous to the ordinary energy of a graph G , which is defined as $\mathcal{E}(G) = \sum_{i=1}^n |\lambda_i|$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are ordinary eigenvalues of G obtained from its adjacency matrix. The studies on the graph energy can be seen in papers [5, 6]. For a detailed survey on applications to the graph energy, see [2–4, 7]. For the ordinary energy, the best known bounds are the Koolen and Moulton upper bound [9, 10] and the McClelland lower bound [12].

For a connected graph G , the Koolen and Moulton upper bound [13] for the distance energy in terms of W , M , and n is

$$\mathcal{E}_D(G) \leq \left(\frac{2W}{n}\right) + \sqrt{(n-1)\left(2M - \left(\frac{2W}{n}\right)^2\right)} \quad \text{for } 2W \geq n, \quad (1.1)$$

where $M = \sum_{i < j}^n d_{ij}^2$.

In this paper, we show that the upper bound (1.1) can be modified to a better bound for all classes of graphs with $n^2 \geq 4m$. Further results on upper bounds can also be seen in [11].

The McClelland bounds [13] for the distance energy of a graph, which is true for any connected graph G , is

$$\sqrt{2M + n(n-1)|\det(A)|^{\frac{2}{n}}} \leq \mathcal{E}_D(G) \leq \sqrt{2Mn}. \quad (1.2)$$

The lower bound obtained in this paper is better than that of McClelland. For more studies on the distance energy, we refer to [1, 8, 14].

We use the following two lemmas, which follow from the properties of distance eigenvalues.

Lemma 1.1 *Let G be a graph with $n \geq 3$ vertices and m edges. Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$ be D -eigenvalues of G . Then*

$$\sum_{i=1}^n \mu_i = 0 \quad \text{and} \quad \sum_{i=1}^n \mu_i^2 = 2M.$$

Lemma 1.2 *If $\mu_1(G)$ is the distance spectral radius of the graph G , then $\mu_1(G) \geq \frac{2W}{n}$.*

Note that $M = \sum_{i < j}^n d_{ij}^2 \geq \sum_{i < j}^n d_{ij} = W$ and $\sqrt{M} = \sqrt{\sum_{i < j}^n d_{ij}^2} \leq \sum_{i < j}^n d_{ij} = W$.

2 The main results

Lemma 2.1 *If $\mu_n(G)$ is the smallest distance eigenvalue of the graph G and $\rho_n(G)$ is the smallest absolute distance eigenvalue, then*

$$\mu_n(G) \leq \sqrt{\frac{2W}{n}} \quad \text{and} \quad \rho_n(G) \leq |\det(A)|^{\frac{1}{n}}.$$

Proof For any nonzero vector X ,

$$\mu_n(G) \leq \min_{X \neq 0} \left(\frac{X'AX}{X'X} \right) \leq \sqrt{\frac{J'AJ}{J'J}} = \sqrt{\frac{2W}{n}},$$

where J is the unit $n \times 1$ matrix $J = [1, 1, 1, \dots, 1]'$. Now consider $\rho_1 \rho_2 \rho_3 \dots \rho_n = |\det(A)|$. Then $\rho_n \rho_n \rho_n \dots \rho_n \leq \rho_1 \rho_2 \rho_3 \dots \rho_n \leq |\det(A)| \Rightarrow \rho_n(G) \leq |\det(A)|^{\frac{1}{n}}$. $\square$

Lemma 2.2 *Let G be a graph with $n \geq 3$ vertices and m edges. For the largest and smallest distance eigenvalues μ_1 and μ_n of G , $\mu_1 + \mu_n \leq 2\sqrt{\frac{M(n-2)}{n}}$.*

Proof For D -eigenvalues $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$ of G , it is well known that $\sum_{i=1}^n \mu_i = 0$ and $\sum_{i=1}^n \mu_i^2 = 2M$. Using the Cauchy–Schwarz inequality for $(\mu_2, \mu_3, \dots, \mu_{n-1})$ and $(\underbrace{1, 1, \dots, 1}_{(n-2) \text{ times}})$ ($n \geq 3$) we have

$$\left(\sum_{i=2}^{n-1} \mu_i \right)^2 \leq \left(\sum_{i=2}^{n-1} 1 \right) \left(\sum_{i=2}^{n-1} \mu_i^2 \right),$$

that is, $(-\mu_1 - \mu_n)^2 \leq (n-2)(2M - \mu_1^2 - \mu_n^2)$.

