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Equivalent property of a half-discrete Hilbert's inequality with parameters

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Abstract

By using the weight functions and the idea of introducing parameters, a half-discrete Hilbert inequality with a nonhomogeneous kernel and its equivalent form are given. The equivalent statements of the constant factor are best possible related to parameters, and some particular cases are considered. The cases of a homogeneous kernel are also deduced.

MSC: 26D15

Keywords: Weight function; Half-discrete Hilbert's inequality; Equivalent form; Best possible constant factor

1 Introduction

If $0<\sum_{m=1}^{\infty}a_m^2<\infty, 0<\sum_{n=1}^{\infty}b_n^2<\infty$, then we have the following discrete Hilbert inequality with the best possible constant factor π [1]:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\pi\left(\sum_{m=1}^{\infty}a_m^2\sum_{n=1}^{\infty}b_n^2\right)^{1/2}. \tag{1}$$

Assuming that $0<\int_0^\infty f^2(x)\,dx<\infty$, $0<\int_0^\infty g^2(y)\,dy<\infty$, we still have the following Hilbert integral inequality [1]:

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y}\,dx\,dy<\pi\left(\int_0^\infty f^2(x)\,dx\int_0^\infty g^2(y)\,dy\right)^{1/2}, \tag{2}$$

where the constant factor π is the best possible. Inequalities (1) and (2) are important in analysis and its applications (cf. [2–13]).

We still have the following half-discrete Hilbert-type inequalities (cf. [1], Theorem 351): If $K(x)(x>0)$ is decreasing, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\phi(s)=\int_0^\infty K(x)x^{s-1}\,dx<\infty$, then

$$\int_0^\infty x^{p-2}\left(\sum_{n=1}^{\infty}K(nx)a_n\right)^p dx<\phi^p\left(\frac{1}{q}\right)\sum_{n=1}^{\infty}a_n^p, \tag{3}$$

$$\sum_{n=1}^{\infty}n^{p-2}\left(\int_0^\infty K(nx)f(x)\,dx\right)^p<\phi^p\left(\frac{1}{q}\right)\int_0^\infty f^p(x)\,dx. \tag{4}$$

In recent years, some new extensions of (3) and (4) were provided by [14–19].

In 2016, Hong [20, 21] also considered some equivalent statements of the extensions of (1) and (2) with a few parameters. For the following work we refer to [22–24].

In this paper, following [20], by the use of the weight functions and the idea of introducing parameters, a half-discrete Hilbert inequality with the nonhomogeneous kernel and its equivalent form are given. The equivalent statements of the constant factor are best possible related to parameters, and some particular cases are considered. The cases of a homogeneous kernel are also deduced.

2 Some lemmas

In what follows, we assume that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, σ, $\sigma_1 \leq 1$, $\sigma, \sigma_1 \in (0, \lambda)$, $f(x)$ is a nonnegative measurable function in $\mathbb{R}_+ = (0, \infty)$, $a_n \geq 0$ ($n \in \mathbb{N} = \{1, 2, \ldots\}$), such that

$$0 < \int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1} f^p(x)\,dx < \infty, \qquad 0 < \sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1} a_n^q < \infty.$$

Lemma 1 *Define the following weight functions*:

$$\omega_\sigma(\sigma_1, n) := n^\sigma \int_0^\infty \frac{1}{(1+xn)^\lambda} x^{\sigma_1 - 1}\,dx \quad (n \in \mathbb{N}), \tag{5}$$

$$\varpi_{\sigma_1}(\sigma, x) := x^{\sigma_1} \sum_{n=1}^\infty \frac{1}{(1+xn)^\lambda} n^{\sigma-1} \quad (x \in \mathbb{R}_+). \tag{6}$$

We have the following equality and inequalities:

$$\omega_\sigma(\sigma_1, n) = B(\sigma_1, \lambda - \sigma_1) n^{\sigma - \sigma_1} \quad (n \in \mathbb{N}), \tag{7}$$

$$\left(B(\sigma, \lambda - \sigma) - \frac{x^\sigma}{\sigma}\right) x^{\sigma_1 - \sigma} < \varpi_{\sigma_1}(\sigma, x) < B(\sigma, \lambda - \sigma) \cdot x^{\sigma_1 - \sigma} \quad (x \in \mathbb{R}_+), \tag{8}$$

where $B(u, v) := \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\,dt$ $(u, v > 0)$ *is the Beta function.*

