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# Central limit theorems for sub-linear expectation under the Lindeberg condition

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**Abstract**

In this paper, we investigate the central limit theorems for sub-linear expectation for a sequence of independent random variables without assumption of identical distribution. We first give a bound on the distance between the normalized sum distribution and G-normal distribution which can be used to derive the central limit theorem for sub-linear expectation under the Lindeberg condition. Then we obtain the central limit theorem for capacity under the Lindeberg condition. We also get the central limit theorem for capacity for summability methods under the Lindeberg condition.

**MSC:** 60F05

**Keywords:** Central limit theorem; Sub-linear expectation; Capacity; Lindeberg condition; G-normal distribution

## 1 Introduction

Peng [15] put forward the theory of sub-linear expectation to describe the probability uncertainties in statistics and economics which are difficult to be handled by classical probability theory. There has been increasing interest in sub-linear expectation (see, for example, [1, 2, 4, 11, 18, 26]).

The classical central limit theorem (CLT for short) is a fundamental result in probability theory. Peng [16] initiated the CLT for sub-linear expectation for a sequence of i.i.d. random variables with finite $(2 + \alpha)$-moments for some $\alpha > 0$. The CLT for sub-linear expectation has gotten considerable development. Hu and Zhang [10] obtained a CLT for capacity. Li and Shi [13] got a CLT for sub-linear expectation without assumption of identical distribution. Hu [9] extended Peng's CLT by weakening the assumptions of test functions. Zhang and Chen [21] derived a weighted CLT for sub-linear expectation. Hu and Zhou [12] presented some multi-dimensional CLTs without assumption of identical distribution. Li [14] proved a CLT for sub-linear expectation for a sequence of m-dependent random variables. Rokhlin [19] gave a CLT under the Lindeberg condition under classical probability with variance uncertainty. Zhang [22] gained a CLT for sub-linear expectation under a moment condition weaker than $(2 + \alpha)$-moments. Zhang [23] established a martingale CLT and functional CLT for sub-linear expectation under the Lindeberg condition.

The purpose of this paper is to investigate the CLTs for sub-linear expectation for a sequence of independent random variables without assumption of identical distribution. We first give a bound on the distance between the normalized sum distribution $\mathbb{E}[\varphi(\frac{S_n}{B_n})]$ and

G-normal distribution $\widetilde{\mathbb{E}}[\varphi(\xi)]$, where $\xi \sim \mathcal{N}(\{0\}; [\underline{\sigma}^2, 1])$. It can be used to derive the CLT for sub-linear expectation under the Lindeberg condition directly, which coincides with the result in Zhang [23]. Different from the classical case, when choosing $\underline{B}_n$ as the normalizing factor, we can also obtain a bound on the distance between the normalized sum distribution $\mathbb{E}[\varphi(\frac{S_n}{\underline{B}_n})]$ and the corresponding G-normal random variable $\mathbb{E}[\varphi(\eta)]$ where $\eta \sim \mathcal{N}(\{0\}; [1, \overline{\sigma}^2])$. Secondly, we obtain a CLT for capacity under the Lindeberg condition which extends the CLT for capacity for a sequence of i.i.d. random variables in Hu and Zhang [10]. We also study the CLT for capacity for summability methods under the Lindeberg condition. The regular summability method is an important subject in functional analysis. In recent years it has been found that summability method plays an important role in the study of statistical convergence (see [5–7, 20]). So it is meaningful to investigate the CLT for capacity for summability methods.

This paper is organized as follows. In Sect. 2, we recall some basic concepts and lemmas related to the main results. In Sect. 3, we give a bound on the distance between the normalized sum distribution and G-normal distribution. In Sect. 4, we prove a CLT for capacity under the Lindeberg condition. In Sect. 5, we show a CLT for capacity for summability methods under the Lindeberg condition.

## 2  Basic concepts and lemmas

This paper is studied under the sub-linear expectation framework established by Peng [15–18]. Let $(\Omega, \mathcal{F})$ be a given measurable space. Let $\mathcal{H}$ be a linear space of real functions defined on $\Omega$ such that if $X_1, X_2, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, X_2, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n)$ where $C_{l,\mathrm{Lip}}(\mathbb{R}^n)$ denotes the linear space of local Lipschitz continuous functions $\varphi$ satisfying

$$\left| \varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y}) \right| \leq C\left(1 + |\boldsymbol{x}|^m + |\boldsymbol{y}|^m\right)|\boldsymbol{x} - \boldsymbol{y}|, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,$$

for some $C > 0$, $m \in \mathbb{N}$ depending on $\varphi$. $\mathcal{H}$ contains all $I_A$ where $A \in \mathcal{F}$. We also denote $C_{b,\mathrm{Lip}}(\mathbb{R}^n)$ as the linear space of bounded Lipschitz continuous functions $\varphi$ satisfying

$$\left| \varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y}) \right| \leq C|\boldsymbol{x} - \boldsymbol{y}|, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,$$

for some $C > 0$.

**Definition 2.1**  A functional $\mathbb{E} \colon \mathcal{H} \to \overline{\mathbb{R}}$ is said to be a sub-linear expectation if it satisfies: for $\forall X, Y \in \mathcal{H}$,

  (a)  Monotonicity: $X \geq Y$ implies $\mathbb{E}[X] \geq \mathbb{E}[Y]$.
  (b)  Constant preserving: $\mathbb{E}[c] = c, \forall c \in \mathbb{R}$.
  (c)  Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0$.
  (d)  Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$ whenever $\mathbb{E}[X] + \mathbb{E}[Y]$ is well defined.
  The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sub-linear expectation space.

*Remark* 2.1  The sub-linear expectation $\mathbb{E}[\cdot]$ satisfies translation invariance: $\mathbb{E}[X + c] = \mathbb{E}[X] + c, \forall c \in \mathbb{R}$.

**Definition 2.2** ([3]) A set function $V : \mathcal{F} \to [0,1]$ is called a capacity if it satisfies

(a) $V(\emptyset) = 0$, $V(\Omega) = 1$.
(b) $V(A) \leq V(B)$, $A \subset B$, $A, B \in \mathcal{F}$.

**Definition 2.3** For a capacity $V$, a set $A$ is a polar set if $V(A) = 0$. And we say a property holds "quasi-surely" (q.s.) if it holds outside a polar set.

**Definition 2.4** A sub-linear expectation $\mathbb{E} : \mathcal{H} \to \overline{\mathbb{R}}$ is said to be continuous if it satisfies:

(1) continuity from below: $X_n \uparrow X$ implies $\mathbb{E}[X_n] \uparrow \mathbb{E}[X]$, where $0 \leq X_n, X \in \mathcal{H}$.
(2) continuity from above: $X_n \downarrow X$ implies $\mathbb{E}[X_n] \downarrow \mathbb{E}[X]$, where $0 \leq X_n, X \in \mathcal{H}$.

A capacity $V : \mathcal{F} \to [0,1]$ is said to be continuous if it satisfies:

(1) continuity from below: $A_n \uparrow A$ implies $V(A_n) \uparrow V(A)$, where $A_n, A \in \mathcal{F}$.
(2) continuity from above: $A_n \downarrow A$ implies $V(A_n) \downarrow V(A)$, where $A_n, A \in \mathcal{F}$.

The conjugate expectation $\mathcal{E}$ of sub-linear expectation $\mathbb{E}$ is defined by

$$\mathcal{E}[X] := -\mathbb{E}[-X], \quad \forall X \in \mathcal{H}.$$

Obviously, for all $X \in \mathcal{H}$, $\mathcal{E}[X] \leq \mathbb{E}[X]$. A pair of capacities can be induced as follows:
$\mathbb{V}(A) := \mathbb{E}[I_A]$, $v(A) := \mathcal{E}[I_A] = 1 - \mathbb{V}(A^c)$, $\forall A \in \mathcal{F}$.

**Definition 2.5** ([15–18]) (Independence) $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ $(Y_i \in \mathcal{H})$ is said to be independent of $\boldsymbol{X} = (X_1, \ldots, X_m)$ $(X_i \in \mathcal{H})$ if, for each test function $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^m \times \mathbb{R}^n)$,

$$\mathbb{E}\big[\varphi(\boldsymbol{X}, \boldsymbol{Y})\big] = \mathbb{E}\big[\mathbb{E}\big[\varphi(\boldsymbol{x}, \boldsymbol{Y})\big]|_{\boldsymbol{x} = \boldsymbol{X}}\big],$$

whenever the sub-linear expectations are finite.

$\{X_n\}_{n=1}^{\infty}$ is said to be a sequence of independent random variables if $X_{n+1}$ is independent of $(X_1, \ldots, X_n)$ for each $n \geq 1$.

Let $\boldsymbol{X}$ be an $n$-dimensional random variable on a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. We define a functional on $C_{l,\mathrm{Lip}}(\mathbb{R}^n)$ such that

$$\mathbb{F}_{\boldsymbol{X}}[\varphi] := \mathbb{E}\big[\varphi(\boldsymbol{X})\big], \quad \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n) \to \mathbb{R}.$$

Then $\mathbb{F}_{\boldsymbol{X}}[\cdot]$ can be regarded as the distribution of $\boldsymbol{X}$ under $\mathbb{E}$ and it characterizes the uncertainty of the distribution of $\boldsymbol{X}$.

**Definition 2.6** ([15–18]) (Identical distribution) Two $n$-dimensional random variables $\boldsymbol{X}_1, \boldsymbol{X}_2$ on respective sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$ are called identically distributed, denoted by $\boldsymbol{X}_1 \overset{d}{=} \boldsymbol{X}_2$, if

$$\mathbb{E}_1\big[\varphi(\boldsymbol{X}_1)\big] = \mathbb{E}_2\big[\varphi(\boldsymbol{X}_2)\big], \quad \forall \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n),$$

whenever the sub-linear expectations are finite.