$$\begin{aligned} \therefore (n-2)2M &\geq (\mu_1 + \mu_n)^2 + (n-2)(\mu_1^2 + \mu_n^2) \\ &= (\mu_1 + \mu_n)^2 + (n-2)((\mu_1 + \mu_n)^2 - 2\mu_1\mu_n) \\ &= (\mu_1 + \mu_n)^2(n-1) - 2(n-2)\mu_1\mu_n. \end{aligned}$$

However, $(\frac{\mu_1 + \mu_n}{2})^2 \geq \mu_1\mu_n$, which implies that $-\mu_1\mu_n \geq -(\frac{\mu_1 + \mu_n}{2})^2$.

Thus $(n-2)2M \geq (\mu_1 + \mu_n)^2(n-1) - 2(n-2)\frac{(\mu_1 + \mu_n)^2}{4} = (\mu_1 + \mu_n)^2\frac{n}{2}$.

Hence $\mu_1 + \mu_n \leq 2\sqrt{\frac{M(n-2)}{n}}$. □

3 Upper bound for the distance energy of a graph

Theorem 3.1 *Let G be a graph with $n \geq 3$ vertices and m edges. If $n^2 \geq 4m$, then*

$$\mathcal{E}_D(G) \leq \frac{2W}{n} + \sqrt{\frac{2W}{n}} + \sqrt{(n-2)\left(2M - \frac{2W}{n} - \frac{4W^2}{n^2}\right)}. \quad (3.1)$$

The equality holds iff G is $\frac{n}{2}K_2$.

Proof Applying the Cauchy–Schwarz inequality for $(|\mu_2|, |\mu_3|, \dots, |\mu_{n-1}|)$ and $(\underbrace{1, 1, \dots, 1}_{(n-2) \text{ times}})$, we have

$$\begin{aligned} \left(\sum_{i=2}^{n-1} |\mu_i| \right)^2 &\leq \left(\sum_{i=2}^{n-1} 1 \right) \left(\sum_{i=2}^{n-1} |\mu_i|^2 \right), \\ (\mathcal{E}_D(G) - |\mu_1| - |\mu_n|)^2 &\leq (n-2)(2M - |\mu_1|^2 - |\mu_n|^2), \\ \mathcal{E}_D(G) &\leq |\mu_1| + |\mu_n| + \sqrt{(n-2)(2M - |\mu_1|^2 - |\mu_n|^2)}. \end{aligned}$$

Let $|\mu_1| = x$ and $|\mu_n| = y$.

We maximize the function $f(x, y) = x + y + \sqrt{(n-2)(2M - x^2 - y^2)}$. Differentiating $f(x, y)$ with respect to x and y , we have

$$\begin{aligned} f_x &= 1 - \frac{x(n-2)}{\sqrt{(n-2)(2M - x^2 - y^2)}}, & f_y &= 1 - \frac{y(n-2)}{\sqrt{(n-2)(2M - x^2 - y^2)}}, \\ f_{xx} &= -\frac{\sqrt{(n-2)(2M - y^2)}}{(2M - x^2 - y^2)^{\frac{3}{2}}}, & f_{yy} &= -\frac{\sqrt{(n-2)(2M - x^2)}}{(2M - x^2 - y^2)^{\frac{3}{2}}} \quad \text{and} \end{aligned}$$

$$f_{xy} = -\frac{\sqrt{(n-2)}(xy)}{(2M - x^2 - y^2)^{\frac{3}{2}}}.$$

For maxima or minima, $f_x = 0$ and $f_y = 0$, which implies

$$x^2(n-1) + y^2 = 2M \quad \text{and} \quad y^2(n-1) + x^2 = 2M.$$

Solving these equations, we obtain that $x = y = \sqrt{\frac{2M}{n}}$. At this point the values of f_{xx}, f_{yy}, f_{xy} , and $\Delta = f_{xx}f_{yy} - (f_{xy})^2$ are

$$\begin{aligned} f_{xx} &= -\frac{\sqrt{(n-2)}(n-1)}{\sqrt{\frac{2M}{n}}(n-2)^{\frac{3}{2}}} \leq 0, & f_{yy} &= -\frac{\sqrt{(n-2)}(n-1)}{\sqrt{\frac{2M}{n}}(n-2)^{\frac{3}{2}}}, \\ f_{xy} &= -\frac{\sqrt{(n-2)}}{\sqrt{\frac{2M}{n}}(n-2)^{\frac{3}{2}}} \quad \text{and} \quad \Delta = \frac{n(n^2 + 3 - 3n)}{2M(n-2)^2} \geq 0. \end{aligned}$$