Proof Setting $u = xn$, we have

$$\begin{aligned}
\omega_\sigma(\sigma_1, n) &= n^\sigma \int_0^\infty \frac{1}{(1+u)^\lambda} \left(\frac{u}{n}\right)^{\sigma_1 - 1} \frac{1}{n}\,du \\
&= n^{\sigma - \sigma_1} \int_0^\infty \frac{u^{\sigma_1 - 1}}{(1+u)^\lambda}\,du \\
&= B(\sigma_1, \lambda - \sigma_1) n^{\sigma - \sigma_1},
\end{aligned}$$

and then (7) follows. In view of the decreasing property, we find

$$\begin{aligned}
\varpi_{\sigma_1}(\sigma, x) &< x^{\sigma_1} \int_0^\infty \frac{t^{\sigma-1}}{(1+xt)^\lambda}\,dt \\
&= x^{\sigma_1 - \sigma} \int_0^\infty \frac{u^{\sigma - 1}}{(1+u)^\lambda}\,du \\
&= x^{\sigma_1 - \sigma} B(\sigma, \lambda - \sigma),
\end{aligned}$$

$$\begin{aligned}
\varpi_{\sigma_1}(\sigma,x) &> x^{\sigma_1}\int_1^\infty \frac{t^{\sigma-1}}{(1+xt)^\lambda}\,dt \\
&= x^{\sigma_1-\sigma}\int_x^\infty \frac{u^{\sigma-1}}{(1+u)^\lambda}\,du \\
&= x^{\sigma_1-\sigma}\left[\int_0^\infty \frac{u^{\sigma-1}\,du}{(1+u)^\lambda} - \int_0^x \frac{u^{\sigma-1}\,du}{(1+u)^\lambda}\right] \geq x^{\sigma_1-\sigma}\left(B(\sigma,\lambda-\sigma) - \int_0^x u^{\sigma-1}\,du\right) \\
&= x^{\sigma_1-\sigma}\left(B(\sigma,\lambda-\sigma) - \frac{x^\sigma}{\sigma}\right).
\end{aligned}$$

Hence, (8) follows. □

Lemma 2 *Setting $k_\lambda(\eta) := B(\eta,\lambda-\eta)(\eta=\sigma,\sigma_1)$, we have the following inequality*:

$$\begin{aligned}
I &:= \int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(1+xn)^\lambda} f(x)\,dx \\
&= \sum_{n=1}^\infty \int_0^\infty \frac{a_n}{(1+xn)^\lambda} f(x)\,dx \\
&< k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\sigma_1)\left\{\int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1} a_n^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{9}$$

Proof By Hölder's inequality (cf. [25]), we have

$$\begin{aligned}
I &= \int_0^\infty \sum_{n=1}^\infty \frac{1}{(1+xn)^\lambda}\left[\frac{x^{(1-\sigma_1)/q}}{n^{(1-\sigma)/p}} f(x)\right]\left[\frac{n^{(1-\sigma)/p}}{x^{(1-\sigma_1)/q}} a_n\right] dx \\
&\leq \left[\int_0^\infty \sum_{n=1}^\infty \frac{1}{(1+xn)^\lambda} \frac{x^{(1-\sigma_1)p/q}}{n^{1-\sigma}} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty \int_0^\infty \frac{1}{(1+xn)^\lambda} \frac{n^{(1-\sigma)q/p}}{x^{1-\sigma_1}}\,dx\, a_n^q\right]^{\frac{1}{q}} \\
&= \left[\int_0^\infty \varpi_{\sigma_1}(\sigma,x) x^{p(1-\sigma_1)-1} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty \omega_\sigma(\sigma_1,n) n^{q(1-\sigma)-1} a_n^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Then, by (7) and (8), we have (9). □

By (9), for $\sigma_1=\sigma$, we find $0<\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx<\infty$, $0<\sum_{n=1}^\infty n^{q(1-\sigma)-1} a_n^q<\infty$, and

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(1+xn)^\lambda} f(x)\,dx < k_\lambda(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{q(1-\sigma)-1} a_n^q\right]^{\frac{1}{q}}. \tag{10}$$

Lemma 3 *The constant factor $k_\lambda(\sigma) = B(\sigma,\lambda-\sigma)$ in (10) is the best possible.*