**Definition 2.7** ([15–18])  A one-dimensional random variable $\xi$ on sub-linear expectation $(\Omega, \mathcal{H}, \mathbb{E})$ is said to be $G$-normal distributed, denoted by $\xi \sim \mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$, if for any $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R})$ the following function defined by

$$u(t,x) := \mathbb{E}[\varphi(x + \sqrt{t}\xi)], \quad (t,x) \in [0, \infty) \times \mathbb{R},$$

is the unique viscosity solution of the following parabolic partial differential equation (PDE) defined on $[0, \infty) \times R$:

$$\begin{cases} \partial_t u - G(\partial_{xx} u) = 0, \\ u|_{t=0} = \varphi, \end{cases}$$

where $G(a) = \frac{1}{2} a^+ \overline{\sigma}^2 - \frac{1}{2} a^- \underline{\sigma}^2, a \in \mathbb{R}$.

*Remark* 2.2  The $G$-normal distributed random variable $\xi$ satisfies: $a\xi + b\overline{\xi} \stackrel{d}{=} \sqrt{a^2 + b^2}\xi$, $\forall a, b \geq 0$, where $\overline{\xi} \stackrel{d}{=} \xi$ and $\overline{\xi}$ is independent of $\xi$. This implies $\mathbb{E}[\xi] = \mathbb{E}[-\xi] = 0$.

Next we recall the definition of G-expectation. Let $\widetilde{\Omega} = C[0, \infty)$ be a space of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t \geq 0}$ with $\omega_0 = 0$, equipped with distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} \left[ \left( \max_{t \in [0,i]} |\omega_t^1 - \omega_t^2| \right) \wedge 1 \right].$$

Denote $W_t(\omega) := \omega_t$ for each $\omega \in \widetilde{\Omega}$ and

$$L_{ip}(\widetilde{\Omega}) := \left\{ \varphi(W_{t_1}, \ldots, W_{t_k}) : k \in \mathbb{N}, t_1, \ldots, t_k \in [0, \infty), \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^k) \right\}.$$

For each given monotonic and sub-linear function $G(a) = \frac{1}{2} a^+ \overline{\sigma}^2 - \frac{1}{2} a^- \underline{\sigma}^2, 0 \leq \underline{\sigma} \leq \overline{\sigma} < \infty$, $a \in \mathbb{R}$, let the canonical process $(W_t)_{t \geq 0}$ be G-Brownian motion on a G-expectation space $(\widetilde{\Omega}, L_{ip}(\widetilde{\Omega}), \widetilde{\mathbb{E}})$. That is,

$$\widetilde{\mathbb{E}}[\varphi(W_{t_1}, \ldots, W_{t_{n-1}}, W_{t_n} - W_{t_{n-1}})] = \widetilde{\mathbb{E}}[\psi(W_{t_1}, \ldots, W_{t_{n-1}})],$$

where $\psi(x_1, \ldots, x_{n-1}) = \widetilde{\mathbb{E}}[\varphi(x_1, \ldots, x_{n-1}, \sqrt{t_n - t_{n-1}} W_1)]$, $W_1 \sim \mathcal{N}(\{0\}; [\underline{\sigma}^2, \overline{\sigma}^2])$.

For each $p \geq 1$, we denote $L_G^p(\widetilde{\Omega})$ as the completion of $L_{ip}(\widetilde{\Omega})$ under the norm $\|X\|_{L_G^p} := (\widetilde{\mathbb{E}}[|X|^p])^{1/p}$. Then the G-expectation $\widetilde{\mathbb{E}}$ can be continuously extended to $(\widetilde{\Omega}, L_G^p(\widetilde{\Omega}))$. We still denote the extended G-expectation space by $(\widetilde{\Omega}, L_G^p(\widetilde{\Omega}), \widetilde{\mathbb{E}})$.

**Proposition 2.1** ([4, 11])  *There exists a weakly compact set of probability measures $\mathcal{P}$ on $(\widetilde{\Omega}, \mathcal{B}(\widetilde{\Omega}))$ such that*

$$\widetilde{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X] \quad \text{for any } X \in L_G^1(\widetilde{\Omega}),$$

*where $\mathcal{B}(\widetilde{\Omega})$ denotes the Borel $\sigma$-algebra of $\widetilde{\Omega}$. We say that $\mathcal{P}$ represents $\widetilde{\mathbb{E}}$.*

Given a G-expectation space $(\widetilde{\Omega}, L_G^1(\widetilde{\Omega}), \widetilde{\mathbb{E}})$, we can define a pair of capacities:

$$\widetilde{\mathbb{V}}(A) := \sup_{P \in \mathcal{P}} P(A) = \widetilde{\mathbb{E}}[I_A], \qquad \widetilde{v}(A) := \inf_{P \in \mathcal{P}} P(A) = 1 - \widetilde{\mathbb{V}}(A^c), \quad A \in \mathcal{B}(\widetilde{\Omega}).$$

Obviously, by Proposition 2.1, $\widetilde{\mathbb{E}}[\cdot]$ and $\widetilde{\mathbb{V}}(\cdot)$ are continuous from below.

**Definition 2.8** A sub-linear expectation $\mathbb{E}$ is said to be regular if, for each sequence $\{X_n\}_{n=1}^\infty$ satisfying $X_n \downarrow 0$, we have $\mathbb{E}[X_n] \downarrow 0$.

**Lemma 2.1** ([4, 11])
  (1) *For any closed sets $F_n \downarrow F$, it holds that $\widetilde{\mathbb{V}}(F_n) \downarrow \widetilde{\mathbb{V}}(F)$.*
  (2) *G-expectation $\widetilde{\mathbb{E}}[\cdot]$ is regular.*

Hu et al. [11] indicated that *G*-Brownian motion does not converge to any single point in probability under capacity $\widetilde{\mathbb{V}}$ as follows.

**Lemma 2.2** *Given a G-expectation space $(\widetilde{\Omega}, L_G^2(\widetilde{\Omega}), \widetilde{\mathbb{E}})$, for any fixed $a \in \mathbb{R}$, it holds that*

$$\lim_{\varepsilon \searrow 0} \sup_{a \in \mathbb{R}} \widetilde{\mathbb{V}}\big(|W_t - a| \le \varepsilon\big) = 0, \quad \forall t > 0.$$

*In particular, the above equation holds for G-normal distribution $W_1$.*

**Lemma 2.3** ([8]) $\mathbb{E}[|X|] < \infty$ *implies* $|X| < \infty$ *q.s., i.e.,* $\mathbb{V}(|X| = \infty) = 0$.

The following Rosenthal's inequality under sub-linear expectation was obtained by Zhang [24].

**Proposition 2.2** *Assume that $\{X_n\}_{n=1}^\infty$ is a sequence of independent random variables. Denote $S_n := \sum_{i=1}^n X_i$. Then, for any $p \ge 2$, we have*

$$\mathbb{E}\Big[\max_{i \le n} |S_i|^p\Big]$$
$$\le C_p\left\{ \sum_{i=1}^n \mathbb{E}\big[|X_i|^p\big] + \left(\sum_{i=1}^n \mathbb{E}\big[|X_i|^2\big]\right)^{\frac{p}{2}} + \left(\sum_{i=1}^n \big[\big(\mathbb{E}[X_i]\big)^+ + \big(\mathcal{E}[X_i]\big)^-\big]\right)^p \right\}, \qquad (2.1)$$

*where $C_p$ is a positive constant depending on p.*

**Lemma 2.4** *Assume that $\mathbb{E}$ is continuous from below and $\lim_{n \to \infty} X_n = X$. Then*

$$\mathbb{E}[X] \le \liminf_{n \to \infty} \mathbb{E}[X_n].$$

*If we further assume that $\mathbb{E}$ is continuous, then*

$$\mathbb{E}[X] = \lim_{n \to \infty} \mathbb{E}[X_n].$$

*Proof* Since $\inf_{i \geq n} X_i$ is non-decreasing in $n$, we have

$$\mathbb{E}[X] = \mathbb{E}\left[\liminf_{n \to \infty} X_n\right] = \mathbb{E}\left[\lim_{n \to \infty} \inf_{i \geq n} X_i\right] = \lim_{n \to \infty} \mathbb{E}\left[\inf_{i \geq n} X_i\right]$$
$$= \liminf_{n \to \infty} \mathbb{E}\left[\inf_{i \geq n} X_i\right] \leq \liminf_{n \to \infty} \mathbb{E}[X_n].$$

If $\mathbb{E}$ is continuous, by noting that $\sup_{i \geq n} X_i$ is non-increasing in $n$, we have

$$\mathbb{E}[X] = \mathbb{E}\left[\limsup_{n \to \infty} X_n\right] = \mathbb{E}\left[\lim_{n \to \infty} \sup_{i \geq n} X_i\right] = \lim_{n \to \infty} \mathbb{E}\left[\sup_{i \geq n} X_i\right] = \limsup_{n \to \infty} \mathbb{E}\left[\sup_{i \geq n} X_i\right]$$
$$\geq \limsup_{n \to \infty} \mathbb{E}[X_n] \geq \liminf_{n \to \infty} \mathbb{E}[X_n] \geq \mathbb{E}[X].$$

Thus $\lim_{n \to \infty} \mathbb{E}[X_n] = \mathbb{E}[X]$. □

**Lemma 2.5** *Assume that $\mathbb{E}$ is continuous from below and regular. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with $\mathbb{E}[X_n] = \mathcal{E}[X_n] = 0$ for any $n \geq 1$ and $\sum_{i=1}^{\infty} \mathbb{E}[X_i^2] < \infty$. Then $S := \sum_{i=1}^{\infty} X_i$ convergence q.s. under capacity $\mathbb{V}$ and, for any $p \geq 2$, we have*

$$\mathbb{E}[|S|^p] \leq C_p \left\{ \sum_{i=1}^{\infty} \mathbb{E}[|X_i|^p] + \left( \sum_{i=1}^{\infty} \mathbb{E}[|X_i|^2] \right)^{\frac{p}{2}} \right\}. \tag{2.2}$$