Therefore $f(x, y)$ attains its maximum value at $x = y = \sqrt{\frac{2M}{n}}$, and this maximum value is $f(\sqrt{\frac{2M}{n}}, \sqrt{\frac{2M}{n}}) = \sqrt{2Mn}$.

However, $f(x, y)$ decreases in the intervals

$$\sqrt{\frac{2M}{n}} \leq x \leq \sqrt{M} \quad \text{and} \quad 0 \leq y \leq \sqrt{\frac{2W}{n}} \leq \sqrt{\frac{2M}{n}} \leq \sqrt{M}.$$

Since $n^2 \geq 4m$, $m \leq W \leq M$, and $\sqrt{M} \leq W$, we have

$$\sqrt{\frac{2m}{n}} \leq \sqrt{\frac{2M}{n}} \leq \frac{2W}{n} \leq |\mu_1| \leq \sqrt{M}, \quad 0 \leq |\mu_n| \leq \sqrt{\frac{2W}{n}} \leq \sqrt{\frac{2M}{n}} \leq \sqrt{M}.$$

Thus $f(|\mu_1|, |\mu_n|) \leq f(\frac{2W}{n}, \sqrt{\frac{2W}{n}}) \leq f(\sqrt{\frac{2M}{n}}, \sqrt{\frac{2M}{n}})$

$$\Rightarrow \quad \mathcal{E}_D(G) \leq \frac{2W}{n} + \sqrt{\frac{2W}{n}} + \sqrt{(n-2)\left(2m - \frac{2W}{n} - \frac{4W^2}{n^2}\right)} \leq \sqrt{2Mn}.$$

For the graph $G \simeq \frac{n}{2}K_2$ ($n = 2m$), $\mathcal{E}_D(G) = n$. Hence the equality holds. $\square$

Now we show that the above bound is an improvement of the Koolen–Moulton bound. Take $g(x, y) = x + y + \sqrt{(n-1)(2M - x^2 - y^2)}$. Then, clearly, $f(x, y) \leq g(x, y)$ for all (x, y) in the given region of x and y .

Along $x = \frac{2W}{n}$, $f(\frac{2W}{n}, y) = \frac{2W}{n} + y + \sqrt{(n-2)(2M - \frac{4W^2}{n^2} - y^2)}$. However, $f(\frac{2W}{n}, y)$ decreases in the interval $0 \leq y \leq \sqrt{2M - \frac{4W^2}{n^2}}$. Since $n^2 \geq 4m$, we also have $0 \leq y \leq \sqrt{\frac{2W}{n}} \leq \sqrt{2M - \frac{4W^2}{n^2}}$. Thus $f(\frac{2W}{n}, \sqrt{\frac{2W}{n}}) \leq f(\frac{2W}{n}, 0)$.

Since $f(\frac{2W}{n}, 0) \leq g(\frac{2W}{n}, 0)$ and $g(\frac{2W}{n}, 0) = \frac{2W}{n} + \sqrt{(n-1)(2M - \frac{4W^2}{n^2})}$, it follows that $f(\frac{2W}{n}, \sqrt{\frac{2W}{n}}) \leq g(\frac{2W}{n}, 0)$. Hence

$$\frac{2W}{n} + \sqrt{\frac{2W}{n}} + \sqrt{(n-2)\left(2M - \frac{2m}{n} - \frac{4W^2}{n^2}\right)} \leq \left(\frac{2W}{n}\right) + \sqrt{(n-1)\left(2M - \left(\frac{2W}{n}\right)^2\right)}.$$

4 Lower bounds for the distance energy of a graph

Theorem 4.1 *If G is a nonsingular graph, then $\mathcal{E}_D(G) \geq n|\det A|^{\frac{1}{n}}$. The equality holds iff G is $\frac{n}{2}K_2$, where $n = 2m$.*

Proof For the eigenvalues $\rho_1 \geq \rho_2 \geq \dots \geq \rho_n$ of G (or its adjacency distance matrix A) it is well known that $|\det(A)| = \rho_1 \rho_2 \dots \rho_n$. Since G is nonsingular, we have $|\det(A)| \neq 0$.