Proof For $0<\varepsilon<q\sigma$, we set

$$\tilde{a}_n = n^{\sigma-\frac{\varepsilon}{q}-1} \quad (n\in\mathbb{N}), \qquad \tilde{f}(x) = \begin{cases} \sigma+\frac{\varepsilon}{p}-1, & 0<x\leq 1, \\ 0, & x>1. \end{cases}$$

If there exists a constant $M \leq k_\lambda(\sigma)$, such that (10) is valid when replacing $k_\lambda(\sigma)$ by M, then, for $a_n = \tilde{a}_n, f = \tilde{f}$, we have

$$\tilde{I} := \int_0^\infty \sum_{n=1}^\infty \frac{\tilde{a}_n}{(1+xn)^\lambda} \tilde{f}(x)\,dx < M\left[\int_0^\infty x^{p(1-\sigma)-1}\tilde{f}^p(x)\,dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\sigma)-1}\tilde{a}_n^q\right]^{\frac{1}{q}}.$$

We obtain

$$\begin{aligned}
\tilde{I} &< M\left[\int_0^1 x^{p(1-\sigma)-1}x^{p(\sigma+\frac{\varepsilon}{p}-1)}\,dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\sigma)-1}n^{q(\sigma-\frac{\varepsilon}{q}-1)}\right]^{\frac{1}{q}} \\
&= M\left(\int_0^1 x^{\varepsilon-1}\,dx\right)^{\frac{1}{p}}\left(1+\sum_{n=2}^\infty n^{-\varepsilon-1}\right)^{\frac{1}{q}} \\
&< M\left(\int_0^1 x^{\varepsilon-1}\,dx\right)^{\frac{1}{p}}\left(1+\int_1^\infty x^{-\varepsilon-1}\,dx\right)^{\frac{1}{q}} \\
&= \frac{M}{\varepsilon}(\varepsilon+1)^{\frac{1}{q}}.
\end{aligned}$$

In view of (8), we find

$$\begin{aligned}
\tilde{I} &= \int_0^1 x^{\varepsilon-1}\left[x^{(\sigma-\frac{\varepsilon}{q})}\sum_{n=1}^\infty \frac{1}{(1+xn)^\lambda} n^{(\sigma-\frac{\varepsilon}{q})-1}\right]dx \\
&> \int_0^1 x^{\varepsilon-1}\left(B\left(\sigma-\frac{\varepsilon}{q},\lambda-\sigma+\frac{\varepsilon}{q}\right)-\frac{x^{\sigma-\frac{\varepsilon}{q}}}{\sigma-\frac{\varepsilon}{q}}\right)dx \\
&= \frac{1}{\varepsilon}B\left(\sigma-\frac{\varepsilon}{q},\lambda-\sigma+\frac{\varepsilon}{q}\right)-\frac{1}{\sigma-\frac{\varepsilon}{q}}\int_0^1 x^{\sigma+\frac{\varepsilon}{p}-1}\,dx \\
&= \frac{1}{\varepsilon}\left[B\left(\sigma-\frac{\varepsilon}{q},\lambda-\sigma+\frac{\varepsilon}{q}\right)-\frac{\varepsilon}{(\sigma-\frac{\varepsilon}{q})(\sigma+\frac{\varepsilon}{p})}\right].
\end{aligned}$$

Then we have

$$B\left(\sigma-\frac{\varepsilon}{q},\lambda-\sigma+\frac{\varepsilon}{q}\right)-\frac{\varepsilon}{(\sigma-\frac{\varepsilon}{q})(\sigma+\frac{\varepsilon}{p})} < M(\varepsilon+1)^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, in view of the continuous property of the Beta function, we find

$$\begin{aligned}
B(\sigma,\lambda-\sigma) &= \lim_{\varepsilon\to 0^+}\left[B\left(\sigma-\frac{\varepsilon}{q},\lambda-\sigma+\frac{\varepsilon}{q}\right)-\frac{\varepsilon}{(\sigma-\frac{\varepsilon}{q})(\sigma+\frac{\varepsilon}{p})}\right] \\
&\leq \lim_{\varepsilon\to 0^+} M(\varepsilon+1)^{\frac{1}{q}} = M.
\end{aligned}$$

Hence, $M = B(\sigma,\lambda-\sigma)$ is the best possible constant factor of (10). □

