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A variational inequality of Kirchhoff-type in $\mathbb{R}^N$

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Abstract

In this paper, we investigate the existence of nontrivial radial solutions for a kind of variational inequalities in $\mathbb{R}^N$. Our main technique is the non-smooth critical point theory, based on the Szulkin-type functionals.

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Keywords: Mountain pass theorem; Kirchhoff-type; Variational inequality; Radial solutions

1 Introduction

Variational inequalities describe a lot of phenomena in the real world and have a wide range of applications in physics, mechanics, engineering etc.; see, for example, [1–3, 5–7, 9, 10, 12–14, 18]. This paper is concerned with a kind of variational inequalities in $\mathbb{R}^N$, the aim is to prove the existence of infinite radial solutions under suitable conditions.

Let $H^1_{O(N)}(\mathbb{R}^N)$ be the Sobolev space of $O(N)$ invariant functions (see the definition in Sect. 3), and B be a closed convex set in $H^1_{O(N)}(\mathbb{R}^N)$ with $0 \in B$. Our problem, denoted by (Q), is to find $u \in B$ such that

$$\begin{aligned}&\left(a + b\int_{\mathbb{R}^N}\left(|\nabla u|^2 + u^2\right)dx\right)\left(\int_{\mathbb{R}^N}\nabla u\cdot\nabla(v-u)\,dx\right.\\&\quad\left.+\int_{\mathbb{R}^N}u(v-u)\,dx\right) - \int_{\mathbb{R}^N}g(x,u)(v-u)\,dx \ge 0, \quad \text{for all } v \in B,\end{aligned}$$

where $a, b > 0, N \ge 2$ and $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

This problem is related to the obstacle problems, extensively studied due to the physical applications (see [15, 17]).

It is well known that the variational inequality is discussed in different ways in the case of regional bounded and unbounded. In [4], on the bounded interval $(0,1)$, a class of variational inequalities of Kirchhoff-type is discussed by applying the non-smooth critical point theory based on Szulkin functionals [16]. In [11], the authors study a kind of variational inequality defined on $(0,\infty)$. Motivated by the above work, in this paper we want to study the radial solutions of the problem (Q) by using two kinds of theorem in [16]. Our research scope is an extension of some problems studied by [4] and [11]. Since the domain is unbounded and the continuous embedding $H^1(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is not compact. We consider the symmetric method of the action of a group, similar to [8], to overcome this difficulty.

Meanwhile, suppose the function g satisfies:
(g_1) $\lim_{|u|\to 0} \frac{g(x,u)}{|u|} = 0$ uniformly for $x \in \mathbb{R}^N$.
(g_2) For $1 < p < 2^* - 1$ and there exists $c > 0$ such that

$$\left|g(x,u)\right| \le c\left(1 + |u|^p\right), \quad \text{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R},$$

where

$$2^* - 1 = \begin{cases} \frac{N+2}{N-2}, & N \ge 3, \\ +\infty, & N = 1, 2. \end{cases}$$

(g_3) There is a constant $\mu > 4$ such that

$$ug(x,u) \ge \mu G(x,u) = \int_0^u g(x,s)\,ds, \quad \text{for all } x \in \mathbb{R}^N, \text{ and } u \in \mathbb{R}^N.$$

(g_4) $\lim_{|u|\to +\infty} \frac{G(x,u)}{u^4} \to +\infty$ uniformly for all $x \in \mathbb{R}^N$.
(g_5) $g(x,u) = g(zx,u)$ for any $z \in O(N)$ and $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.
(g_6) $g(x,u) = -g(x,-u)$ for any $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.
We state the main result of this paper.

Theorem 1.1 *If assumptions* (g_1)–(g_5) *hold, then the problem* (Q) *has a nontrivial radial solution in* B. *Furthermore, if the condition* (g_6) *holds, then the problem* (Q) *has infinitely many pairs of nontrivial radial solutions in* B.

The structure of the paper is as follows. In Sect. 2, we review some preliminaries. Section 3 gives the proof of our main result.