*Proof* One can refer to Zhang and Lin [25] for the proof of the convergence of $S$. Now we prove (2.2). By $\mathbb{E}[X_n] = \mathcal{E}[X_n] = 0$, taking $\limsup_{n \to \infty}$ on both sides of (2.1), we have

$$\limsup_{n \to \infty} \mathbb{E}\left[\max_{i \leq n} |S_i|^p\right] \leq C_p \left\{ \sum_{i=1}^{\infty} \mathbb{E}[|X_i|^p] + \left( \sum_{i=1}^{\infty} \mathbb{E}[|X_i|^2] \right)^{\frac{p}{2}} \right\}.$$

On the other hand,

$$\mathbb{E}\left[\max_{i \leq n} |S_i|^p\right] \geq \mathbb{E}[|S_n|^p].$$

Note that $\lim_{n \to \infty} S_n = S$. By Lemma 2.4 we have

$$\liminf_{n \to \infty} \mathbb{E}\left[\max_{i \leq n} |S_i|^p\right] \geq \liminf_{n \to \infty} \mathbb{E}[|S_n|^p] \geq \mathbb{E}[|S|^p].$$

Combining the above inequalities, we have

$$\mathbb{E}[|S|^p] \leq C_p \left\{ \sum_{i=1}^{\infty} \mathbb{E}[|X_i|^p] + \left( \sum_{i=1}^{\infty} \mathbb{E}[|X_i|^2] \right)^{\frac{p}{2}} \right\}. □$$

Throughout the rest of this paper, let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables on a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ with $\mathbb{E}[X_n] = \mathcal{E}[X_n] = 0$, $\mathbb{E}[X_n^2] = \overline{\sigma}_n^2$, $\mathcal{E}[X_n^2] = \underline{\sigma}_n^2$, $0 < \underline{\sigma}_n \leq \overline{\sigma}_n < \infty$. Denote $S_n := \sum_{i=1}^{n} X_i$, $\overline{B}_n^2 := \sum_{i=1}^{n} \overline{\sigma}_i^2$, and $\underline{B}_n^2 := \sum_{i=1}^{n} \underline{\sigma}_i^2$. The symbol $C$ presents an arbitrary positive constant and may take different values in different positions.

Zhang [23] obtained the following CLT for sub-linear expectation under the Lindeberg condition as a corollary of the martingale CLT for sub-linear expectation.

**Theorem 2.1** *Let $\xi$ be G-normal distributed on a G-expectation space $(\widetilde{\Omega}, L_G^2(\widetilde{\Omega}), \widetilde{\mathbb{E}})$ with $\xi \sim \mathcal{N}(\{0\}; [\underline{\sigma}^2, 1])$, $0 < \underline{\sigma} \leq 1$. Assume that*

(1)

$$\lim_{n \to \infty} \frac{1}{\overline{B}_n^2} \sum_{i=1}^n \left| \underline{\sigma}^2 \cdot \overline{\sigma}_i^2 - \underline{\sigma}_i^2 \right| = 0. \tag{2.3}$$

(2) *For any $\varepsilon > 0$,*

$$\lim_{n \to \infty} \frac{1}{\overline{B}_n^2} \sum_{i=1}^n \mathbb{E}\left[ |X_i|^2 I\left( |X_i| > \varepsilon \overline{B}_n \right) \right] = 0. \tag{2.4}$$

*Then, for any $\varphi \in C_{b,\mathrm{Lip}}(\mathbb{R})$,*

$$\lim_{n \to \infty} \mathbb{E}\left[ \varphi\left( \frac{S_n}{\overline{B}_n} \right) \right] = \widetilde{\mathbb{E}}\left[ \varphi(\xi) \right]. \tag{2.5}$$

## 3 The bound on the distance between the normalized sum distribution and G-normal distribution

The following theorem gives a bound on the distance between the normalized sum distribution $\mathbb{E}[\varphi(\frac{S_n}{\overline{B}_n})]$ and G-normal distribution $\widetilde{\mathbb{E}}[\varphi(\xi)]$ where $\xi \sim \mathcal{N}(\{0\}; [\underline{\sigma}^2, 1])$.

**Theorem 3.1** *Let $\xi$ be G-normal distributed on a G-expectation space $(\widetilde{\Omega}, L_G^2(\widetilde{\Omega}), \widetilde{\mathbb{E}})$ with $\xi \sim \mathcal{N}(\{0\}; [\underline{\sigma}^2, 1])$, $0 < \underline{\sigma} \leq 1$.*

*Then, for any fixed $\varphi \in C_{b,\mathrm{Lip}}(\mathbb{R})$ and any $h > 0$, $0 < \varepsilon < 1$, there exist some $0 < \alpha < 1$, $C > 0$, and $C_h > 0$ (a positive constant depending on h) such that*

$$\left| \mathbb{E}\left[ \varphi\left( \frac{S_n}{\overline{B}_n} \right) \right] - \widetilde{\mathbb{E}}\left[ \varphi(\xi) \right] \right|$$

$$\leq \frac{C_h}{\overline{B}_n^2} \sum_{i=1}^n \left| \underline{\sigma}^2 \cdot \overline{\sigma}_i^2 - \underline{\sigma}_i^2 \right| + \frac{C_h}{\overline{B}_n^2} \sum_{i=1}^n \mathbb{E}\left[ |X_i|^2 I\left( |X_i| > \varepsilon \overline{B}_n \right) \right]$$

$$+ C_h \left( \frac{1}{\overline{B}_n^2} \sum_{i=1}^n \mathbb{E}\left[ |X_i|^2 I\left( |X_i| > \varepsilon \overline{B}_n \right) \right] \right)^{1 + \frac{\alpha}{2}} + C_h \varepsilon^\alpha + C(\sqrt{h} + \sqrt{1+h} - 1). \tag{3.1}$$

*Remark* 3.1 By Theorem 3.1 we can derive Theorem 2.1. If (2.3) and (2.4) hold, taking $n \to \infty$, $\varepsilon \to 0$, and $h \to 0$ in turn on both sides of (3.1), we can get (2.5).

*Proof* For any fixed $\varphi \in C_{b,\mathrm{Lip}}(\mathbb{R})$ and any $h > 0$, let $V(t,x) = \widetilde{\mathbb{E}}[\varphi(x + \sqrt{1 + h - t}\xi)]$. By Definition 2.7, we have that $V$ is the unique viscosity solution of the following parabolic PDE:

$$\begin{cases} \partial_t V + \frac{1}{2}(\partial_{xx} V)^+ - \frac{1}{2}(\partial_{xx} V)^- \underline{\sigma}^2 = 0, \\ V|_{t=1+h} = \varphi. \end{cases} \tag{3.2}$$

Let $X_i^{(n)} = (-\overline{B}_n) \vee (X_i \wedge \overline{B}_n)$, $S_i^{(n)} = \frac{1}{\overline{B}_n} \sum_{j=1}^{i} X_j^{(n)}$, $S_0^{(n)} = 0$, $\delta_i^{(n)} = \frac{1}{\overline{B}_n^2} \sum_{j=1}^{i} \overline{\sigma}_j^2$, $\delta_0^{(n)} = 0$ for each $i = 1, 2, \ldots, n$. Then we have

$$
\begin{aligned}
&\left| \mathbb{E}\left[ \varphi\left( \frac{S_n}{\overline{B}_n} \right) \right] - \widetilde{\mathbb{E}}[\varphi(\xi)] \right| \\
&\quad \leq \left| \mathbb{E}\left[ \varphi\left( \frac{S_n}{\overline{B}_n} \right) \right] - \mathbb{E}\left[ \varphi\left( S_n^{(n)} \right) \right] \right| + \left| \mathbb{E}\left[ \varphi\left( S_n^{(n)} \right) \right] - \widetilde{\mathbb{E}}[\varphi(\xi)] \right| \\
&\quad = \left| \mathbb{E}\left[ \varphi\left( \frac{S_n}{\overline{B}_n} \right) \right] - \mathbb{E}\left[ \varphi\left( S_n^{(n)} \right) \right] \right| + \left| \mathbb{E}\left[ V\left( 1 + h, S_n^{(n)} \right) \right] - V(h, 0) \right| \\
&\quad \leq \left| \mathbb{E}\left[ \varphi\left( \frac{S_n}{\overline{B}_n} \right) \right] - \mathbb{E}\left[ \varphi\left( S_n^{(n)} \right) \right] \right| + \left| \mathbb{E}\left[ V\left( 1 + h, S_n^{(n)} \right) \right] - \mathbb{E}\left[ V\left( 1, S_n^{(n)} \right) \right] \right| \\
&\qquad + \left| \mathbb{E}\left[ V\left( 1, S_n^{(n)} \right) \right] - V(0, 0) \right| + \left| V(0, 0) - V(h, 0) \right|.
\end{aligned}
$$

Since $\varphi \in C_{b,\mathrm{Lip}}(\mathbb{R})$, for any $0 < \varepsilon < 1$, it holds that

$$
\begin{aligned}
&\left| \mathbb{E}\left[ \varphi\left( \frac{S_n}{\overline{B}_n} \right) \right] - \mathbb{E}\left[ \varphi\left( S_n^{(n)} \right) \right] \right| \\
&\quad \leq \frac{C}{\overline{B}_n} \sum_{i=1}^{n} \mathbb{E}\left[ |X_i - X_i^{(n)}| \right] \leq \frac{C}{\overline{B}_n} \sum_{i=1}^{n} \mathbb{E}\left[ |X_i| I\left( |X_i| > \overline{B}_n \right) \right] \\
&\quad \leq \frac{C}{\overline{B}_n^2} \sum_{i=1}^{n} \mathbb{E}\left[ |X_i|^2 I\left( |X_i| > \overline{B}_n \right) \right] \leq \frac{C}{\overline{B}_n^2} \sum_{i=1}^{n} \mathbb{E}\left[ |X_i|^2 I\left( |X_i| > \varepsilon \overline{B}_n \right) \right], \\
&\left| \mathbb{E}\left[ V\left( 1 + h, S_n^{(n)} \right) \right] - \mathbb{E}\left[ V\left( 1, S_n^{(n)} \right) \right] \right| \\
&\quad \leq \sup_x \left| V(1 + h, x) - V(1, x) \right| = \sup_x \left| \varphi(x) - \widetilde{\mathbb{E}}\left[ \varphi(x + \sqrt{h}\xi) \right] \right| \\
&\quad \leq \sup_x \widetilde{\mathbb{E}}\left[ |\varphi(x) - \varphi(x + \sqrt{h}\xi)| \right] \leq C\sqrt{h} \widetilde{\mathbb{E}}\left[ |\xi| \right] \leq C\sqrt{h},
\end{aligned}
$$

and

$$
\begin{aligned}
\left| V(0, 0) - V(h, 0) \right| &= \left| \widetilde{\mathbb{E}}\left[ \varphi(\sqrt{1 + h}\xi) \right] - \widetilde{\mathbb{E}}\left[ \varphi(\xi) \right] \right| \\
&\leq C(\sqrt{1 + h} - 1) \widetilde{\mathbb{E}}\left[ |\xi| \right] \leq C(\sqrt{1 + h} - 1).
\end{aligned}
$$