Applying the Cauchy–Schwarz inequality for n terms $a_i = \sqrt{\rho_i}$ and $b_i = 1$ for all $i = 1, 2, \dots, n$, We have

$$\sum_{i=1}^n \sqrt{\rho_i} \leq \sqrt{\left(\sum_{i=1}^n \rho_i\right)n},$$

$$\sqrt{\mathcal{E}_D(G)} \geq \frac{\sum_{i=1}^n \sqrt{\rho_i}}{\sqrt{n}}.$$

However, $\frac{\sqrt{\rho_1} + \sqrt{\rho_2} + \dots + \sqrt{\rho_n}}{n} \geq (\sqrt{\rho_1 \rho_2 \dots \rho_n})^{\frac{1}{n}}$,

$$\sqrt{\mathcal{E}_D(G)} \geq \frac{n(\sqrt{\rho_1 \rho_2 \dots \rho_n})^{\frac{1}{n}}}{\sqrt{n}} \Rightarrow \mathcal{E}_D(G) \geq n|\det A|^{\frac{1}{n}}.$$

For the graph $G \simeq \frac{n}{2}K_2$ with $n = 2m$, $|\det(A)| = 1$. Hence the equality holds. $\square$

Theorem 4.2 *Let G be a graph with $n > 1$ vertices and m edges, and let $2W \geq n$. Then*

$$\mathcal{E}_D(G) \geq \frac{2W}{n} + \frac{(n-1)|\det(A)|^{\frac{1}{(n-1)}}}{\left(\frac{2W}{n}\right)^{\frac{1}{(n-1)}}}. \quad (4.1)$$

The equality holds if G is isomorphic to K_n and $\frac{n}{2}K_2$ with $n = 2m$.

Proof Using the Cauchy–Schwarz inequality for $\sqrt{\rho_2}, \sqrt{\rho_3}, \dots, \sqrt{\rho_n}$ and $\underbrace{(1, 1, \dots, 1)}_{(n-1) \text{ times}}$, we have

$$\sum_{i=2}^n \sqrt{\rho_i} \leq \sqrt{\left(\sum_{i=2}^n \rho_i\right)(n-1)},$$

$$\sum_{i=2}^n \sqrt{\rho_i} \leq \sqrt{(\mathcal{E}_D(G) - \rho_1)(n-1)},$$

$$\sqrt{\mathcal{E}_D(G) - \rho_1} \geq \frac{\sum_{i=2}^n \sqrt{\rho_i}}{\sqrt{n-1}}.$$

However, $\frac{\sqrt{\rho_2} + \sqrt{\rho_3} + \dots + \sqrt{\rho_n}}{n-1} \geq (\sqrt{\rho_2 \rho_3 \dots \rho_n})^{\frac{1}{n-1}}$, and therefore

$$\sqrt{\mathcal{E}_D(G) - \rho_1} \geq \frac{(n-1)(\sqrt{\rho_2 \rho_3 \dots \rho_n})^{\frac{1}{(n-1)}}}{\sqrt{(n-1)}},$$

$$\mathcal{E}_D(G) \geq \rho_1 + (n-1)(\rho_2 \rho_3 \dots \rho_n)^{\frac{1}{(n-1)}} = \rho_1 + (n-1) \left(\frac{|\det(A)|}{\rho_1} \right)^{\frac{1}{(n-1)}}.$$

Let $\rho_1 = x$ and $f(x) = x + (n-1)\left(\frac{|\det(A)|}{x}\right)^{\frac{1}{(n-1)}}$. Then $f'(x) = 1 - \frac{|\det(A)|^{\frac{1}{(n-1)}}}{x^{\frac{n}{(n-1)}}}$ and $f''(x) = \frac{n|\det(A)|^{\frac{1}{(n-1)}}}{(n-1)x^{\frac{n}{(n-1)}}}$.