2 Szulkin-type functionals

Let X be a real Banach space and denote by X^* its dual. Let $T = \Phi + \psi$ with $\Phi \in C^1(X,\mathbb{R})$ and let $\psi : X \to \mathbb{R} \cup \{+\infty\}$ be convex, lower semicontinuous. Then $T = \Phi + \psi$ is a Szulkin-type functional. A point $u \in X$ is called critical if $\psi(u) \ne +\infty$ and

$$\Phi'(u)(v-u) + \psi(v) - \psi(u) \ge 0 \quad \text{for all } v \in X,$$

or equivalently

$$0 \in \Phi'(u) + \partial\psi(u) \quad \text{in } X^*,$$

where $\partial\psi(u)$ is called the subdifferential of ψ at u.

Definition 2.1 ([16]) The functional $T = \Phi + \psi$ fulfills the (PS) condition at level $c \in \mathbb{R}$; it can be written as $(PSZ)_c$ if every sequence $\{u_n\} \subset X$ such that $\lim_{n\to\infty} T(u_n) = c$ and

$$\left\langle \Phi'(u_n), (v-u_n)\right\rangle_X + \psi(v) - \psi(u_n) \ge \varepsilon_n \|v-u_n\| \quad \text{for all } v \in X,$$

where $\varepsilon_n \to 0$, possesses a convergent subsequence.

Lemma 2.2 ([16], Mountain pass theorem) *Suppose that $T = \Phi + \psi : X \to \mathbb{R} \cup \{+\infty\}$ be a Szulkin-type functional and that*

(i) $T(0) = 0$ *and there exist* $\alpha, \rho > 0$ *such that* $T(u) \geq \alpha$ *for all* $\|u\| = \rho$;

(ii) $T(e) \leq 0$ *for some* $e \in X$ *with* $\|e\| > \rho$.

If T satisfies the $(PSZ)_c$-condition, then T has a critical value $c \geq \alpha$ which may be characterized by

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} T\bigl(\gamma(t)\bigr),$$

where $\Gamma = \{\gamma \in c([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$.

Lemma 2.3 ([16], Corollary 4.8) *Suppose that $T = \Phi + \psi : E \to \mathbb{R} \cup \{+\infty\}$ is an even Szulkin-type functional and satisfies the $(PSZ)_c$-condition with $T(0) = 0$. If $E = X \oplus Y$, where X is a finite dimensional, and assume also that*

(A_1) *there are constants* $\alpha, \rho > 0$ *such that* $T|_{\partial F_\rho \cap Y} \geq \alpha$;

(A_2) *for any positive integer k, there is k-dimensional subspace $E_k \subset E$, such that $T(u) \to -\infty$ as $\|u\| \to +\infty$, $u \in E_k$.*

Then T has infinitely many pairs of nontrivial critical points, where $F_\rho = \{u \in E : \|u\| < \rho\}$.

3 The proof of the main result

Let

$$H := H^1\bigl(\mathbb{R}^N\bigr) = \bigl\{u \in L^2\bigl(\mathbb{R}^N\bigr) : \nabla u \in L^2\bigl(\mathbb{R}^N\bigr)\bigr\}$$

be the Sobolev space with inner product and corresponding norm

$$\langle u, v\rangle := \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)\, dx, \qquad \|u\| = \left(\int_{\mathbb{R}^N} \bigl(|\nabla u|^2 + u^2\bigr)\, dx\right)^{\frac{1}{2}}.$$

Denote by $\|\cdot\|_p$ the norm of $L^p(\mathbb{R}^N)$, i.e. $\|u\|_p = (\int_{\mathbb{R}^N} |u|^p\, dx)^{\frac{1}{p}}$.

Let $O(N)$ is an orthogonal transformation group on $\mathbb{R}^N$. We have that

$$E = H^1_{O(N)}\bigl(\mathbb{R}^N\bigr) := \bigl\{u \in H \mid zu(x) := u\bigl(z^{-1}x\bigr) = u(x), \forall z \in O(N)\bigr\}$$

is a subspace of $H^1(\mathbb{R}^N)$, and it is invariant. We note that the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ is compact when $s \in (2, 2^*)$ by Corollary 1.26 of [19]. Define the functional $\Phi : E \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}b\|u\|^4 - \Psi(u), \tag{3.1}$$

where $\Psi(u) := \int_{\mathbb{R}^N} G(x, u)\, dx$, and the indicator function of the set B as follows:

$$\psi_B(u) := \begin{cases} 0, & \text{if } u \in B, \\ +\infty, & \text{otherwise.} \end{cases}$$

The function $\psi_B(u)$ is convex, proper, even, and lower semicontinuous. In order to show that $T = \Phi + \psi_B$ is a Szulkin-type functional, we need the following proposition.