Then

$$
\begin{aligned}
\left| \mathbb{E}\left[ \varphi\left( \frac{S_n}{\overline{B}_n} \right) \right] - \widetilde{\mathbb{E}}[\varphi(\xi)] \right| &\leq \left| \mathbb{E}\left[ V\left( 1, S_n^{(n)} \right) \right] - V(0, 0) \right| + \frac{C}{\overline{B}_n^2} \sum_{i=1}^{n} \mathbb{E}\left[ |X_i|^2 I\left( |X_i| > \varepsilon \overline{B}_n \right) \right] \\
&\quad + C(\sqrt{h} + \sqrt{1 + h} - 1).
\end{aligned} \tag{3.3}
$$

So it is sufficient to get the bound of $|\mathbb{E}[V(1, S_n^{(n)})] - V(0, 0)|$.

$$
\begin{aligned}
&V\left( 1, S_n^{(n)} \right) - V(0, 0) \\
&\quad = \sum_{i=0}^{n-1} \left[ V\left( \delta_{i+1}^{(n)}, S_{i+1}^{(n)} \right) - V\left( \delta_i^{(n)}, S_i^{(n)} \right) \right]
\end{aligned}
$$

$$= \sum_{i=0}^{n-1} \left[ \left( V\big(\delta_{i+1}^{(n)}, S_{i+1}^{(n)}\big) - V\big(\delta_i^{(n)}, S_{i+1}^{(n)}\big) \right) + \left( V\big(\delta_i^{(n)}, S_{i+1}^{(n)}\big) - V\big(\delta_i^{(n)}, S_i^{(n)}\big) \right) \right]$$

$$:= \sum_{i=0}^{n-1} \big( I_i^{(n)} + J_i^{(n)} \big),$$

where $I_i^{(n)}$ and $J_i^{(n)}$ are obtained by Taylor expansion:

$$I_i^{(n)} = \frac{\overline{\sigma}_{i+1}^2}{\overline{B}_n^2} \partial_t V\big(\delta_i^{(n)}, S_i^{(n)}\big) + \frac{1}{2\overline{B}_n^2} \partial_{xx} V\big(\delta_i^{(n)}, S_i^{(n)}\big) \big|X_{i+1}^{(n)}\big|^2 + \frac{1}{\overline{B}_n} \partial_x V\big(\delta_i^{(n)}, S_i^{(n)}\big) X_{i+1}^{(n)}$$

$$= \left( \frac{\overline{\sigma}_{i+1}^2}{\overline{B}_n^2} \partial_t V\big(\delta_i^{(n)}, S_i^{(n)}\big) + \frac{1}{2\overline{B}_n^2} \partial_{xx} V\big(\delta_i^{(n)}, S_i^{(n)}\big) X_{i+1}^2 + \frac{1}{\overline{B}_n} \partial_x V\big(\delta_i^{(n)}, S_i^{(n)}\big) X_{i+1} \right)$$

$$+ \left( \frac{1}{2\overline{B}_n^2} \partial_{xx} V\big(\delta_i^{(n)}, S_i^{(n)}\big) \big( \big|X_{i+1}^{(n)}\big|^2 - X_{i+1}^2 \big) + \frac{1}{\overline{B}_n} \partial_x V\big(\delta_i^{(n)}, S_i^{(n)}\big) \big( X_{i+1}^{(n)} - X_{i+1} \big) \right)$$

$$:= I_{1,i}^{(n)} + I_{2,i}^{(n)}.$$

$$J_i^{(n)} = \frac{\overline{\sigma}_{i+1}^2}{\overline{B}_n^2} \big[ \partial_t V\big(\delta_i^{(n)}, S_{i+1}^{(n)}\big) - \partial_t V\big(\delta_i^{(n)}, S_i^{(n)}\big) \big]$$

$$+ \frac{\overline{\sigma}_{i+1}^2}{\overline{B}_n^2} \int_0^1 \left[ \partial_t V\left( \delta_i^{(n)} + \beta \cdot \frac{\overline{\sigma}_{i+1}^2}{\overline{B}_n^2}, S_{i+1}^{(n)} \right) - \partial_t V\big(\delta_i^{(n)}, S_{i+1}^{(n)}\big) \right] d\beta$$

$$+ \int_0^1 \int_0^1 \left[ \partial_{xx} V\left( \delta_i^{(n)}, S_i^{(n)} + \gamma\beta \frac{X_{i+1}^{(n)}}{\overline{B}_n} \right) - \partial_{xx} V\big(\delta_i^{(n)}, S_i^{(n)}\big) \right] \frac{\big|X_{i+1}^{(n)}\big|^2}{\overline{B}_n^2} \gamma \, d\beta \, d\gamma.$$

Since $X_{i+1}$ is independent of $S_i^{(n)}$, we have

$$\mathbb{E}\big[ \partial_x V\big(\delta_i^{(n)}, S_i^{(n)}\big) X_{i+1} \big] = \mathbb{E}\big[ \mathbb{E}[r \cdot X_{i+1}]\big|_{r = \partial_x V(\delta_i^{(n)}, S_i^{(n)})} \big]$$

$$= \mathbb{E}\big[ \big(\partial_x V\big(\delta_i^{(n)}, S_i^{(n)}\big)\big)^+ \mathbb{E}[X_{i+1}] - \big(\partial_x V\big(\delta_i^{(n)}, S_i^{(n)}\big)\big)^- \mathcal{E}[X_{i+1}] \big]$$

$$= 0.$$

Similarly, we can also have $\mathbb{E}[-\partial_x V(\delta_i^{(n)}, S_i^{(n)}) X_{i+1}] = 0$. It follows that

$$\mathbb{E}\big[ I_{1,i}^{(n)} \big] = \mathbb{E}\left[ \frac{\overline{\sigma}_{i+1}^2}{\overline{B}_n^2} \partial_t V\big(\delta_i^{(n)}, S_i^{(n)}\big) + \frac{1}{2\overline{B}_n^2} \partial_{xx} V\big(\delta_i^{(n)}, S_i^{(n)}\big) X_{i+1}^2 \right]$$

$$= \frac{1}{\overline{B}_n^2} \mathbb{E}\left[ \mathbb{E}\left[ \overline{\sigma}_{i+1}^2 p + \frac{1}{2} q X_{i+1}^2 \right] \Big|_{p = \partial_t V(\delta_i^{(n)}, S_i^{(n)}), q = \partial_{xx} V(\delta_i^{(n)}, S_i^{(n)})} \right]$$

$$= \frac{1}{\overline{B}_n^2} \mathbb{E}\left[ \overline{\sigma}_{i+1}^2 p + \frac{1}{2} q^+ \mathbb{E}[X_{i+1}^2] - \frac{1}{2} q^- \mathcal{E}[X_{i+1}^2] \right] \Big|_{p = \partial_t V(\delta_i^{(n)}, S_i^{(n)}), q = \partial_{xx} V(\delta_i^{(n)}, S_i^{(n)})}$$

$$= \frac{1}{\overline{B}_n^2} \mathbb{E}\left[ \partial_t V\big(\delta_i^{(n)}, S_i^{(n)}\big) \overline{\sigma}_{i+1}^2 + \frac{1}{2} \big( \partial_{xx} V\big(\delta_i^{(n)}, S_i^{(n)}\big) \big)^+ \overline{\sigma}_{i+1}^2 \right.$$

$$\left. - \frac{1}{2} \big( \partial_{xx} V\big(\delta_i^{(n)}, S_i^{(n)}\big) \big)^- \underline{\sigma}_{i+1}^2 \right]$$

$$\le \frac{\overline{\sigma}_{i+1}^2}{\overline{B}_n^2} \mathbb{E}\left[ \partial_t V\big(\delta_i^{(n)}, S_i^{(n)}\big) + \frac{1}{2} \big( \partial_{xx} V\big(\delta_i^{(n)}, S_i^{(n)}\big) \big)^+ - \frac{1}{2} \big( \partial_{xx} V\big(\delta_i^{(n)}, S_i^{(n)}\big) \big)^- \underline{\sigma}^2 \right]$$

$$+ \frac{1}{2\overline{B}_n^2} \mathbb{E}\big[\partial_{xx} V\big(\delta_i^{(n)}, S_i^{(n)}\big)^{-} \underline{\sigma}^2 \cdot \overline{\sigma}_{i+1}^2 - \partial_{xx} V\big(\delta_i^{(n)}, S_i^{(n)}\big)^{-} \underline{\sigma}_{i+1}^2\big]$$

$$\leq \frac{1}{2\overline{B}_n^2} \big|\underline{\sigma}^2 \cdot \overline{\sigma}_{i+1}^2 - \underline{\sigma}_{i+1}^2\big| \cdot \mathbb{E}\big[\big|\partial_{xx} V\big(\delta_i^{(n)}, S_i^{(n)}\big)\big|\big].$$