For maxima or minima, $f'(x) = 0$, which gives the value $x = |\det(A)|^{\frac{1}{n}}$.

At this point, $f''(x) = \frac{n}{(n-1)}|\det(A)|^{\frac{1}{n}} \geq 0$ for all $n > 1$. Thus the function $f(x)$ attains its minimum at $x = |\det(A)|^{\frac{1}{n}}$, and the minimum value is $f(|\det(A)|^{\frac{1}{n}}) = n|\det(A)|^{\frac{1}{n}}$. However, $\frac{2M}{n} = \frac{\rho_1^2 + \rho_2^2 + \dots + \rho_n^2}{n} \geq \frac{2W}{n} \geq \frac{\rho_1 + \rho_2 + \dots + \rho_n}{n} \geq (\rho_1 \rho_2 \dots \rho_n)^{\frac{1}{n}}$. This implies $|\det(A)|^{\frac{1}{n}} \leq \frac{2W}{n}$. Since $2W \geq n$, we have $\frac{2W}{n} \leq \rho_1$.

Therefore, the function is increasing in the interval $|\det(A)|^{\frac{1}{n}} \leq \frac{2W}{n} \leq \rho_1 \leq \sqrt{2M}$, and therefore $f(\rho_1) \geq f(\frac{2W}{n})$, and

$$\mathcal{E}_D(G) \geq \frac{2W}{n} + \frac{(n-1)|\det(A)|^{\frac{1}{(n-1)}}}{\left(\frac{2W}{n}\right)^{\frac{1}{(n-1)}}}.$$

- (i) If G is isomorphic to K_n , then $|\det(A)| = n-1$, $\frac{2W}{n} = n-1$, and hence $\mathcal{E}_D(G) = 2(n-1)$.
- (ii) If G isomorphic to $\frac{n}{2}K_2$ with $n = 2m$, then the eigenvalues are ± 1 (each with multiplicity $\frac{n}{2}$), and hence $\mathcal{E}_D(G) = n$. $\square$

Theorem 4.3 *Let G be a graph with m edges and $n (> 3)$ vertices, and let $W \geq n$. Then*

$$\mathcal{E}_D(G) \geq \left(\frac{W}{n}\right) + \left(\frac{2W}{n}\right) + \frac{(n-2)|\det(A)|^{\frac{1}{(n-2)}}}{\left(\left(\frac{W}{n}\right)\left(\frac{2W}{n}\right)\right)^{\frac{1}{(n-2)}}}. \quad (4.2)$$

Proof For $(n-2)$ entries of eigenvalues $\sqrt{\rho_2}, \sqrt{\rho_3}, \dots, \sqrt{\rho_{n-1}}$ and $\underbrace{(1, 1, \dots, 1)}_{(n-2) \text{ times}}$, applying the

Cauchy–Schwarz inequality, we have

$$\sum_{i=2}^{n-1} \sqrt{\rho_i} \leq \sqrt{\left(\sum_{i=2}^{n-1} \rho_i\right)(n-2)},$$

that is,

$$\sum_{i=2}^{n-1} \sqrt{\rho_i} \leq \sqrt{(\mathcal{E}_D(G) - \rho_1 - \rho_n)(n-2)},$$

so that

$$\sqrt{\mathcal{E}_D(G) - \rho_1 - \rho_n} \geq \frac{\sum_{i=2}^{n-1} \sqrt{\rho_i}}{\sqrt{n-2}}.$$

Since the arithmetic mean is greater than or equal to the geometric mean, we get

$$\sqrt{\mathcal{E}_D(G) - \rho_1 - \rho_n} \geq \frac{(n-2)(\sqrt{\rho_2 \rho_3 \dots \rho_{n-1}})^{\frac{1}{(n-2)}}}{\sqrt{(n-2)}},$$

$$\mathcal{E}_D(G) \geq \rho_1 + \rho_n + (n-2) \left(\frac{|\det(A)|}{\rho_1 \rho_n} \right)^{\frac{1}{(n-2)}}.$$

The equality holds if G is $\frac{n}{2}K_2$ with $n = 2m$ or $K_{n,n}$.

