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Bonnesen-style inequalities on surfaces of constant curvature

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Abstract

In this paper, some Bonnesen-style inequalities on a surface $\mathbb{X}_\kappa$ of constant curvature κ (i.e., the Euclidean plane $\mathbb{R}^2$, projective plane $\mathbb{R}P^2$, or hyperbolic plane $\mathbb{H}^2$) are proved. The method is integral geometric and gives a uniform proof of some Bonnesen-style inequalities alone with equality conditions.

MSC: 52A22

Keywords: Isoperimetric inequality; Bonnesen-style inequality; Isoperimetric deficit; Surface of constant curvature

1 Introduction

The classical isoperimetric problem dates back to antique literature and geometry. The problem can be stated as: Among all closed curves of given length in the Euclidean plane $\mathbb{R}^2$, which one maximizes the area of its enclosed region?

The solution to the problem is usually expressed in the form of an inequality that relates the length P_K of a rectifiable simple closed curve and the area A_K of the planar region K that the curve encloses in $\mathbb{R}^2$. The solution to the classical isoperimetric problem is characterized as the following isoperimetric inequality:

$$P_K^2 - 4\pi A_K \geq 0, \tag{1.1}$$

with equality if and only if K is a Euclidean disc.

The history of geometric proofs for the classical isoperimetric problem goes back to Ancient Greeks and was recorded by Pappus of Alexandria in the fourth century AD, but their arguments were incomplete. The first progress towards the solution was made by Steiner [22] in 1838 by a geometric method later named Steiner symmetrization. His proof contained a flaw that later was fixed by analytic approach. In 1870, Weierstrass gave the first rigorous proof as a corollary of his theory of calculus of several variables. Since then, many other proofs have been discovered. In 1902, Hurwitz [10] published a short proof using the Fourier series that applies to arbitrary rectifiable curves (not assumed to be smooth). An elegant direct proof based on comparison of a simple closed curve of length L with the circle of radius $\frac{L}{2\pi}$ was given by Schmidt [20]. See [4, 15, 16, 21] for more references.

In 1920s, Bonnesen proved a series of inequalities of the form [3]

$$P_K^2 - 4\pi A_K \geq B_K, \tag{1.2}$$

where B_K is a non-negative invariant and vanishes if and only if the domain K is a Euclidean disc.

A well-known Bonnesen-style inequality is

$$P_K^2 - 4\pi A_K \geq \pi^2 (R_K - r_K)^2, \tag{1.3}$$

where R_K and r_K respectively denote the circumradius and inradius of K, with equality if and only if K is a Euclidean disc.

Many inequalities of style (1.2), called Bonnesen-style inequalities, were found along with variations and generalizations in the past decades [2, 5, 6, 8, 11, 19, 29–34]. On the other hand, the classical isoperimetric inequality has been extended to higher dimensions and a surface of constant curvature κ, i.e., the Euclidean plane $\mathbb{R}^2$, projective plane $\mathbb{R}P^2$, or hyperbolic plane $\mathbb{H}^2$.

Let K be a compact set bounded by a rectifiable simple closed curve with the area A_K and perimeter P_K in $\mathbb{X}_\kappa$. Then [1, 7, 9, 14, 17, 18, 23–26]

$$P_K^2 - (4\pi - \kappa A_K) A_K \geq 0, \tag{1.4}$$

with equality if and only if K is a geodesic disc.

The geodesic disc of radius r with center x is defined as

$$B_\kappa(x, r) = \bigl\{ y \in \mathbb{X}_\kappa : d(x, y) \leq r \bigr\},$$

where d is the geodesic distance function in $\mathbb{X}_\kappa$. The area, perimeter of $B_\kappa(x,r)$ in $\mathbb{X}_\kappa$ are respectively [12]

$$A\bigl(B_\kappa(x,r)\bigr) = \frac{2\pi}{\kappa}\bigl(1 - \mathrm{cn}_\kappa(r)\bigr), \qquad P\bigl(B_\kappa(x,r)\bigr) = 2\pi \, \mathrm{sn}_\kappa(r). \tag{1.5}$$

The limiting cases of as $\kappa \to 0$ yield the Euclidean formulas $A(B(x,r)) = \pi r^2$ and $P(B(x,r)) = 2\pi r$.