Setting $\tilde{\sigma} = \frac{\sigma}{p} + \frac{\sigma_1}{q}$ $(\sigma,\sigma_1 \leq 1, \sigma,\sigma_1 \in (0,\lambda))$, we may rewrite (9) as follows:

$$I < k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)\left[\int_0^\infty x^{p(1-\tilde{\sigma})-1}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\tilde{\sigma})-1}a_n^q\right]^{\frac{1}{q}}. \tag{11}$$

The parameter $\tilde{\sigma}$ in (11) also satisfies

$$\begin{aligned}
0 < k_\lambda(\tilde{\sigma}) &= k_\lambda\left(\frac{\sigma}{p} + \frac{\sigma_1}{q}\right) \\
&= \int_0^\infty \frac{1}{(1+u)^\lambda}\left(u^{\frac{\sigma-1}{p}}\right)\left(u^{\frac{\sigma_1-1}{q}}\right) du \\
&\le \left[\int_0^\infty \frac{u^{\sigma-1}}{(1+u)^\lambda} du\right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\sigma_1-1}}{(1+u)^\lambda} du\right]^{\frac{1}{q}} \\
&= k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\sigma_1) \\
&< \infty, \qquad (12)
\end{aligned}$$

by Hölder's inequality, and $\tilde{\sigma} \le \frac{1}{p} + \frac{1}{q} = 1$, $\tilde{\sigma} \in (0, \lambda)$, such that

$$k_\lambda(\tilde{\sigma}) - \frac{x^{\tilde{\sigma}}}{\tilde{\sigma}} < x^{\tilde{\sigma}} \sum_{n=1}^\infty \frac{n^{\tilde{\sigma}-1}}{(1+xn)^\lambda} < k_\lambda(\tilde{\sigma}).$$

Lemma 4 *If the constant factor* $k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\sigma_1)$ *in* (11) *is the best possible, then we have* $\sigma_1 = \sigma$.

Proof If the constant factor $k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\sigma_1)$ in (11) is the best possible, then, by (10), the unique best possible constant factor must be $k_\lambda(\tilde{\sigma})$, namely, $k_\lambda(\tilde{\sigma}) = k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\sigma_1)$. Since the condition of (12) keeps the form of equality is that there exist constants A and B, such that they are not all zero and $Au^{\sigma-1} = Bu^{\sigma_1-1}$ a.e. in $\mathbb{R}_+$. Assuming that $A \ne 0$, it follows that $u^{\sigma-\sigma_1} = \frac{B}{A}$ a.e. in $\mathbb{R}_+$, and then $\sigma - \sigma_1 = 0$, namely, $\sigma_1 = \sigma$. □

3 Main results and some corollaries

Theorem 1 *Inequality* (9) *is equivalent to the following inequalities*:

$$\begin{aligned}
J_1 &:= \left\{\sum_{n=1}^\infty n^{p(\frac{\sigma}{p}+\frac{\sigma_1}{q})-1}\left[\int_0^\infty \frac{f(x)}{(1+xn)^\lambda} dx\right]^p\right\}^{\frac{1}{p}} \\
&< k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\sigma_1) \left\{\int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1} f^p(x)\, dx\right\}^{\frac{1}{p}}, \qquad (13)
\end{aligned}$$

$$\begin{aligned}
J_2 &:= \left\{\int_0^\infty x^{q(\frac{\sigma}{p}+\frac{\sigma_1}{q})-1}\left[\sum_{n=1}^\infty \frac{a_n}{(1+xn)^\lambda}\right]^q dx\right\}^{\frac{1}{q}} \\
&< k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\sigma_1) \left\{\sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1} a_n^q\right\}^{\frac{1}{q}}. \qquad (14)
\end{aligned}$$

If the constant factor in (9) *is the best possible, then so is the constant factor in* (13) *and* (14).