Put $\rho_1 = x$ and $\rho_n = y$. We minimize the right side of the above function. Let $f(x, y) = x + y + (n-2)\left(\frac{|\det(A)|}{xy}\right)^{\frac{1}{n-2}}$. Then $f_x = 1 - |\det(A)|^{\frac{1}{n-2}} y(xy)^{\frac{-(n-1)}{n-2}}$,

$$f_y = 1 - |\det(A)|^{\frac{1}{n-2}} x(xy)^{\frac{-(n-1)}{n-2}},$$

$$f_{xx} = |\det(A)|^{\frac{1}{n-2}} \left(\frac{n-1}{n-2}\right) y^2 (xy)^{\frac{-(2n-3)}{n-2}},$$

$$f_{yy} = |\det(A)|^{\frac{1}{n-2}} \left(\frac{n-1}{n-2}\right) x^2 (xy)^{\frac{-(2n-3)}{n-2}} \quad \text{and}$$

$$f_{xy} = |\det(A)|^{\frac{1}{n-2}} \frac{(xy)^{\frac{-(n-1)}{n-2}}}{n-2}.$$

For maxima or minima, $f_x = 0$ and $f_y = 0$, which gives $xy^{\frac{1}{n-1}} = |\det(A)|^{\frac{1}{n-1}}$ and $yx^{\frac{1}{n-1}} = |\det(A)|^{\frac{1}{n-1}}$. Solving, we get $x = |\det(A)|^{\frac{1}{n}}$ and $y = |\det(A)|^{\frac{1}{n}}$. At this point, the values of f_{xx} , f_{yy} , f_{xy} , and $\Delta = f_{xx}f_{yy} - (f_{xy})^2$ are $f_{xx} = f_{yy} = \left(\frac{n-1}{n-2}\right) |\det(A)|^{\frac{-1}{n}}$, $f_{xy} = \frac{1}{n-2} |\det(A)|^{\frac{-1}{n}}$, and $\Delta = \left(\frac{n}{n-2}\right) |\det(A)|^{\frac{-2}{n}} \geq 0$ for all $n \neq 2$. The minimum value is $f(|\det(A)|^{\frac{1}{n}}, |\det(A)|^{\frac{1}{n}}) = n |\det(A)|^{\frac{1}{n}}$. However, $|\det(A)|^{\frac{1}{n}} \leq \frac{2W}{n} \leq \rho_1 \leq \sqrt{2M}$ and $0 \leq \rho_n \leq |\det(A)|^{\frac{1}{n}} \leq \frac{2W}{n} \leq \sqrt{2M}$.

For $W \geq n$, $f(x, y)$ increases in the intervals $|\det(A)|^{\frac{1}{n}} \leq \frac{2W}{n} \leq x \leq \sqrt{2M}$ and $0 \leq y \leq |\det(A)|^{\frac{1}{n}} \leq \frac{W}{n} \leq \frac{2W}{n} \leq \sqrt{2M}$, that is, $f(x, y)$ increases in the intervals $|\det(A)|^{\frac{1}{n}} \leq \frac{2W}{n} \leq \rho_1 \leq \sqrt{2M}$ and $0 \leq \rho_n \leq |\det(A)|^{\frac{1}{n}} \leq \frac{W}{n} \leq \sqrt{2M}$. At $\rho_n = \frac{W}{n}$, we have

$$f(\rho_1, \rho_n) \geq f\left(\frac{2W}{n}, \frac{W}{n}\right) \geq f\left(\frac{2W}{n}, |\det(A)|^{\frac{1}{n}}\right) \geq f(|\det(A)|^{\frac{1}{n}}, |\det(A)|^{\frac{1}{n}}).$$

$$\text{Therefore } \mathcal{E}_D(G) \geq \left(\frac{W}{n}\right) + \left(\frac{2W}{n}\right) + \frac{(n-2)|\det(A)|^{\frac{1}{n-2}}}{\left(\left(\frac{W}{n}\right)\left(\frac{2W}{n}\right)\right)^{\frac{1}{n-2}}}.$$

□

5 Brief summary and conclusion

In this paper, we established lower and upper bounds for the distance energy of a graph. Across the globe, attempts are being made by researchers to improve these bounds. The lower and upper bounds obtained in this paper improve the McClelland and Koolen–Moulton bounds for the distance energy of a graph.

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Authors' contributions

GS and MRR drafted the manuscript. RPK and DSN revised it. All authors read and approved the final manuscript.

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