A Bonnesen-type inequality in $\mathbb{X}_\kappa$ is of the form

$$P_K^2 - (4\pi - \kappa A_K) A_K \geq B_K, \tag{1.6}$$

where B_K vanishes if and only if K is a geodesic disc [15, 28].

Bonnesen [3] established an inequality of the type (1.6) in the sphere of radius $1/\sqrt{\kappa}$:

$$P_K^2 - (4\pi - \kappa A_K) A_K \geq 4\pi^2 \, \mathrm{tn}_\kappa^2\left(\frac{R_K - r_K}{2}\right), \tag{1.7}$$

where R_K and r_K are respectively the minimum circumscribed radius and the maximum inscribed radius of K.

Let $\mathbb{X}_\kappa$ be the surface of constant curvature κ, specifically:

$$\mathbb{X}_\kappa = \begin{cases} \mathbb{P}R^2, \text{Euclidean 2-sphere of radius } 1/\sqrt{\kappa}, & \text{if } \kappa > 0; \\ \mathbb{R}^2, \text{Euclidean plane}, & \text{if } \kappa = 0; \\ \mathbb{H}^2, \text{Hyperbolic plane of constant curvature } \kappa, & \text{if } \kappa < 0. \end{cases}$$

Let

$$\Delta_\kappa(K) = P_K^2 - (4\pi - \kappa A_K)A_K \tag{1.8}$$

denote the isoperimetric deficit of K in $\mathbb{X}_\kappa$. The trigonometric functions appearing in (1.7) are defined by

$$\mathrm{sn}_\kappa(t) = \begin{cases} \frac{1}{\sqrt{-\kappa}}\sinh(\sqrt{-\kappa}t), & \kappa < 0, \\ t, & \kappa = 0, \\ \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t), & \kappa > 0; \end{cases}$$

$$\mathrm{cn}_\kappa(t) = \begin{cases} \cosh(\sqrt{-\kappa}t), & \kappa < 0, \\ 1, & \kappa = 0, \\ \cos(\sqrt{\kappa}t), & \kappa > 0; \end{cases}$$

$$\mathrm{tn}_\kappa(t) = \frac{\mathrm{sn}_\kappa(t)}{\mathrm{cn}_\kappa(t)}; \qquad \mathrm{ct}_\kappa(t) = \frac{\mathrm{cn}_\kappa(t)}{\mathrm{sn}_\kappa(t)};$$

and

$$\kappa \cdot \mathrm{sn}_\kappa^2(t) + \mathrm{cn}_\kappa^2(t) = 1. \tag{1.9}$$

The following Bonnesen-type inequality is obtained in [31]:

$$\Delta_\kappa(K) \ge \left(2\pi - \frac{\kappa}{2}A_K\right)^2 \left(\mathrm{tn}_\kappa \frac{R_K}{2} - \mathrm{tn}_\kappa \frac{r_K}{2}\right)^2 \tag{1.10}$$

for a convex set K, with equality if K is a geodesic disc.

Inequality (1.10) was strengthened [28] as

$$\begin{aligned} \Delta_\kappa(K) \ge{}& \left(2\pi - \frac{\kappa}{2}A_K\right)^2 \left(\mathrm{tn}_\kappa \frac{R_K}{2} - \mathrm{tn}_\kappa \frac{r_K}{2}\right)^2 \\ &+ \left(2\pi - \frac{\kappa}{2}A_K\right)^2 \left(\mathrm{tn}_\kappa \frac{R_K}{2} + \mathrm{tn}_\kappa \frac{r_K}{2} - \frac{2P_K}{4\pi - \kappa A_K}\right)^2, \end{aligned} \tag{1.11}$$

with equality if K is a geodesic disc.