By the interior regularity of $V$ (see Peng [18]), it holds that

$$\|V\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}([0,1] \times \mathbb{R})} < \infty \quad \text{for some } \alpha \in (0,1),$$

which implies $\partial_t V$, $\partial_x V$, and $\partial_{xx} V$ are uniformly $\frac{\alpha}{2}$-Hölder continuous in $t$ and $\alpha$-Hölder continuous in $x$ on $[0,1] \times \mathbb{R}$. For any $n \geq 1$ and $i \leq n$, it holds that

$$\mathbb{E}\big[\big|\partial_{xx} V\big(\delta_i^{(n)}, S_i^{(n)}\big)\big|\big] \leq \big|\partial_{xx} V(0,0)\big| + C_h\big(\big|\delta_i^{(n)}\big|^{\frac{\alpha}{2}} + \mathbb{E}\big[\big|S_i^{(n)}\big|^\alpha\big]\big)$$

$$\leq C_h\big(1 + \big(\mathbb{E}\big[\big|S_i^{(n)}\big|^2\big]\big)^{\frac{\alpha}{2}}\big).$$

By Proposition 2.2, we have

$$\mathbb{E}\big[\big|S_i^{(n)}\big|^2\big] \leq \frac{C}{\overline{B}_n^2} \sum_{i=1}^{n} \mathbb{E}\big[\big|X_i^{(n)}\big|^2\big] + \frac{C}{\overline{B}_n^2} \left(\sum_{i=1}^{n} \big[\big(\mathbb{E}\big[X_i^{(n)}\big]\big)^+ + \big(\mathbb{E}\big[-X_i^{(n)}\big]\big)^+\big]\right)^2$$

$$\leq \frac{C}{\overline{B}_n^2} \sum_{i=1}^{n} \mathbb{E}\big[|X_i|^2\big] + \frac{C}{\overline{B}_n^2} \left(\sum_{i=1}^{n} \mathbb{E}\big[\big|X_i - X_i^{(n)}\big|\big]\right)^2$$

$$\leq C + C\left(\frac{1}{\overline{B}_n} \sum_{i=1}^{n} \mathbb{E}\big[|X_i| I\big(|X_i| > \overline{B}_n\big)\big]\right)^2$$

$$\leq C + C\left(\frac{1}{\overline{B}_n^2} \sum_{i=1}^{n} \mathbb{E}\big[|X_i|^2\big]\right)^2$$

$$\leq C.$$

So we have $\mathbb{E}[|\partial_{xx} V(\delta_i^{(n)}, S_i^{(n)})|] \leq C_h$. Similarly $\mathbb{E}[|\partial_x V(\delta_i^{(n)}, S_i^{(n)})|] \leq C_h$. Then

$$\mathbb{E}\left[\sum_{i=0}^{n-1} I_{1,i}^{(n)}\right] \leq \sum_{i=0}^{n-1} \mathbb{E}\big[I_{1,i}^{(n)}\big] \leq \frac{C_h}{\overline{B}_n^2} \sum_{i=1}^{n} \big|\underline{\sigma}^2 \cdot \overline{\sigma}_i^2 - \underline{\sigma}_i^2\big|.$$

On the other hand,

$$\mathbb{E}\left[\sum_{i=0}^{n-1} I_{1,i}^{(n)}\right]$$

$$= \mathbb{E}\left[\sum_{i=0}^{n-2} I_{1,i}^{(n)} + \frac{\overline{\sigma}_n^2}{\overline{B}_n^2} \partial_t V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big) + \frac{1}{2\overline{B}_n^2} \partial_{xx} V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big) X_n^2\right.$$

$$\left. + \frac{1}{\overline{B}_n} \partial_x V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big) X_n\right]$$

$$= \mathbb{E}\left[\sum_{i=0}^{n-2} I_{1,i}^{(n)} + \frac{\overline{\sigma}_n^2}{\overline{B}_n^2} \partial_t V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big) + \frac{1}{2\overline{B}_n^2} \partial_{xx} V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big) X_n^2\right]$$

$$= \mathbb{E}\left[\sum_{i=0}^{n-2} I_{1,i}^{(n)} + \frac{\overline{\sigma}_n^2}{\overline{B}_n^2}\partial_t V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big)\right.$$

$$\left. + \frac{1}{2\overline{B}_n^2}\big(\partial_{xx}V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big)\big)^+\overline{\sigma}_n^2 - \frac{1}{2\overline{B}_n^2}\big(\partial_{xx}V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big)\big)^-\underline{\sigma}_n^2\right]$$

$$= \mathbb{E}\left[\sum_{i=0}^{n-2} I_{1,i}^{(n)} + \frac{\overline{\sigma}_n^2}{\overline{B}_n^2}\left(\partial_t V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big) + \frac{1}{2}\big(\partial_{xx}V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big)\big)^+\right.\right.$$

$$\left. - \frac{1}{2}\big(\partial_{xx}V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big)\big)^-\underline{\sigma}^2\right)$$

$$\left. + \frac{1}{2\overline{B}_n^2}\big(\partial_{xx}V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big)\big)^-\underline{\sigma}^2\cdot\overline{\sigma}_n^2 - \frac{1}{2\overline{B}_n^2}\big(\partial_{xx}V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big)\big)^-\underline{\sigma}_n^2\right]$$

$$= \mathbb{E}\left[\sum_{i=0}^{n-2} I_{1,i}^{(n)} + \frac{1}{2\overline{B}_n^2}\big(\partial_{xx}V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big)\big)^-\big(\underline{\sigma}^2\cdot\overline{\sigma}_n^2 - \underline{\sigma}_n^2\big)\right]$$

$$\geq \mathbb{E}\left[\sum_{i=0}^{n-2} I_{1,i}^{(n)}\right] - \mathbb{E}\left[-\frac{1}{2\overline{B}_n^2}\big(\partial_{xx}V\big(\delta_{n-1}^{(n)}, S_{n-1}^{(n)}\big)\big)^-\big(\underline{\sigma}^2\cdot\overline{\sigma}_n^2 - \underline{\sigma}_n^2\big)\right]$$

$$\geq \mathbb{E}\left[\sum_{i=0}^{n-2} I_{1,i}^{(n)}\right] - \frac{C_h}{\overline{B}_n^2}\big|\underline{\sigma}^2\cdot\overline{\sigma}_n^2 - \underline{\sigma}_n^2\big|$$

$$\geq -\frac{C_h}{\overline{B}_n^2}\sum_{i=1}^{n}\big|\underline{\sigma}^2\cdot\overline{\sigma}_i^2 - \underline{\sigma}_i^2\big|.$$

So we have

$$\left|\mathbb{E}\left[\sum_{i=0}^{n-1} I_{1,i}^{(n)}\right]\right| \leq \frac{C_h}{\overline{B}_n^2}\sum_{i=1}^{n}\big|\underline{\sigma}^2\cdot\overline{\sigma}_i^2 - \underline{\sigma}_i^2\big|. \tag{3.4}$$

Since $|X_{i+1}^{(n)}|^2 - X_{i+1}^2$ is independent of $\partial_{xx}V(\delta_i^{(n)}, S_i^{(n)})$ and $X_{i+1}^{(n)} - X_{i+1}$ is independent of $\partial_x V(\delta_i^{(n)}, S_i^{(n)})$, we have

$$\sum_{i=0}^{n-1}\mathbb{E}\big[\big|I_{2,i}^{(n)}\big|\big] \leq \sum_{i=0}^{n-1}\left\{\frac{1}{2\overline{B}_n^2}\mathbb{E}\big[\big|\partial_{xx}V\big(\delta_i^{(n)}, S_i^{(n)}\big)\big|\big]\mathbb{E}\big[\big||X_{i+1}^{(n)}|^2 - X_{i+1}^2\big|\big]\right.$$

$$\left. + \frac{1}{\overline{B}_n}\mathbb{E}\big[\big|\partial_x V\big(\delta_i^{(n)}, S_i^{(n)}\big)\big|\big]\mathbb{E}\big[\big|X_{i+1}^{(n)} - X_{i+1}\big|\big]\right\}$$

$$\leq \frac{C_h}{\overline{B}_n^2}\sum_{i=1}^{n}\mathbb{E}\big[|X_i|^2 I\big(|X_i| > \overline{B}_n\big)\big] + \frac{C_h}{\overline{B}_n}\sum_{i=1}^{n}\mathbb{E}\big[|X_i| I\big(|X_i| > \overline{B}_n\big)\big]$$

$$\leq \frac{C_h}{\overline{B}_n^2}\sum_{i=1}^{n}\mathbb{E}\big[|X_i|^2 I\big(|X_i| > \varepsilon\overline{B}_n\big)\big]. \tag{3.5}$$

On the other hand,

$$\left|\frac{\overline{\sigma}_{i+1}^2}{\overline{B}_n^2}\big[\partial_t V\big(\delta_i^{(n)}, S_{i+1}^{(n)}\big) - \partial_t V\big(\delta_i^{(n)}, S_i^{(n)}\big)\big]\right| \leq C_h\cdot\frac{\overline{\sigma}_{i+1}^2}{\overline{B}_n^2}\cdot\left(\frac{|X_{i+1}^{(n)}|}{\overline{B}_n}\right)^\alpha,$$

$$
\left| \frac{\overline{\sigma}_{i+1}^2}{\overline{B}_n^2} \int_0^1 \left[ \partial_t V\left( \delta_i^{(n)} + \beta \cdot \frac{\overline{\sigma}_{i+1}^2}{\overline{B}_n^2}, S_{i+1}^{(n)} \right) - \partial_t V\left( \delta_i^{(n)}, S_{i+1}^{(n)} \right) \right] d\beta \right| \leq C_h \cdot \frac{\overline{\sigma}_{i+1}^2}{\overline{B}_n^2} \cdot \left( \frac{\overline{\sigma}_{i+1}^2}{\overline{B}_n^2} \right)^{\frac{\alpha}{2}},
$$

$$
\left| \int_0^1 \int_0^1 \left[ \partial_{xx} V\left( \delta_i^{(n)}, S_i^{(n)} + \gamma\beta \frac{X_{i+1}^{(n)}}{\overline{B}_n} \right) - \partial_{xx} V\left( \delta_i^{(n)}, S_i^{(n)} \right) \right] \frac{|X_{i+1}^{(n)}|^2}{\overline{B}_n^2} \gamma \, d\beta \, d\gamma \right|
$$

$$
\leq C_h \cdot \frac{|X_{i+1}^{(n)}|^2}{\overline{B}_n^2} \cdot \left( \frac{|X_{i+1}^{(n)}|}{\overline{B}_n} \right)^{\alpha}.
$$