Proof Suppose that (13) (or (14)) is valid. By Hölder's inequality, we have

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\left[n^{\frac{-1}{p}+(\frac{\sigma}{p}+\frac{\sigma_1}{q})}\int_0^{\infty}\frac{f(x)}{(1+xn)^{\lambda}}\,dx\right]\left[n^{\frac{1}{p}-(\frac{\sigma}{p}+\frac{\sigma_1}{q})}a_n\right] \\ &\le J_1\left\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}a_n^q\right\}^{\frac{1}{q}}, \end{aligned} \tag{15}$$

$$\begin{aligned} I &= \int_0^{\infty}\left[x^{\frac{1}{q}-(\frac{\sigma}{p}+\frac{\sigma_1}{q})}f(x)\right]\left[x^{\frac{-1}{q}+(\frac{\sigma}{p}+\frac{\sigma_1}{q})}\sum_{n=1}^{\infty}\frac{1}{(1+xn)^{\lambda}}a_n\right]dx \\ &\le \left\{\int_0^{\infty}x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}f^p(x)\,dx\right\}^{\frac{1}{p}}J_2. \end{aligned} \tag{16}$$

Then, by (13) (or (14)), we have (9). On the other hand, assuming (9) is valid, we set

$$a_n := n^{p(\frac{\sigma}{p}+\frac{\sigma_1}{q})-1}\left[\int_0^{\infty}\frac{f(x)}{(1+xn)^{\lambda}}\,dx\right]^{p-1} \quad (n\in\mathbb{N}).$$

If $J_1 = 0$, then (13) is naturally valid; if $J_1 = \infty$, then it is impossible that it makes (13) valid. Suppose that $0 < J_1 < \infty$. By (9) we have

$$\begin{aligned} &\sum_{n=1}^{\infty}n^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}a_n^q \\ &\quad = J_1^p = I < k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)\left\{\int_0^{\infty}x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}a_n^q\right\}^{\frac{1}{q}}, \end{aligned}$$

$$\left\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}a_n^q\right\}^{\frac{1}{p}} = J_1 < k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)\left\{\int_0^{\infty}x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}f^p(x)\,dx\right\}^{\frac{1}{p}},$$

namely, (13) follows.

In the same way, assuming (9) is valid, we set

$$f(x) := x^{q(\frac{\sigma}{p}+\frac{\sigma_1}{q})-1}\left[\sum_{n=1}^{\infty}\frac{1}{(1+xn)^{\lambda}}a_n\right]^{q-1} \quad (x\in\mathbb{R}).$$

If $J_2 = 0$, then (14) is naturally valid; if $J_2 = \infty$, then it is impossible that makes (14) valid. Suppose that $0 < J_2 < \infty$. By (9) we have

$$\begin{aligned} &\int_0^{\infty}x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}f^p(x)\,dx \\ &\quad = J_2^q = I < k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)\left\{\int_0^{\infty}x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}a_n^q\right\}^{\frac{1}{q}}, \end{aligned}$$

$$\left\{\int_0^{\infty}x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}f^p(x)\,dx\right\}^{\frac{1}{q}} = J_2 < k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)\left\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}a_n^q\right\}^{\frac{1}{q}},$$

namely, (14) follows. Hence, inequalities (9), (13) and (14) are equivalent.

If the constant factor in (9) is the best possible, then so is constant factor in (13) (or (14)). Otherwise, by (15) (or (16)), we would reach the contradiction that the constant factor in (9) is not the best possible. □

Theorem 2 *The statements* (i), (ii), (iii) *and* (iv) *are equivalent*:

(i) $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)$ *is independent of* p, q;

(ii) $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)$ *is expressed as a single integral*;

(iii) $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)$ *in* (9) *is the best possible constant*;

(iv) $\sigma_1 = \sigma$.

If the statement (iv) *follows, then we have the following equivalent inequalities with the best possible constant factor* $B(\sigma, \lambda - \sigma)$:

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(1+xn)^\lambda} f(x)\,dx < B(\sigma, \lambda-\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{q(1-\sigma)-1} a_n^q\right]^{\frac{1}{q}}. \tag{17}$$

$$\left[\sum_{n=1}^\infty n^{p\sigma-1}\left(\int_0^\infty \frac{f(x)}{(1+xn)^\lambda}\,dx\right)^p\right]^{\frac{1}{p}} < B(\sigma, \lambda-\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}, \tag{18}$$

$$\left[\int_0^\infty x^{q\sigma-1}\left(\sum_{n=1}^\infty \frac{a_n}{(1+xn)^\lambda}\right)^q dx\right]^{\frac{1}{q}} < B(\sigma, \lambda-\sigma)\left[\sum_{n=1}^\infty n^{q(1-\sigma)-1} a_n^q\right]^{\frac{1}{q}}. \tag{19}$$

Proof (i)⇒(ii). By (i) we have

$$k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1) = \lim_{p\to 1^+} k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1) = k_\lambda(\sigma),$$

namely, $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)$ is expressed as a single integral.