For a convex set K in $\mathbb{X}_\kappa$ such that $(2\pi - \kappa A_K)^2 + \kappa P_K^2 \ge 0$ if $\kappa < 0$, Klain [12] obtained the following Bonnesen-style inequality:

$$\Delta_\kappa(K) \ge \frac{((2\pi - \kappa A_K)^2 + \kappa P_K^2)^2}{4(2\pi - \kappa A_K)^2}\big(\mathrm{sn}_\kappa(R_K) - \mathrm{sn}_\kappa(r_K)\big)^2, \tag{1.12}$$

with equality if K is a geodesic disc.

For more results on Bonnesen-style inequality, see, e.g., [1, 2, 5–9, 11, 14, 17, 19, 23–26, 29–34].

The purpose of this paper is to find a new Bonnesen-style inequality with equality condition on surfaces $\mathbb{X}_\kappa$ of constant curvature, especially on the hyperbolic plane $\mathbb{H}^2$ by integral geometric method. We are going to seek the following Bonnesen-style inequality for a convex set K in $\mathbb{X}_\kappa$:

$$\Delta_\kappa(K) \geq \pi^2\big(\mathrm{tn}_\kappa(R_K) - \mathrm{tn}_\kappa(r_K)\big)^2,$$

with equality if and only if K is a hyperbolic disc.

Finally, we give some special cases of these Bonnesen-style inequalities that strengthen some known Bonnesen-style inequalities in the Euclidean plane including the Bonnesen isoperimetric inequality (1.3).

2 Preliminaries

Let $\mathcal{C}(\mathbb{X}_\kappa)$ be the set of all convex sets with perimeter $P_K \leq \frac{2\pi}{\sqrt{\kappa}}$ if $\kappa > 0$ in $\mathbb{X}_\kappa$. For a fixed point $x_0 \in \mathbb{X}_\kappa$, the geodesic disc of radius r with center x_0 is the set of points that lie at most a distance r from x_0 in $\mathbb{X}_\kappa$. For $K \in \mathbb{X}_\kappa$, let A_K and P_K denote the area and the perimeter of K, respectively. Let r_K and R_K be the maximum inscribed radius and the minimum circumscribed radius of K, respectively. We always assume that K lies in an open hemisphere of $\mathbb{P}R^2$ such that $R_K < \frac{\pi}{2\sqrt{\kappa}}$.

A set K is convex if, for points $x, y \in K$, the shortest geodesic curve connecting x, y belongs to K. It should be noted that, for a convex set K in $\mathbb{P}R^2$, $2\pi - \kappa A_K > 0$.

Let G_κ be the group of isometries in $\mathbb{X}_\kappa$, and let dg be the kinematic density (Harr measure) on G_κ. Let K be fixed and gL as moving via the isometry $g \in G_\kappa$. For $K, L \in \mathbb{X}_\kappa$, let $\chi(K \cap gL)$ and $\sharp(\partial K \cap \partial(gL))$ be the Euler–Poincaré characteristic of $K \cap gL$ and the number of points of the intersection $\partial K \cap \partial(gL)$, respectively.

The following fundamental kinematic formula is due to Blaschke [19]:

$$\int_{\{g:K\cap gL\neq\emptyset\}} \chi(K \cap gL)\,dg = 2\pi(A_K + A_L) + P_K P_L - \kappa A_K A_L. \tag{2.1}$$

As the limiting case, when K, L degenerate to curves ∂K, ∂L, respectively, then $A_K = A_L = 0$ and the perimeters are $2P_K$, $2P_L$. Hence we have the following kinematic formula of Poincaré [19]:

$$\int_{\{g:\partial K\cap\partial(gL)\neq\emptyset\}} \sharp\big(\partial K \cap \partial(gL)\big)\,dg = 4P_K P_L. \tag{2.2}$$