Then

$$
\sum_{i=0}^{n-1} \mathbb{E}\left[ |J_i^{(n)}| \right] \leq \frac{C_h}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \overline{\sigma}_i^2 \mathbb{E}\left[ |X_i^{(n)}|^{\alpha} \right] + \frac{C_h}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \overline{\sigma}_i^{2+\alpha} + \frac{C_h}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \mathbb{E}\left[ |X_i^{(n)}|^{2+\alpha} \right]. \quad (3.6)
$$

For any $0 < \varepsilon < 1$, we have

$$
\frac{1}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \mathbb{E}\left[ |X_i^{(n)}|^{2+\alpha} \right]
$$

$$
= \frac{1}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \mathbb{E}\left[ |X_i|^{2+\alpha} I\left( |X_i| \leq \overline{B}_n \right) + \overline{B}_n^{2+\alpha} I\left( |X_i| > \overline{B}_n \right) \right]
$$

$$
\leq \frac{1}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \mathbb{E}\left[ |X_i|^{2+\alpha} I\left( |X_i| \leq \varepsilon \overline{B}_n \right) \right] + \frac{1}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \mathbb{E}\left[ |X_i|^{2+\alpha} I\left( \varepsilon \overline{B}_n < |X_i| \leq \overline{B}_n \right) \right]
$$

$$
+ \frac{1}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \mathbb{E}\left[ |X_i|^2 \overline{B}_n^{\alpha} I\left( |X_i| > \overline{B}_n \right) \right]
$$

$$
\leq \frac{1}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \varepsilon^{\alpha} \overline{B}_n^{\alpha} \mathbb{E}\left[ |X_i|^2 \right] + \frac{1}{\overline{B}_n^2} \sum_{i=1}^n \mathbb{E}\left[ |X_i|^2 I\left( |X_i| > \varepsilon \overline{B}_n \right) \right]
$$

$$
+ \frac{1}{\overline{B}_n^2} \sum_{i=1}^n \mathbb{E}\left[ |X_i|^2 I\left( |X_i| > \overline{B}_n \right) \right]
$$

$$
\leq \varepsilon^{\alpha} + \frac{C}{\overline{B}_n^2} \sum_{i=1}^n \mathbb{E}\left[ |X_i|^2 I\left( |X_i| > \varepsilon \overline{B}_n \right) \right]. \quad (3.7)
$$

By Hölder's inequality under sub-linear expectation, we have

$$
\frac{1}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \overline{\sigma}_i^{2+\alpha}
$$

$$
= \frac{1}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \left( \mathbb{E}\left[ |X_i|^2 I\left( |X_i| \leq \varepsilon \overline{B}_n \right) + |X_i|^2 I\left( |X_i| > \varepsilon \overline{B}_n \right) \right] \right)^{1+\frac{\alpha}{2}}
$$

$$
\leq \frac{C}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \left( \mathbb{E}\left[ |X_i|^2 I\left( |X_i| \leq \varepsilon \overline{B}_n \right) \right] \right)^{1+\frac{\alpha}{2}} + \frac{C}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \left( \mathbb{E}\left[ |X_i|^2 I\left( |X_i| > \varepsilon \overline{B}_n \right) \right] \right)^{1+\frac{\alpha}{2}}
$$

$$
\leq \frac{C}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \mathbb{E}\left[ |X_i|^{2+\alpha} I\left( |X_i| \leq \varepsilon \overline{B}_n \right) \right] + \frac{C}{\overline{B}_n^{2+\alpha}} \sum_{i=1}^n \left( \mathbb{E}\left[ |X_i|^2 I\left( |X_i| > \varepsilon \overline{B}_n \right) \right] \right)^{1+\frac{\alpha}{2}}
$$

$$\leq C\varepsilon^\alpha + C\left(\frac{1}{\overline{B}_n^2}\sum_{i=1}^n \mathbb{E}\big[|X_i|^2 I\big(|X_i| > \varepsilon\overline{B}_n\big)\big]\right)^{1+\frac{\alpha}{2}}. \tag{3.8}$$

$$\frac{1}{\overline{B}_n^{2+\alpha}}\sum_{i=1}^n \overline{\sigma}_i^2 \mathbb{E}\big[|X_i^{(n)}|^\alpha\big]$$

$$\leq \frac{1}{\overline{B}_n^{2+\alpha}}\sum_{i=1}^n \overline{\sigma}_i^2 \mathbb{E}\big[|X_i|^\alpha\big] \leq \frac{1}{\overline{B}_n^{2+\alpha}}\sum_{i=1}^n \overline{\sigma}_i^{2+\alpha}$$

$$\leq C\varepsilon^\alpha + C\left(\frac{1}{\overline{B}_n^2}\sum_{i=1}^n \mathbb{E}\big[|X_i|^2 I\big(|X_i| > \varepsilon\overline{B}_n\big)\big]\right)^{1+\frac{\alpha}{2}}. \tag{3.9}$$

Combining (3.7), (3.8), and (3.9), we have

$$\sum_{i=0}^{n-1}\mathbb{E}\big[|J_i^{(n)}|\big] \leq C_h\varepsilon^\alpha + \frac{C_h}{\overline{B}_n^2}\sum_{i=1}^n \mathbb{E}\big[|X_i|^2 I\big(|X_i| > \varepsilon\overline{B}_n\big)\big]$$

$$+ C_h\left(\frac{1}{\overline{B}_n^2}\sum_{i=1}^n \mathbb{E}\big[|X_i|^2 I\big(|X_i| > \varepsilon\overline{B}_n\big)\big]\right)^{1+\frac{\alpha}{2}}. \tag{3.10}$$

By (3.4), (3.5), and (3.10), it holds that

$$\big|\mathbb{E}\big[V\big(1, S_n^{(n)}\big)\big] - V(0,0)\big| \leq \left|\mathbb{E}\left[\sum_{i=0}^{n-1} I_{1,i}^{(n)}\right]\right| + \sum_{i=0}^{n-1}\mathbb{E}\big[|I_{2,i}^{(n)}|\big] + \sum_{i=0}^{n-1}\mathbb{E}\big[|J_i^{(n)}|\big]$$

$$\leq \frac{C_h}{\overline{B}_n^2}\sum_{i=1}^n \big|\underline{\sigma}^2 \cdot \overline{\sigma}_i^2 - \underline{\sigma}_i^2\big| + C_h\varepsilon^\alpha$$

$$+ \frac{C_h}{\overline{B}_n^2}\sum_{i=1}^n \mathbb{E}\big[|X_i|^2 I\big(|X_i| > \varepsilon\overline{B}_n\big)\big]$$

$$+ C_h\left(\frac{1}{\overline{B}_n^2}\sum_{i=1}^n \mathbb{E}\big[|X_i|^2 I\big(|X_i| > \varepsilon\overline{B}_n\big)\big]\right)^{1+\frac{\alpha}{2}}.$$

Thus we obtain (3.1). □

By a similar method, we can obtain a bound on the distance between the normalized sum distribution $\mathbb{E}[\varphi(\frac{S_n}{\underline{B}_n})]$ and the corresponding G-normal distribution $\mathbb{E}[\varphi(\eta)]$. And it can also be used to derive the CLT for normalizing factor $\underline{B}_n$. We only give the theorem and omit the proof.

**Theorem 3.2** *Let $\eta$ be G-normal distributed on a G-expectation space $(\widetilde{\Omega}, L_G^2(\widetilde{\Omega}), \widetilde{\mathbb{E}})$ with $\eta \sim \mathcal{N}(\{0\}; [1, \overline{\sigma}^2]), \overline{\sigma} \geq 1$.*

*Then, for any fixed $\varphi \in C_{b,\mathrm{Lip}}(\mathbb{R})$ and any $h > 0$, $0 < \varepsilon < 1$, there exist some $0 < \alpha < 1$, $C > 0$, and $C_h > 0$ (a constant depending on $h$) such that*

$$\left|\mathbb{E}\left[\varphi\left(\frac{S_n}{\underline{B}_n}\right)\right] - \widetilde{\mathbb{E}}\big[\varphi(\eta)\big]\right|$$

$$\leq \frac{C_h}{\underline{B}_n^2}\sum_{i=1}^n \big|\overline{\sigma}^2 \cdot \underline{\sigma}_i^2 - \overline{\sigma}_i^2\big| + \frac{C_h}{\underline{B}_n^2}\sum_{i=1}^n \mathbb{E}\big[|X_i|^2 I\big(|X_i| > \varepsilon\underline{B}_n\big)\big]$$

$$+ C_h \left( \frac{1}{\underline{B}_n^2} \sum_{i=1}^n \mathbb{E}\left[ |X_i|^2 I\left( |X_i| > \varepsilon \underline{B}_n \right) \right] \right)^{1+\frac{\alpha}{2}}$$

$$+ C_h \varepsilon^\alpha + C(\sqrt{h} + \sqrt{1+h} - 1).$$

*If we further assume that*

(1)

$$\lim_{n \to \infty} \frac{1}{\underline{B}_n^2} \sum_{i=1}^n \left| \overline{\sigma}^2 \cdot \underline{\sigma}_i^2 - \overline{\sigma}_i^2 \right| = 0.$$

(2) *For any $\varepsilon > 0$,*

$$\lim_{n \to \infty} \frac{1}{\underline{B}_n^2} \sum_{i=1}^n \mathbb{E}\left[ |X_i|^2 I\left( |X_i| > \varepsilon \underline{B}_n \right) \right] = 0.$$

*Then, for any $\varphi \in C_{b,\mathrm{Lip}}(\mathbb{R})$,*

$$\lim_{n \to \infty} \mathbb{E}\left[ \varphi\left( \frac{S_n}{\underline{B}_n} \right) \right] = \widetilde{\mathbb{E}}\left[ \varphi(\eta) \right].$$

## 4 Central limit theorem for capacity

The following theorem is the CLT for capacity under the Lindeberg condition.