Proposition 3.1 *Every critical point* $u \in E$ *of* $T = \Phi + \psi_B$ *is a solution of* (Q).

Proof Since $u \in E$ of $T = \Phi + \psi_B$ is a critical point, we have

$$\Phi'(u)(v-u) + \psi_B(v) - \psi_B(u) \geq 0 \quad \text{for all } v \in E.$$

It is clear that u belongs to B. If not, we get $\psi_B = +\infty$, and in the inequality above, setting $v = 0 \in B$ we get a contradiction. We fix $v \in B$. Since

$$\begin{aligned}\Phi'(u)(v-u) = {} & \bigl(a + b\|u\|^2\bigr)\left(\int_{\mathbb{R}^N} \nabla u \nabla (v-u)\,dx + \int_{\mathbb{R}^N} u(v-u)\,dx\right) \\ & - \int_{\mathbb{R}^N} g\bigl(x, u(x)\bigr)(v-u)\,dx \geq 0,\end{aligned}$$

u is a solution of (Q). □

Proposition 3.2 *Suppose that* g *satisfies the conditions* (g_1) *and* (g_2) *and* $\langle \Psi'(u), v\rangle = \int_{\mathbb{R}^N} g(x,u)v\,dx$, *then* $\Phi \in C^1(E, \mathbb{R})$,

$$\bigl\langle \Phi'(u), v\bigr\rangle = \left(a + b\int_{\mathbb{R}^N} \bigl(|\nabla u|^2 + u^2\bigr)\,dx\right)\int_{\mathbb{R}^N} (\nabla u \nabla v + uv)\,dx - \bigl\langle \Psi'(u), v\bigr\rangle.$$

Proof By (3.1), we only need to prove that

$$\Psi \in C^1(H, \mathbb{R}), \quad \bigl\langle \Psi'(u), v\bigr\rangle = \int_{\mathbb{R}^N} g(x,u)v\,dx, \quad \forall u, v \in H.$$

Thus, we divide the whole proof into the following two steps.

Step 1. We verify that Ψ is a Gateaux derivative.

For small enough $\varepsilon > 0$, using (g_1) and (g_2), there is a positive constant c depend on ε such that

$$\bigl|g(x,u)\bigr| \leq \varepsilon|u| + c(\varepsilon)|u|^p \tag{3.2}$$

for every $(x,u) \in \mathbb{R}^N \times \mathbb{R}$. For any $u(x), v(x) \in H$ and $0 < |t| < 1$, according to (3.2) and using the mean value theorem, there exists $\theta \in (0,1)$ such that

$$\begin{aligned}\frac{|G(x, u+tv) - G(x,u)|}{|t|} &= \bigl|g(x, u+\theta tv)v\bigr| \\ &\leq \varepsilon|u||v| + \varepsilon|v|^2 + c(\varepsilon)\bigl(|u+\theta tv|\bigr)^p|v| \\ &\leq \varepsilon|u||v| + \varepsilon|v|^2 + 2^p c(\varepsilon)\bigl(|u|^p|v| + |v|^{p+1}\bigr).\end{aligned}$$

By the Hölder inequality, it follows that

$$h := \varepsilon|u||v| + \varepsilon|v|^2 + 2^p c(\varepsilon)\bigl(|u|^p|v| + |v|^{p+1}\bigr) \in L^1\bigl(R^N\bigr).$$

So, by the Lebesgue dominated convergence theorem, we have

$$\bigl\langle \Psi'(u), v\bigr\rangle = \int_{\mathbb{R}^N} g(x,u)v\,dx.$$

Step 2. We show that $\Psi'(\cdot): H \to H^*$ is continuous.