Since the compact sets are assumed to be simply connected and enclosed by simple curves, $\chi(K \cap gL) = n(g) \equiv$ (the number of connected components of the intersection $K \cap gL$). Let $\mu = \{g \in G_\kappa : K \subset gL \text{ or } K \supset gL\}$, then the fundamental kinematic formula of Blaschke (2.1) can be rewritten as [31]:

$$\int_\mu dg + \int_{\{g:\partial K\cap\partial(gL)\neq\emptyset\}} n(g)\,dg = 2\pi(A_K + A_L) + P_K P_L - \kappa A_K A_L. \tag{2.3}$$

When $\partial K \cap \partial(gL) \neq \emptyset$, each component of $K \cap gL$ is bounded by at least an arc of ∂K and an arc of $\partial(gL)$, and $n(g) \leq \sharp(\partial K \cap \partial(gL))/2$. Then the following containment measure inequality is an immediate consequence of Poincaré 's formula (2.2) and Blaschke's formula (2.3) [12, 13, 19].

Lemma 2.1 *Let K, L be two compact sets bounded by rectifiable simple closed curves in $\mathbb{X}_\kappa$, then*

$$\int_\mu dg \geq 2\pi(A_K + A_L) - P_K P_L - \kappa A_K A_L. \tag{2.4}$$

If $K \equiv L$, then there is no $g \in G_\kappa$ such that $gK \supset K$ nor $gK \subset K$. Hence $\int_\mu dg = 0$ and inequality (2.4) immediately leads to the isoperimetric inequality (1.2).

The following Bonnesen inequality in $\mathbb{X}_\kappa$ is important for our main results [27].

Lemma 2.2 *For $K \in \mathcal{C}(\mathbb{X}_\kappa)$, let R_K and r_K be respectively the maximum inscribed radius and the minimum circumscribed radius of K. Then, for $r_K \leq r \leq R_K$,*

$$\left[(2\pi - \kappa A_K)^2 + \kappa P_K^2\right] \mathrm{sn}_\kappa^2(r) - 4\pi P_K \,\mathrm{sn}_\kappa(r) - A_K(\kappa A_K - 4\pi) \leq 0. \tag{2.5}$$

Proof Let L be a geodesic disc of radius r. Then neither $gB_\kappa(r) \subset K$ nor $gB_\kappa(r) \supset K$ for any $g \in G_\kappa$ and hence the measure $\int_\mu dg = 0$. By (1.5) and (2.4) we have

$$P_K \,\mathrm{sn}_\kappa(r) - \left(\frac{2\pi}{\kappa} - A_K\right)\left(1 - \mathrm{cn}_\kappa(r)\right) - A_K \geq 0. \tag{2.6}$$

Identity (1.9) shows $1 - \kappa \cdot \mathrm{sn}_\kappa^2(r) = \mathrm{cn}_\kappa^2(r) > 0$ and inequality (2.6) can be rewritten as

$$P_K \,\mathrm{sn}_\kappa(r) - \frac{2\pi}{\kappa} \geq \left(A_K - \frac{2\pi}{\kappa}\right)\sqrt{1 - \kappa \cdot \mathrm{sn}_\kappa^2(r)}. \tag{2.7}$$

For $\kappa \geq 0$,

$$P_K \,\mathrm{sn}_\kappa(r) - \frac{2\pi}{\kappa} \leq 0,$$

hence by squaring both sides of (2.7) we have

$$\left(P_K \,\mathrm{sn}_\kappa(r) - \frac{2\pi}{\kappa}\right)^2 \leq \left(A_K - \frac{2\pi}{\kappa}\right)^2 \left(1 - \kappa \cdot \mathrm{sn}_\kappa^2(r)\right),$$

that is,

$$\left((2\pi - \kappa A_K)^2 + \kappa P_K^2\right) \mathrm{sn}_\kappa^2(r) - 4\pi P_K \,\mathrm{sn}_\kappa(r) - A_K(\kappa A_K - 4\pi) \leq 0.$$