**Theorem 4.1** *Assume that*

(1) *$\xi$ is G-normal distributed on a G-expectation space $(\widetilde{\Omega}, L_G^2(\widetilde{\Omega}), \widetilde{\mathbb{E}})$ with $\xi \sim \mathcal{N}(\{0\}; [\underline{\sigma}^2, 1]), 0 < \underline{\sigma} \leq 1$, and*

$$\lim_{n \to \infty} \frac{1}{\overline{B}_n^2} \sum_{i=1}^n \left| \underline{\sigma}^2 \cdot \overline{\sigma}_i^2 - \underline{\sigma}_i^2 \right| = 0. \tag{4.1}$$

(2) *For any $\varepsilon > 0$,*

$$\lim_{n \to \infty} \frac{1}{\overline{B}_n^2} \sum_{i=1}^n \mathbb{E}\left[ |X_i|^2 I\left( |X_i| > \varepsilon \overline{B}_n \right) \right] = 0. \tag{4.2}$$

*Then, for any $a \in \mathbb{R}$,*

$$\lim_{n \to \infty} \mathbb{V}\left( \frac{S_n}{\overline{B}_n} \leq a \right) = \widetilde{\mathbb{V}}(\xi \leq a), \qquad \lim_{n \to \infty} v\left( \frac{S_n}{\overline{B}_n} \leq a \right) = \widetilde{v}(\xi \leq a). \tag{4.3}$$

*Proof* For any fixed $\varepsilon > 0$, define

$$f(x) = \begin{cases} 1, & x \leq a, \\ -\frac{1}{\varepsilon}(x - a - \varepsilon), & a < x \leq a + \varepsilon, \\ 0, & x > a + \varepsilon, \end{cases}$$

and

$$g(x) = \begin{cases} 1, & x \le a - \varepsilon, \\ -\frac{1}{\varepsilon}(x-a), & a - \varepsilon < x \le a, \\ 0, & x > a. \end{cases}$$

It is easy to verify that $f, g \in C_{b,\mathrm{Lip}}(\mathbb{R})$ and $g(x) \le I(x \le a) \le f(x)$. It follows that

$$\left| \mathbb{V}\left( \frac{S_n}{\overline{B}_n} \le a \right) - \widetilde{\mathbb{V}}(\xi \le a) \right|$$

$$= \left| \mathbb{E}\left[ I\left( \frac{S_n}{\overline{B}_n} \le a \right) \right] - \widetilde{\mathbb{E}}[I(\xi \le a)] \right|$$

$$\le \left| \mathbb{E}\left[ f\left( \frac{S_n}{\overline{B}_n} \right) \right] - \widetilde{\mathbb{E}}[g(\xi)] \right| \vee \left| \mathbb{E}\left[ g\left( \frac{S_n}{\overline{B}_n} \right) \right] - \widetilde{\mathbb{E}}[f(\xi)] \right|$$

$$\le \left\{ \left| \mathbb{E}\left[ f\left( \frac{S_n}{\overline{B}_n} \right) \right] - \widetilde{\mathbb{E}}[f(\xi)] \right| + \left| \widetilde{\mathbb{E}}[f(\xi)] - \widetilde{\mathbb{E}}[g(\xi)] \right| \right\}$$

$$\vee \left\{ \left| \mathbb{E}\left[ g\left( \frac{S_n}{\overline{B}_n} \right) \right] - \widetilde{\mathbb{E}}[g(\xi)] \right| + \left| \widetilde{\mathbb{E}}[g(\xi)] - \widetilde{\mathbb{E}}[f(\xi)] \right| \right\}$$

$$\le \left| \mathbb{E}\left[ f\left( \frac{S_n}{\overline{B}_n} \right) \right] - \widetilde{\mathbb{E}}[f(\xi)] \right| \vee \left| \mathbb{E}\left[ g\left( \frac{S_n}{\overline{B}_n} \right) \right] - \widetilde{\mathbb{E}}[g(\xi)] \right| + \widetilde{\mathbb{E}}[|f(\xi) - g(\xi)|].$$

By Theorem 2.1 we have

$$\lim_{n \to \infty} \left| \mathbb{E}\left[ f\left( \frac{S_n}{\overline{B}_n} \right) \right] - \widetilde{\mathbb{E}}[f(\xi)] \right| = 0, \qquad \lim_{n \to \infty} \left| \mathbb{E}\left[ g\left( \frac{S_n}{\overline{B}_n} \right) \right] - \widetilde{\mathbb{E}}[g(\xi)] \right| = 0.$$

Then

$$\limsup_{n \to \infty} \left| \mathbb{V}\left( \frac{S_n}{\overline{B}_n} \le a \right) - \widetilde{\mathbb{V}}(\xi \le a) \right| \le \widetilde{\mathbb{E}}[|f(\xi) - g(\xi)|].$$

Note that

$$0 \le f(x) - g(x) \le I(a - \varepsilon < x \le a + \varepsilon),$$

which implies

$$\widetilde{\mathbb{E}}[|f(\xi) - g(\xi)|] \le \widetilde{\mathbb{V}}(a - \varepsilon < \xi \le a + \varepsilon).$$

By the arbitrariness of $\varepsilon > 0$ and Lemma 2.2, we have

$$\lim_{n \to \infty} \left| \mathbb{V}\left( \frac{S_n}{\overline{B}_n} \le a \right) - \widetilde{\mathbb{V}}(\xi \le a) \right| = 0,$$

which implies

$$\lim_{n \to \infty} \mathbb{V}\left( \frac{S_n}{\overline{B}_n} \le a \right) = \widetilde{\mathbb{V}}(\xi \le a).$$

Similarly, we can also obtain

$$\lim_{n\to\infty} \mathbb{V}\left(\frac{S_n}{\overline{B}_n} \geq a\right) = \widetilde{\mathbb{V}}(\xi \geq a).$$

That is,

$$\lim_{n\to\infty} v\left(\frac{S_n}{\overline{B}_n} \leq a\right) = \widetilde{v}(\xi \leq a). \qquad \square$$

*Remark* 4.1  By a similar method, we can also obtain the CLT for capacity for the normalized sum $S_n/\underline{B}_n$. We omit the details here.

## 5  Central limit theorem for summability methods

Let $c_i(\lambda)$ be continuous functions on $(0,\infty)$ or $\lambda$ only valued in $\mathbb{N}^*$. Assume that $0 \leq c_i(\lambda) \leq 1$ and, for any $\lambda > 0$,

$$\sum_{i=1}^{\infty} c_i(\lambda) = 1 + \theta(\lambda),$$

where $\lim_{\lambda\to\infty} \theta(\lambda) = 0$. Denote $\overline{B}_\lambda^2 = \sum_{i=1}^{\infty} c_i(\lambda)^2 \overline{\sigma}_i^2$, $\underline{B}_\lambda^2 = \sum_{i=1}^{\infty} c_i(\lambda)^2 \underline{\sigma}_i^2$, where $\overline{B}_\lambda^2 < \infty$ for any $\lambda > 0$. Assume that $\mathbb{E}$ is continuous from below and regular, then by Lemma 2.5, $S_\lambda := \sum_{i=1}^{\infty} c_i(\lambda) X_i$ is well defined q.s. under capacity $\mathbb{V}$.

**Theorem 5.1**  *Given a sub-linear expectation space* $(\Omega, \mathcal{H}, \mathbb{E})$, $\mathbb{E}$ *is continuous from below and regular. Assume that*

(1)  $\xi$ *is G-normal distributed on a G-expectation space* $(\widetilde{\Omega}, L_G^2(\widetilde{\Omega}), \widetilde{\mathbb{E}})$ *with*
   $\xi \sim \mathcal{N}(\{0\}; [\underline{\sigma}^2, 1])$, $0 < \underline{\sigma} \leq 1$, *and*

$$\lim_{\lambda\to\infty} \frac{1}{\overline{B}_\lambda^2} \sum_{i=1}^{\infty} c_i^2(\lambda) \left|\underline{\sigma}^2 \cdot \overline{\sigma}_i^2 - \underline{\sigma}_i^2\right| = 0. \tag{5.1}$$

(2)  *For any* $\varepsilon > 0$,

$$\lim_{\lambda\to\infty} \frac{1}{\overline{B}_\lambda^2} \sum_{i=1}^{\infty} c_i^2(\lambda) \mathbb{E}\left[|X_i|^2 I\left(c_i(\lambda)|X_i| > \varepsilon \overline{B}_\lambda\right)\right] = 0. \tag{5.2}$$

*Then, for any* $a \in \mathbb{R}$,

$$\lim_{\lambda\to\infty} \mathbb{V}\left(\frac{S_\lambda}{\overline{B}_\lambda} \leq a\right) = \widetilde{\mathbb{V}}(\xi \leq a), \qquad \lim_{\lambda\to\infty} v\left(\frac{S_\lambda}{\overline{B}_\lambda} \leq a\right) = \widetilde{v}(\xi \leq a). \tag{5.3}$$

*Proof*  Denote

$$S_{\lambda,N} = \sum_{i=1}^{N} c_i(\lambda) X_i, \qquad S_{\lambda,N}^* = \sum_{i=N+1}^{\infty} c_i(\lambda) X_i, \qquad \overline{B}_{\lambda,N}^2 = \sum_{i=1}^{N} c_i^2(\lambda) \overline{\sigma}_i^2.$$

Note that $\lim_{N\to\infty} \overline{B}_{\lambda,N}^2 = \overline{B}_\lambda^2$. For any $k > 1$, we can choose $N$ sufficiently large such that $\overline{B}_{\lambda,N}^2 \geq \overline{B}_\lambda^2 / k^2$.