Suppose that $u_n \to u$ in H. Since the imbedding $H \hookrightarrow L^s(\mathbb{R}^N)(2 \le s \le 2^*)$ is continuous, we see that, for each $s \in [2, 2^*]$, there is a constant $\eta_s > 0$ such that

$$\|w\|_s \le \eta_s \|w\|, \quad \forall w \in H^1(\mathbb{R}^N), \qquad u_n \to u \quad \text{in } L^s(\mathbb{R}^N).$$

Note that

$$\begin{aligned}\left\|\Psi'(u_n) - \Psi'(u)\right\| &= \sup_{\|v\|\le 1} \left| \int_{\mathbb{R}^N} \big(g(x,u_n) - g(x,u)\big) v \, dx \right| \\ &\le \sup_{\|v\|\le 1} \int_{\mathbb{R}^N} \left|\big(g(x,u_n) - g(x,u)\big)\right| |v| \, dx.\end{aligned}$$

According to the Hölder inequality, and Theorem A.4 in [19], we have

$$\sup_{\|v\|\le 1} \int_{\mathbb{R}^N} \left|\big(g(x,u_n) - g(x,u)\big)\right| |v| \, dx \to 0$$

as $n \to \infty$. So, we obtain $\|\Psi'(u_n) - \Psi'(u)\| \to 0$, and thus the claim is proven. Consequently, $T = \Phi + \psi_B$ is a Szulkin-type functional. □

It follows from (g_5) that T is $O(N)$-invariant, i.e. for all $(z,u) \in O(N) \times H$, $T(u) = T(zu)$, and the action of the group $O(N)$ on H is isometric, i.e. for all $(z,u) \in O(N) \times H$, $\|u\| = \|zu\|$. Furthermore, because of Lemma 2.2 and Theorem 1.28 of [19], we notice that u is a critical point of $T|_E$ if and only if u is a critical point of T in H. We will use the symmetric mountain pass theorem to obtain the critical points of the functional $T|_E$.

Proposition 3.3 *If the continuous function f fulfills (g_3) and (g_4), then $T = \Phi + \psi_B$ fulfills $(PSZ)_c$-condition for every $c \in \mathbb{R}$.*

Proof Fix $c \in \mathbb{R}$. Set $\{u_n\} \subset E$ such that

$$T(u_n) = \Phi(u_n) + \psi_B(u_n) \to c, \tag{3.3}$$

$$\Phi'(u_n)(v - u_n) + \psi_B(v) - \psi_B(u_n) \ge -\varepsilon_n \|v - u_n\|, \quad \forall v \in E, \tag{3.4}$$

where $\varepsilon_n \to 0$ in $[0, \infty)$. According to (3.3), obviously, we notice that the sequence $\{u_n\} \subset B$. Setting $v = 2u_n$ in (3.4) we have

$$\Phi'(u_n)(u_n) \ge -\varepsilon_n \|u_n\|.$$

Thus

$$a\|u_n\|^2 + b\|u_n\|^4 - \int_{\mathbb{R}^N} g\big(x, u_n(x)\big) u_n(x) \, dx \ge -\varepsilon_n \|u_n\|. \tag{3.5}$$

On the basis of (3.3), for large enough $n \in N$, we get

$$c + 1 \ge \frac{1}{2} a\|u_n\|^2 + \frac{1}{4} b\|u_n\|^4 - \int_{\mathbb{R}^N} G(x, u_n) \, dx. \tag{3.6}$$

Multiply both sides of inequality (3.5) by μ^{-1}, adding it to another inequality (3.6), and applying the condition (g_3). When $n \in N$ is sufficiently large, we have

$$\begin{aligned}
&c + 1 + \frac{1}{\mu}\|u_n\| \\
&\quad \geq a\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 + b\left(\frac{1}{4} - \frac{1}{\mu}\right)\|u_n\|^4 \\
&\qquad - \int_{\mathbb{R}^N}\left(G\bigl(x, u_n(x)\bigr) - \frac{1}{\mu}g\bigl(x, u_n(x)\bigr)u_n(x)\right)dx \\
&\quad = a\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 + b\left(\frac{1}{4} - \frac{1}{\mu}\right)\|u_n\|^4 \\
&\qquad - \frac{1}{\mu}\int_{\mathbb{R}^N}\bigl(\mu G\bigl(x, u_n(x)\bigr) - g\bigl(x, u_n(x)\bigr)u_n(x)\bigr)\,dx \\
&\quad \geq a\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 + b\left(\frac{1}{4} - \frac{1}{\mu}\right)\|u_n\|^4.
\end{aligned}$$