For $\kappa < 0$, then $A_K - \frac{2\pi}{\kappa} > 0$. Squaring both sides of (2.7) leads to

$$\left(P_K \,\mathrm{sn}_\kappa(r) - \frac{2\pi}{\kappa}\right)^2 \geq \left(A_K - \frac{2\pi}{\kappa}\right)^2 \left(1 - \kappa \cdot \mathrm{sn}_\kappa^2(r)\right),$$

i.e.,

$$\left((2\pi - \kappa A_K)^2 + \kappa P_K^2\right) \mathrm{sn}_\kappa^2(r) - 4\pi P_K \,\mathrm{sn}_\kappa(r) - A_K(\kappa A_K - 4\pi) \leq 0. \qquad \square$$

We are now in the position to prove our Bonnesen-style inequalities.

Theorem 2.1 *Let $K \in \mathcal{C}(\mathbb{X}_\kappa)$. If $r_K \le r \le R_K$, then*

$$\Delta_\kappa(K) \ge \frac{(P_K - 2\pi\,\mathrm{sn}_\kappa(r))^2}{\mathrm{cn}_\kappa^2(r)}, \tag{2.8}$$

with equality if K is a geodesic disc.

Proof Inequality (2.5) can be rewritten as

$$P_K^2 - A_K(4\pi - \kappa A_K) \ge P_K^2 + \big((2\pi - \kappa A_K)^2 + \kappa P_K^2\big)\,\mathrm{sn}_\kappa^2(r) - 4\pi P_K\,\mathrm{sn}_\kappa(r).$$

Since $(2\pi - \kappa A_K)^2 + \kappa P_K^2 = 4\pi^2 + \kappa\Delta_\kappa(K)$, we have

$$\big(1 - \kappa\ \mathrm{sn}_\kappa^2(r)\big)\Delta_\kappa(K) \ge \big(P_K - 2\pi\,\mathrm{sn}_\kappa(r)\big)^2.$$

Via (1.9), that is, $1 - \kappa \cdot \mathrm{sn}_\kappa^2(r) = \mathrm{cn}_\kappa^2(r)$, then the previous inequality results in (2.8) and we complete the proof. □

Theorem 2.2 *Let $K \in \mathcal{C}(\mathbb{X}_\kappa)$, then*

$$\begin{aligned}\Delta_\kappa(K) \ge c\Big\{&2\pi^2\big[\mathrm{sn}_\kappa(R_K) - \mathrm{sn}_\kappa(r_K)\big]^2 + 2\big[P_K - \pi\,\mathrm{sn}_\kappa(R_K) - \pi\,\mathrm{sn}_\kappa(r_K)\big]^2\\ &- \kappa\big[\mathrm{sn}_\kappa^2(R_K)\big(P_K - 2\pi\,\mathrm{sn}_\kappa(r_K)\big)^2 + \mathrm{sn}_\kappa^2(r_K)\big(P_K - 2\pi\,\mathrm{sn}_\kappa(R_K)\big)^2\big]\Big\},\end{aligned}$$

where $c = 1/(2\,\mathrm{cn}_\kappa^2(R_K)\,\mathrm{cn}_\kappa^2(r_K))$, and the equality holds if K is a geodesic disc.

Proof Inequality (2.8) holds for $r = R_K$ and $r = r_K$, respectively:

$$\Delta_\kappa(K) \ge \frac{(P_K - 2\pi\,\mathrm{sn}_\kappa(R_K))^2}{\mathrm{cn}_\kappa^2(R_K)};$$
$$\Delta_\kappa(K) \ge \frac{(P_K - 2\pi\,\mathrm{sn}_\kappa(r_K))^2}{\mathrm{cn}_\kappa^2(r_K)}.$$