(ii)⇒(iv). In (12), if $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)$ is expressed as a single integral $k_\lambda(\frac{\sigma}{p} + \frac{\sigma_1}{q})$, then (12) keeps the form of equality. In view of the proof of Lemma 4, if and only if $\sigma_1 = \sigma$, (12) keeps the form of equality.

(iv)⇒(i). If $\sigma_1 = \sigma$, then $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1) = k_\lambda(\sigma)$, which is independent of p, q. Hence, we have (i) ↔ (ii) ⇔ (iv).

(iii)⇒(iv). By Lemma 4, we have $\sigma_1 = \sigma$. (iv)⇒(iii). By Lemma 3, $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1) = k_\lambda(\sigma)$ in (9) (for $\sigma_1 = \sigma$) is the best possible constant. Therefore, we have (iii) ⇔ (iv).

Hence, the statements (i), (ii), (iii) and (iv) are equivalent. □

Replacing x by $\frac{1}{x}$, and then $x^{\lambda-2}f(\frac{1}{x})$ by $f(x)$ in Theorem 1, setting $\sigma_1 = \lambda - \mu$, we have the following.

Corollary 1 *The following inequalities with the homogeneous kernel are equivalent*:

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} f(x)\,dx$$
$$< k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\lambda-\mu) \left\{ \int_0^\infty x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x)\,dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1} a_n^q \right\}^{\frac{1}{q}}. \tag{20}$$

$$\left\{ \sum_{n=1}^\infty n^{p(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})-1} \left[\int_0^\infty \frac{f(x)}{(x+n)^\lambda}\,dx \right]^p \right\}^{\frac{1}{p}}$$
$$< k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\lambda-\mu) \left\{ \int_0^\infty x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x)\,dx \right\}^{\frac{1}{p}}, \tag{21}$$

$$\left\{ \int_0^\infty x^{q(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})-1} \left[\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} \right]^q dx \right\}^{\frac{1}{q}}$$
$$< k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\lambda-\mu) \left\{ \sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1} a_n^q \right\}^{\frac{1}{q}}. \tag{22}$$

If the constant factor in (20) is the best possible, then so is the constant factor in (21) and (22).

Corollary 2 *The statements* (I), (II), (III) *and* (IV) *are equivalent*:

(I) $k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\lambda-\mu)$ *is independent of* p,q;

(II) $k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\lambda-\mu)$ *is expressed as a single integral*;

(III) $k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\lambda-\mu)$ *in* (20) *is the best possible constant*;

(IV) $\mu+\sigma=\lambda$.

If the statement (IV) *follows, then we have the following equivalent inequalities with the best possible constant factor* $B(\mu,\sigma)$:

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} f(x)\,dx < B(\mu,\sigma) \left[\int_0^\infty x^{p(1-\mu)-1} f^p(x)\,dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}. \tag{23}$$

$$\left\{ \sum_{n=1}^\infty n^{p\sigma-1} \left[\int_0^\infty \frac{f(x)}{(x+n)^\lambda}\,dx \right]^p \right\}^{\frac{1}{p}} < B(\mu,\sigma) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx \right]^{\frac{1}{p}}, \tag{24}$$

$$\left\{ \int_0^\infty x^{q\mu-1} \left[\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} \right]^q dx \right\}^{\frac{1}{q}} < B(\mu,\sigma) \left[\sum_{n=1}^\infty n^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}. \tag{25}$$

Remark 1 (i) For $\sigma = \frac{1}{p}(<\lambda)$ in (17), (18) and (19), we have the following equivalent inequalities with the nonhomogeneous kernel and the best possible constant factor

$B(\frac{1}{p}, \lambda - \frac{1}{p})$:

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(1+xn)^\lambda} f(x)\,dx < B\left(\frac{1}{p}, \lambda - \frac{1}{p}\right) \left(\int_0^\infty x^{p-2} f^p(x)\,dx\right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty a_n^q\right)^{\frac{1}{q}}. \tag{26}$$

$$\left\{\sum_{n=1}^\infty \left[\int_0^\infty \frac{f(x)}{(1+xn)^\lambda}\,dx\right]^p\right\}^{\frac{1}{p}} < B\left(\frac{1}{p}, \lambda - \frac{1}{p}\right) \left(\int_0^\infty x^{p-2} f^p(x)\,dx\right)^{\frac{1}{p}}, \tag{27}$$

$$\left\{\int_0^\infty x^{q-2} \left[\sum_{n=1}^\infty \frac{a_n}{(1+xn)^\lambda}\right]^q dx\right\}^{\frac{1}{q}} < B\left(\frac{1}{p}, \lambda - \frac{1}{p}\right) \left(\sum_{n=1}^\infty a_n^q\right)^{\frac{1}{q}}. \tag{28}$$

(ii) For $\sigma = \frac{1}{q} (< \lambda)$ in (17), (18) and (19), we have the following equivalent inequalities with the best possible constant factor $B(\frac{1}{q}, \lambda - \frac{1}{q})$:

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(1+xn)^\lambda} f(x)\,dx < B\left(\frac{1}{q}, \lambda - \frac{1}{q}\right) \left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{q-2} a_n^q\right)^{\frac{1}{q}}. \tag{29}$$

$$\left\{\sum_{n=1}^\infty n^{p-2} \left[\int_0^\infty \frac{f(x)}{(1+xn)^\lambda}\,dx\right]^p\right\}^{\frac{1}{p}} < B\left(\frac{1}{q}, \lambda - \frac{1}{q}\right) \left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}}, \tag{30}$$

$$\left\{\int_0^\infty \left[\sum_{n=1}^\infty \frac{a_n}{(1+xn)^\lambda}\right]^q dx\right\}^{\frac{1}{q}} < B\left(\frac{1}{q}, \lambda - \frac{1}{q}\right) \left(\sum_{n=1}^\infty n^{q-2} a_n^q\right)^{\frac{1}{q}}. \tag{31}$$

(iii) For $\lambda = 1$, $\sigma = \frac{1}{p}$, $\mu = \frac{1}{q}$ in (23), (24) and (25), we have the following equivalent inequalities with the homogeneous kernel and the best possible constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$:

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{x+n} f(x)\,dx < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty a_n^q\right)^{\frac{1}{q}}, \tag{32}$$

$$\left[\sum_{n=1}^\infty \left(\int_0^\infty \frac{f(x)}{x+n}\,dx\right)^p\right]^{\frac{1}{p}} < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}}, \tag{33}$$

$$\left[\int_0^\infty \left(\sum_{n=1}^\infty \frac{a_n}{x+n}\right)^q dx\right]^{\frac{1}{q}} < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{n=1}^\infty a_n^q\right)^{\frac{1}{q}}. \tag{34}$$

(iv) For $\lambda = 1$, $\sigma = \frac{1}{q}$, $\mu = \frac{1}{p}$ in (23), (24) and (25), we have the following equivalent inequalities with the best possible constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$:

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{x+n} f(x)\,dx < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\int_0^\infty x^{p-2} f^p(x)\,dx\right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{q-2} a_n^q\right)^{\frac{1}{q}}, \tag{35}$$

$$\left[\sum_{n=1}^{\infty} n^{p-2}\left(\int_0^{\infty} \frac{f(x)}{x+n}\,dx\right)^p\right]^{\frac{1}{p}} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\int_0^{\infty} x^{p-2} f^p(x)\,dx\right)^{\frac{1}{p}}, \tag{36}$$

$$\left[\int_0^{\infty} x^{q-2}\left(\sum_{n=1}^{\infty} \frac{a_n}{x+n}\right)^q dx\right]^{\frac{1}{q}} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{n=1}^{\infty} n^{q-2} a_n^q\right)^{\frac{1}{q}}. \tag{37}$$

4 Conclusions

In this paper, by using the weight functions and the idea of introducing parameters, a half-discrete Hilbert inequality with the nonhomogeneous kernel and its equivalent form are given in Theorem 1. The equivalent statements of the constant factor being best possible related to parameters, and some particular cases are considered in Theorem 2 and Remark 1. The cases of homogeneous kernel are deduced in Corollary 1 and Corollary 2. The lemmas and theorems provide an extensive account of this type of inequalities.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. ZH participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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References