Since $\mu > 4$, the sequence $\{u_n\}$ is bounded in B. Then there exists a subsequence converging weakly in E. According to the compactness embedding $E \hookrightarrow\hookrightarrow L^s(\mathbb{R}^N)$. Without loss of generality, assume

$$u_n \rightharpoonup u \quad \text{in } E; \tag{3.7}$$

$$u_n \to u \quad \text{in } L^s\bigl(\mathbb{R}^N\bigr), s \in \bigl(2, 2^*\bigr). \tag{3.8}$$

By observing that B is weakly closed, we get $u \in B$. Let again $v = u$ in (3.4), we have

$$\bigl(a + b\|u_n\|^2\bigr)\langle u_n, u - u_n\rangle_E + \int_{\mathbb{R}^N} g\bigl(x, u_n(x)\bigr)\bigl(u_n(x) - u(x)\bigr)\,dx \geq -\varepsilon_n\|u - u_n\|. \tag{3.9}$$

We use

$$\bigl(a + b\|u_n\|^2\bigr)\|u - u_n\|^2 = \bigl(a + b\|u_n\|^2\bigr)\langle u - u_n, u - u_n\rangle_E. \tag{3.10}$$

So, for large enough n and any $\varepsilon > 0$, it follows from (3.9) and (3.10) that

$$\begin{aligned}
&\bigl(a + b\|u_n\|^2\bigr)\|u - u_n\|^2 \\
&\quad \leq \bigl(a + b\|u_n\|^2\bigr)\langle u, u - u_n\rangle_E + \int_{\mathbb{R}^N} g(x, u_n)(u_n - u)\,dx + \varepsilon_n\|u - u_n\| \\
&\quad \leq \bigl(a + b\|u_n\|^2\bigr)\langle u, u - u_n\rangle_E + \int_{\mathbb{R}^N}\bigl(\varepsilon|u_n| + c(\varepsilon)|u_n|^p\bigr)|u - u_n|\,dx + \varepsilon_n\|u - u_n\| \\
&\quad \leq \bigl(a + b\|u_n\|^2\bigr)\langle u, u - u_n\rangle_E + \varepsilon c_1 + c(\varepsilon)\|u_n - u\|_{p+1}\|u_n\|_{p+1}^p + \varepsilon_n\|u - u_n\| \\
&\quad \leq \bigl(a + b\|u_n\|^2\bigr)\langle u, u - u_n\rangle_E + \varepsilon c_1 + c_2c(\varepsilon)\|u_n - u\|_{p+1} + \varepsilon_n\|u - u_n\|,
\end{aligned}$$

where the constants c_1 and c_2 are independent of n and ε. By (3.7) and the fact that $\{u_n\}$

is bounded in E, we obtain

$$\lim_n \big(a + b\|u_n\|^2\big)\langle u, u - u_n\rangle_E = 0.$$

Taking into account (3.8), $\|u_n - u\|_{p+1} \to 0$. Setting $\varepsilon_n \to 0^+$, then we have proved that

$$\big(a + b\|u_n\|^2\big)\|u - u_n\|^2 \to 0.$$

Consequently, we get $u_n \to u$ in E. This means that the proof of this conclusion has been completed. □

Now we give the proof of Theorem 1.1.