By adding the above two inequalities side by side, we have

$$\begin{aligned}\Delta_\kappa(K) &\ge c\big[\mathrm{cn}_\kappa^2(r_K)\big(P_K - 2\pi\,\mathrm{sn}_\kappa(R_K)\big)^2 + \mathrm{cn}_\kappa^2(R_K)\big(P_K - 2\pi\,\mathrm{sn}_\kappa(r_K)\big)^2\big]\\ &= c\big[\big(1 - \kappa\,\mathrm{sn}_\kappa^2(r_K)\big)\big(P_K - 2\pi\,\mathrm{sn}_\kappa(R_K)\big)^2 + \big(1 - \kappa\,\mathrm{sn}_\kappa^2(R_K)\big)\big(P_K - 2\pi\,\mathrm{sn}_\kappa(r_K)\big)^2\big]\\ &= c\Big\{\big(P_K - 2\pi\,\mathrm{sn}_\kappa(R_K)\big)^2 + \big(P_K - 2\pi\,\mathrm{sn}_\kappa(r_K)\big)^2\\ &\quad - \kappa\big[\mathrm{sn}_\kappa^2(R_K)\big(P_K - 2\pi\,\mathrm{sn}_\kappa(r_K)\big)^2 + \mathrm{sn}_\kappa^2(r_K)\big(P_K - 2\pi\,\mathrm{sn}_\kappa(R_K)\big)^2\big]\Big\}.\end{aligned}$$

Via elementary calculations we obtain the desired Bonnesen-style inequality. □

For $\kappa < 0$, the following Bonnesen-style inequalities are immediate consequences of Theorem 2.2 with equality conditions.

Corollary 2.1 *Let $K \in \mathcal{C}(\mathbb{H}^2)$, then*

$$\Delta_\kappa(K) \ge 2c\Big\{\pi^2\big[\mathrm{sn}_\kappa(R_K) - \mathrm{sn}_\kappa(r_K)\big]^2 + \big[P_K - \pi\,\mathrm{sn}_\kappa(R_K) - \pi\,\mathrm{sn}_\kappa(r_K)\big]^2\Big\}, \tag{2.9}$$

where $c = 1/(2\,\mathrm{cn}_\kappa^2(R_K)\,\mathrm{cn}_\kappa^2(r_K))$, and the equality holds if K is a hyperbolic disc.

Corollary 2.2 *Let $K \in \mathcal{C}(\mathbb{H}^2)$, then*

$$\Delta_\kappa(K) \geq 2c\bigl(P_K - \pi\ \mathrm{sn}_\kappa(R_K) - \pi\ \mathrm{sn}_\kappa(r_K)\bigr)^2, \tag{2.10}$$

where $c = 1/(2\,\mathrm{cn}_\kappa^2(R_K)\,\mathrm{cn}_\kappa^2(r_K))$, with equality if and only if K is a hyperbolic disc.

Proof Since

$$\begin{aligned}
&\pi^2\bigl[\mathrm{sn}_\kappa(R_K) - \mathrm{sn}_\kappa(r_K)\bigr]^2 + \bigl[P_K - \pi\ \mathrm{sn}_\kappa(R_K) - \pi\ \mathrm{sn}_\kappa(r_K)\bigr]^2 \\
&\quad \geq \bigl[P_K - \pi\ \mathrm{sn}_\kappa(R_K) - \pi\ \mathrm{sn}_\kappa(r_K)\bigr]^2,
\end{aligned}$$

with equality holds if and only if $R_K = r_K$, that is, K must be a hyperbolic disc, the Bonnesen-style inequality (2.10) follows from inequality (2.9) immediately. □

3 Bonnesen-style inequalities in $\mathbb{H}^2$

We are seeking more Bonnesen-style inequalities in $\mathbb{H}^2$.

Theorem 3.1 *Let $K \in \mathcal{C}(\mathbb{H}^2)$, then*

$$\Delta_\kappa(K) \geq \pi^2\bigl(\mathrm{tn}_\kappa(R_K) - \mathrm{tn}_\kappa(r_K)\bigr)^2 + \frac{P_K^2}{4}\left(\frac{1}{\mathrm{cn}_\kappa(r_K)} - \frac{1}{\mathrm{cn}_\kappa(R_K)}\right)^2, \tag{3.1}$$

with equality if K is a hyperbolic disc.