1. Hardy, G.H., Littlewood, J.E., Polya, G.: Inequalities. Cambridge University Press, Cambridge (1934)
2. Yang, B.C.: The Norm of Operator and Hilbert-Type Inequalities. Science Press, Beijing, China (2009)
3. Yang, B.C.: Hilbert-Type Integral Inequalities. Bentham Science Publishers Ltd., The United, Arab Emirates (2009)
4. Yang, B.C.: On the norm of an integral operator and applications. J. Math. Anal. Appl. **321**, 182–192 (2006)
5. Xu, J.S.: Hardy-Hilbert's inequalities with two parameters. Adv. Math. **36**(2), 63–76 (2007)
6. Yang, B.C.: On the norm of a Hilbert's type linear operator and applications. J. Math. Anal. Appl. **325**, 529–541 (2007)
7. Xie, Z.T., Zeng, Z., Sun, Y.F.: A new Hilbert-type inequality with the homogeneous kernel of degree –2. Adv. Appl. Math. Sci. **12**(7), 391–401 (2013)
8. Zhen, Z., Raja Rama Gandhi, K., Xie, Z.T.: A new Hilbert-type inequality with the homogeneous kernel of degree –2 and with the integral. Bull. Math. Sci. Appl. **3**(1), 11–20 (2014)
9. Xin, D.M.: A Hilbert-type integral inequality with the homogeneous kernel of zero degree. Math. Theory Appl. **30**(2), 70–74 (2010)
10. Azar, L.E.: The connection between Hilbert and Hardy inequalities. J. Inequal. Appl. **2013**, Article ID 452 (2013)
11. Batbold, T., Sawano, Y.: Sharp bounds for m-linear Hilbert-type operators on the weighted Morrey spaces. Math. Inequal. Appl. **20**, 263–283 (2017)
12. Adiyasuren, V., Batbold, T., Krnic, M.: Multiple Hilbert-type inequalities involving some differential operators. Banach J. Math. Anal. **10**, 320–337 (2016)
13. Adiyasuren, V., Batbold, T., Krni´c, M.: Hilbert-type inequalities involving differential operators, the best constants and applications. Math. Inequal. Appl. **18**, 111–124 (2015)
14. Rassias, M.T., Yang, B.C.: On half-discrete Hilbert's inequality. Appl. Math. Comput. **220**, 75–93 (2013)
15. Yang, B.C., Krnic, M.: A half-discrete Hilbert-type inequality with a general homogeneous kernel of degree 0. J. Math. Inequal. **6**(3), 401–417 (2012)
16. MTh, R., Yang, B.C.: A multidimensional half-discrete Hilbert-type inequality and the Riemann zeta function. Appl. Math. Comput. **225**, 263–277 (2013)
17. MTh, R., Yang, B.C.: On a multidimensional half -discrete Hilbert - type inequality related to the hyperbolic cotangent function. Appl. Math. Comput. **242**, 800–813 (2013)

18. Huang, Z.X., Yang, B.C.: On a half-discrete Hilbert-type inequality similar to Mulholland's inequality. J. Inequal. Appl. **2013**, Article ID 290 (2013)
19. Yang, B.C., Lebnath, L.: Half-Discrete Hilbert-Type Inequalities. World Scientific Publishing, Singapore (2014)
20. Hong, Y., Wen, Y.: A necessary and sufficient condition of that Hilbert type series inequality with homogeneous kernel has the best constant factor. Ann. Math. **37A**(3), 329–336 (2016)
21. Hong, Y.: On the structure character of Hilbert's type integral inequality with homogeneous kernel and applications. J. Jilin Univ. Sci. Ed. **55**(2), 189–194 (2017)
22. Hong, Y., Huang, Q.L., Yang, B.C., Liao, J.L.: The necessary and sufficient conditions for the existence of a kind of Hilbert-type multiple integral inequality with the non-homogeneous kernel and its applications. J. Inequal. Appl. **2017**, Article ID 316 (2017)
23. Xin, D.M., Yang, B.C., Wang, A.Z.: Equivalent property of a Hilbert-type integral inequality related to the beta function in the whole plane. J. Funct. Spaces **2018**, Article ID ID2691816 (2018)
24. Hong, Y., He, B., Yang, B.C.: Necessary and sufficient conditions for the validity of Hilbert type integral inequalities with a class of quasi-homogeneous kernels and its application in operator theory. JIPAM. J. Inequal. Pure Appl. Math. **12**(3), 777–788 (2018)
25. Kuang, J.C.: Applied Inequalities. Shangdong Science and Technology Press, Jinan, China (2004)