Proof By (3.2), for any $0 < \varepsilon < \frac{a}{\eta_2^2}$ (η_2 is continuous imbedding constant $E \hookrightarrow L^2(\mathbb{R}^N)$), we obtain

$$\big|G(x,u)\big| \le \int_0^1 \big|g(x,tu)u\big|\,dt \le \frac{\varepsilon}{2}|u|^2 + \frac{c(\varepsilon)}{p+1}|u|^{p+1}, \quad \text{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

The condition (g_4) implies $p > 4$. Therefore, for small enough $\rho > 0$, we have

$$\begin{aligned} T(u) &\ge \frac{1}{2}a\|u\|^2 + \frac{1}{4}b\|u\|^4 - \frac{\varepsilon}{2}\|u\|_2^2 - \frac{c(\varepsilon)}{p+1}\|u\|_{p+1}^{p+1} \\ &\ge \frac{1}{2}\big(a - \eta_2^2\varepsilon\big)\|u\|^2 + \frac{1}{4}b\|u\|^4 - \frac{c(\varepsilon)}{p+1}c_{p+1}^{p+1}\|u\|^{p+1} \\ &\ge \frac{1}{4}\big(a - \eta_2^2\varepsilon\big)\|u\|^2 + \frac{1}{4}b\|u\|^4, \end{aligned}$$

for all $u \in \overline{F}_\rho$. Thus,

$$T|_{\partial F_\rho} \ge \frac{1}{4}\big(a - \eta_2^2\varepsilon\big)\rho^2 + \frac{1}{4}b\rho^4 := \alpha > 0.$$

Let $\{e_i\}$ be a complete normal orthogonal basis of E. Take $X = \operatorname{span}\{e_1, e_2, \dots, e_n\}$ and $Y = X^\perp$. Then $E = X \oplus Y$. Thus,

$$T|_{\partial F_\rho \cap Y} \ge \alpha > 0.$$

For every finite dimensional subspace $\widetilde{E} \subset E$, there exists $k \in N^+$ such that $\widetilde{E} \subset E_k$. Due to the equivalence of all norms in a finite dimensional space, for some positive constant c_4 we have

$$\|u\|_4 \ge c_4\|u\|, \quad \text{for all } u \in E_k.$$

According to the conditions (g_1), (g_2), and (g_4), we note that, for $D > \frac{b}{4c_4^4}$, there exists a positive constant $C(D)$ such that

$$G(x,u) \ge D|u|^4 - C(D)|u|^2, \quad \text{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

So, fixing $u_0 \in B\backslash\{0\} \subset E_k$, and taking $u = su_0 (s > 0)$, we get

$$\begin{aligned} T(su_0) &\leq \frac{1}{2}as^2\|u_0\|^2 + \frac{1}{4}bs^4\|u_0\|^4 - Ds^4\|u_0\|_4^4 + C(D)s^2\|u_0\|_2^2 \\ &\leq \frac{1}{2}as^2\|u_0\|^2 - \left(Dc_4^4s^4 - \frac{1}{4}bs^4\right)\|u_0\|^4 + C(D)\eta_2^2s^2\|u_0\|^2. \end{aligned}$$

Obviously, we have $T(su_0) \to -\infty$ as $s \to +\infty$. Therefore, we take s ($e = su_0$) large enough such that $\|e\| > \rho$ and $T(e) < 0$.

By Proposition 3.3, we know that T satisfies the $(PSZ)_c$-condition($c \in \mathbb{R}$), and $T(0) = 0$. So T has a critical value according to Lemma 2.2. We remark that the critical point $u_1 \in E$ associated to the critical value η is nontrivial due to $T(u_1) = \eta > 0 = T(0)$. From Proposition 3.1, we notice that $u_1 \in B$ and it is a radial solution of (Q).

If the condition (g_6) holds, then T is even. Similar to the previous discussion, we see that all conditions of Lemma 2.3 are satisfied. Therefore, the second conclusion of Theorem 1.1 is obtained. □

Example 3.4 For $n = 1, 2, 3, \ldots$, considering $g(x, u) = u^{2n+1}|u|^{\frac{2n+1}{2}}$, it is satisfied with all assumptions of Theorem 1.1.

4 Conclusion

In this article, the existence of nontrivial radial solutions to problem (Q) is established by using the variational methods under suitable conditions. We consider a variational inequality of Kirchhoff-type in $\mathbb{R}^N$, which improves the previous results. In order to overcome new difficulties, we need to adopt symmetric method of the action of a group in our paper.

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Each of the authors contributed to each part of this study equally, all authors read and approved the final manuscript.

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