Proof By Theorem 2.1, $r = R_K$ and $r = r_K$ respectively lead to

$$\Delta_\kappa(K) \geq \frac{(2\pi\ \mathrm{sn}_\kappa(R_K) - P_K)^2}{\mathrm{cn}_\kappa^2(R_K)} \quad \text{and} \quad \Delta_\kappa(K) \geq \frac{(P_K - 2\pi\ \mathrm{sn}_\kappa(r_K))^2}{\mathrm{cn}_\kappa^2(r_K)}.$$

Adding two inequalities side by side and by the inequality $x^2 + y^2 \geq \frac{(x+y)^2}{2}$, we have

$$\begin{aligned}
2\Delta_\kappa(K) &\geq \frac{(2\pi\ \mathrm{sn}_\kappa(R_K) - P_K)^2}{\mathrm{cn}_\kappa^2(R_K)} + \frac{(P_K - 2\pi\ \mathrm{sn}_\kappa(r_K))^2}{\mathrm{cn}_\kappa^2(r_K)} \\
&\geq \frac{1}{2}\left\{2\pi\left(\mathrm{tn}_\kappa(R_K) - \mathrm{tn}_\kappa(r_K)\right) + \left(\frac{P_K}{\mathrm{cn}_\kappa(r_K)} - \frac{P_K}{\mathrm{cn}_\kappa(R_K)}\right)\right\}^2.
\end{aligned}$$

For $K \in \mathcal{C}(\mathbb{H}^2)$, tn_κ and sn_κ are respectively hyperbolic tangent $\tanh(x)$ and hyperbolic cosine $\cosh(x)$ that are strictly increasing on $[0,\infty)$. Therefore, for $r_K \leq R_K$,

$$\mathrm{tn}_\kappa(R_K) - \mathrm{tn}_\kappa(r_K) \geq 0; \qquad \frac{1}{\mathrm{cn}_\kappa(r_K)} - \frac{1}{\mathrm{cn}_\kappa(R_K)} \geq 0.$$

By inequality $(x+y)^2 \geq x^2 + y^2$ $(x \geq 0, y \geq 0)$, we have

$$2\Delta_\kappa(K) \geq 2\pi^2\bigl(\mathrm{tn}_\kappa(R_K) - \mathrm{tn}_\kappa(r_K)\bigr)^2 + \frac{P_K^2}{2}\left(\frac{1}{\mathrm{cn}_\kappa(r_K)} - \frac{1}{\mathrm{cn}_\kappa(R_K)}\right)^2. \qquad \square$$

The following Bonnesen-style inequality with equality condition for $K \in \mathcal{C}(\mathbb{H}^2)$ is a direct consequence of Theorem 3.1.

Corollary 3.1 *Let $K \in \mathcal{C}(\mathbb{H}^2)$, then*

$$\Delta_\kappa(K) \geq \pi^2 \big(\mathrm{tn}_\kappa(R_K) - \mathrm{tn}_\kappa(r_K)\big)^2, \tag{3.2}$$

with equality if and only if K is a hyperbolic disc.

Proof By slightly complicated elementary calculations, we have

$$\pi^2 \big(\mathrm{tn}_\kappa(R_K) - \mathrm{tn}_\kappa(r_K)\big)^2 + \frac{P_K^2}{4}\left(\frac{1}{\mathrm{cn}_\kappa(r_K)} - \frac{1}{\mathrm{cn}_\kappa(R_K)}\right)^2 \geq \pi^2 \big(\mathrm{tn}_\kappa(R_K) - \mathrm{tn}_\kappa(r_K)\big)^2. \tag{3.3}$$

Then inequality (3.2) follows from (3.1) and (3.3) immediately.

Equality holds in (3.2) and (3.3) if and only if either $P_K = 0$, which implies that K is a single point, or $R_K = r_K$, which means that K is a hyperbolic disc. □

Since

$$\pi^2 \big(\mathrm{tn}_\kappa(R_K) - \mathrm{tn}_\kappa(r_K)\big)^2 + \frac{P_K^2}{4}\left(\frac{1}{\mathrm{cn}_\kappa(r_K)} - \frac{1}{\mathrm{cn}_\kappa(R_K)}\right)^2 \geq \frac{P_K^2}{4}\left(\frac{1}{\mathrm{cn}_\kappa(r_K)} - \frac{1}{\mathrm{cn}_\kappa(R_K)}\right)^2,$$

with equality if and only if $R_K = r_K$, which implies that K must be a hyperbolic disc.

Combining this inequality with inequality (3.1) immediately leads to the following Bonnesen-style inequality.

Corollary 3.2 *Let $K \in \mathcal{C}(\mathbb{H}^2)$, then*

$$\Delta_\kappa(K) \geq \frac{P_K^2}{4}\left(\frac{1}{\mathrm{cn}_\kappa(r_K)} - \frac{1}{\mathrm{cn}_\kappa(R_K)}\right)^2, \tag{3.4}$$

with equality if and only if K is a hyperbolic disc.

4 The limiting cases on the Euclidean plane $\mathbb{R}^2$

The limiting cases of Bonnesen-style inequalities obtained in the previous sections are known as Bonnesen-style inequalities in the Euclidean plane $\mathbb{R}^2$.

Corollary 4.1 *Let K be a compact convex set in $\mathbb{R}^2$. If $r_K \leq r \leq R_K$, then*

$$P_K^2 - 4\pi A_K \geq (P_K - 2\pi r)^2, \tag{4.1}$$

with equality if K is a Euclidean disc.

Proof For $\kappa < 0$, let $\kappa = -\frac{1}{R^2}$. Then inequality (2.8) becomes

$$P_K^2 - 4\pi A_K - \frac{A_K^2}{R^2} \geq \frac{(P_K - 2\pi R \sinh(\frac{r}{R}))^2}{\cosh^2(\frac{r}{R})}.$$

As $R \to \infty$, the inequality above leads to the following inequality by L'Hôpital's rule:

$$P_K^2 - 4\pi A_K \geq \lim_{R\to\infty}\left(P_K - 2\pi R \sinh\left(\frac{r}{R}\right)\right)^2 = (P_K - 2\pi r)^2.$$

□

Corollary 4.2 *Let K be a compact convex set in $\mathbb{R}^2$, then*

$$P_K^2 - 4\pi A_K \geq \pi^2(R_K - r_K)^2, \tag{4.2}$$

with equality if and only if K is a Euclidean disc.

Proof For $\kappa = -\frac{1}{R^2}$, inequality (3.2) becomes

$$P_K^2 - 4\pi A_K - \frac{A_K^2}{R^2} \geq \pi^2 \left(\frac{R\sinh(\frac{R_K}{R})}{\cosh(\frac{R_K}{R})} - \frac{R\sinh(\frac{r_K}{R})}{\cosh(\frac{r_K}{R})} \right)^2.$$

As $R \to \infty$, the inequality above becomes

$$P_K^2 - 4\pi A_K \geq \lim_{R\to\infty} \left(R\sinh\left(\frac{R_K}{R}\right) - R\sinh\left(\frac{r_K}{R}\right) \right)^2 = \pi^2(R_K - r_K)^2.$$

The inequality holds as an equality if and only if $R_K = r_K$, that is, K is a Euclidean disc. □

The limiting case of Theorem 2.2 is the following strengthening inequality of (4.2).

Corollary 4.3 *Let K be a compact convex set in $\mathbb{R}^2$. Then*

$$P_K^2 - 4\pi A_K \geq \pi^2(R_K - r_K)^2 + (P_K - \pi R_K - \pi r_K)^2,$$

with equality if K is a Euclidean disc.

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Authors' contributions
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