For any $a > 0$, $t > 0$, and $X, Y \in \mathcal{H}$, it holds that

$$\{X + Y \leq a\} \subset \{X \leq a + t\} \cup \big\{|Y| \geq t\big\}.$$

Then

$$\mathbb{V}(X + Y \leq a) \leq \mathbb{V}(X \leq a + t) + \mathbb{V}\big(|Y| \geq t\big).$$

Hence

$$\mathbb{V}\left(\frac{S_\lambda}{\overline{B}_\lambda} \leq a\right) = \mathbb{V}\left(\frac{S_{\lambda,N}}{\overline{B}_\lambda} + \frac{S_{\lambda,N}^*}{\overline{B}_\lambda} \leq a\right)$$

$$\leq \mathbb{V}\left(\frac{S_{\lambda,N}}{\overline{B}_\lambda} \leq a + t\right) + \mathbb{V}\left(\frac{|S_{\lambda,N}^*|}{\overline{B}_\lambda} \geq t\right)$$

$$\leq \mathbb{V}\left(\frac{S_{\lambda,N}}{\overline{B}_{\lambda,N}} \leq k(a + t)\right) + \mathbb{V}\big(|S_{\lambda,N}^*| \geq t\overline{B}_\lambda\big).$$

For any $\eta > 0$, let

$$f(x) = \begin{cases} 1, & x \leq k(a + t), \\ -\frac{1}{\eta}(x - k(a + t) - \eta), & k(a + t) < x \leq k(a + t) + \eta, \\ 0, & x > k(a + t) + \eta, \end{cases}$$

and

$$g(x) = \begin{cases} 1, & x \leq k(a + t) - \eta, \\ -\frac{1}{\eta}(x - k(a + t)), & k(a + t) - \eta < x \leq k(a + t), \\ 0, & x > k(a + t). \end{cases}$$

It is easy to verify that $f, g \in C_{b,\mathrm{Lip}}(\mathbb{R})$ and $g(x) \leq I(x \leq k(a + t)) \leq f(x)$. By the proof process of Theorem 4.1, we have

$$\mathbb{V}\left(\frac{S_{\lambda,N}}{\overline{B}_{\lambda,N}} \leq k(a + t)\right)$$

$$\leq \widetilde{\mathbb{V}}\big(\xi \leq k(a + t)\big) + \left|\mathbb{E}\left[f\left(\frac{S_{\lambda,N}}{\overline{B}_{\lambda,N}}\right)\right] - \widetilde{\mathbb{E}}[f(\xi)]\right| \vee \left|\mathbb{E}\left[g\left(\frac{S_{\lambda,N}}{\overline{B}_{\lambda,N}}\right)\right] - \widetilde{\mathbb{E}}[g(\xi)]\right|$$

$$+ \widetilde{\mathbb{V}}\big(k(a + t) - \eta < \xi \leq k(a + t) + \eta\big).$$

By (3.1), for any $0 < \varepsilon < 1$, we have

$$\left|\mathbb{E}\left[f\left(\frac{S_{\lambda,N}}{\overline{B}_{\lambda,N}}\right)\right] - \widetilde{\mathbb{E}}[f(\xi)]\right| \vee \left|\mathbb{E}\left[g\left(\frac{S_{\lambda,N}}{\overline{B}_{\lambda,N}}\right)\right] - \widetilde{\mathbb{E}}[g(\xi)]\right|$$

$$\leq \frac{C_h}{\overline{B}_{\lambda,N}^2} \sum_{i=1}^N c_i^2(\lambda)\big|\underline{\sigma}^2 \cdot \overline{\sigma}_i^2 - \underline{\sigma}_i^2\big| + \frac{C_h}{\overline{B}_{\lambda,N}^2} \sum_{i=1}^N c_i^2(\lambda)\mathbb{E}\big[|X_i|^2 I\big(c_i(\lambda)|X_i| > \varepsilon\overline{B}_{\lambda,N}\big)\big]$$

$$
+ C_h\left(\frac{1}{\overline{B}_{\lambda,N}^2}\sum_{i=1}^{N}c_i^2(\lambda)\mathbb{E}\big[|X_i|^2 I\big(c_i(\lambda)|X_i| > \varepsilon\overline{B}_{\lambda,N}\big)\big]\right)^{1+\frac{\alpha}{2}}
$$
$$
+ C_h\varepsilon^\alpha + C(\sqrt{h} + \sqrt{1+h} - 1).
$$

Note that $\overline{B}_{\lambda,N}^2 \geq \overline{B}_\lambda^2/k^2$, we have

$$
\left|\mathbb{E}\left[f\left(\frac{S_{\lambda,N}}{\overline{B}_{\lambda,N}}\right)\right] - \widetilde{\mathbb{E}}[f(\xi)]\right| \vee \left|\mathbb{E}\left[g\left(\frac{S_{\lambda,N}}{\overline{B}_{\lambda,N}}\right)\right] - \widetilde{\mathbb{E}}[g(\xi)]\right|
$$
$$
\leq \frac{k^2 C_h}{\overline{B}_\lambda^2}\sum_{i=1}^{\infty}c_i^2(\lambda)\big|\underline{\sigma}^2\cdot\overline{\sigma}_i^2 - \underline{\sigma}_i^2\big| + \frac{k^2 C_h}{\overline{B}_\lambda^2}\sum_{i=1}^{\infty}c_i^2(\lambda)\mathbb{E}\left[|X_i|^2 I\left(c_i(\lambda)|X_i| > \frac{\varepsilon}{k}\overline{B}_\lambda\right)\right]
$$
$$
+ C_h\left(\frac{k^2}{\overline{B}_\lambda^2}\sum_{i=1}^{\infty}c_i^2(\lambda)\mathbb{E}\left[|X_i|^2 I\left(c_i(\lambda)|X_i| > \frac{\varepsilon}{k}\overline{B}_\lambda\right)\right]\right)^{1+\frac{\alpha}{2}}
$$
$$
+ C_h\varepsilon^\alpha + C(\sqrt{h} + \sqrt{1+h} - 1).
$$

In addition, by Lemma 2.5 we have

$$
\mathbb{E}\big[|S_{\lambda,N}^*|^2\big] = \mathbb{E}\left[\left|\sum_{i=N+1}^{\infty}c_i(\lambda)X_i\right|^2\right] \leq C\sum_{i=N+1}^{\infty}c_i^2(\lambda)\mathbb{E}\big[|X_i|^2\big] \searrow 0 \quad \text{as } N \to \infty.
$$

Let $t = (\mathbb{E}[|S_{\lambda,N}^*|^2])^{\frac{1}{3}}$. We have $\lim_{N\to\infty}t = 0$ and

$$
\mathbb{V}\big(|S_{\lambda,N}^*| \geq t\overline{B}_\lambda\big) \leq \frac{1}{t^2\overline{B}_\lambda^2}\mathbb{E}\big[|S_{\lambda,N}^*|^2\big] = \frac{1}{\overline{B}_\lambda^2}\big(\mathbb{E}\big[|S_{\lambda,N}^*|^2\big]\big)^{\frac{1}{3}} \to 0 \quad \text{as } N \to \infty.
$$

So we have

$$
\mathbb{V}\left(\frac{S_\lambda}{\overline{B}_\lambda} \leq a\right)
$$
$$
\leq \widetilde{\mathbb{V}}\big(\xi \leq k(a+t)\big) + \widetilde{\mathbb{V}}\big(k(a+t) - \eta < \xi \leq k(a+t) + \eta\big) + \frac{1}{t^2\overline{B}_\lambda^2}\mathbb{E}\big[|S_{\lambda,N}^*|^2\big]
$$
$$
+ \frac{k^2 C_h}{\overline{B}_\lambda^2}\sum_{i=1}^{\infty}c_i^2(\lambda)\big|\underline{\sigma}^2\cdot\overline{\sigma}_i^2 - \underline{\sigma}_i^2\big| + \frac{k^2 C_h}{\overline{B}_\lambda^2}\sum_{i=1}^{\infty}c_i^2(\lambda)\mathbb{E}\left[|X_i|^2 I\left(c_i(\lambda)|X_i| > \frac{\varepsilon}{k}\overline{B}_\lambda\right)\right]
$$
$$
+ C_h\left(\frac{k^2}{\overline{B}_\lambda^2}\sum_{i=1}^{\infty}c_i^2(\lambda)\mathbb{E}\left[|X_i|^2 I\left(c_i(\lambda)|X_i| > \frac{\varepsilon}{k}\overline{B}_\lambda\right)\right]\right)^{1+\frac{\alpha}{2}} + C_h\varepsilon^\alpha
$$
$$
+ C(\sqrt{h} + \sqrt{1+h} - 1).
$$

By (5.1), (5.2), Lemma 2.1, and Lemma 2.2, letting $t = (\mathbb{E}[|S_{\lambda,N}^*|^2])^{\frac{1}{3}}$, $N \to \infty$, $\lambda \to \infty$, $\varepsilon \to 0$, and $\eta \to 0$ in turn, we have

$$
\limsup_{\lambda\to\infty}\mathbb{V}\left(\frac{S_\lambda}{\overline{B}_\lambda} \leq a\right) \leq \widetilde{\mathbb{V}}(\xi \leq ka) + C(\sqrt{h} + \sqrt{1+h} - 1).
$$

By the arbitrariness of $k > 1$ and $h > 0$, we get

$$\limsup_{\lambda \to \infty} \mathbb{V}\left(\frac{S_\lambda}{\overline{B}_\lambda} \le a\right) \le \widetilde{\mathbb{V}}(\xi \le a). \tag{5.4}$$

On the other hand, for any $a > 0$, $0 < t < a$, and $X, Y \in \mathcal{H}$,

$$\{X + Y \le a\} \supset \{X \le a - t\} \setminus \{|Y| \ge t\}.$$

Then

$$\mathbb{V}(X + Y \le a) \ge \mathbb{V}(X \le a - t) - \mathbb{V}(|Y| \ge t).$$

Hence

$$\begin{aligned}
\mathbb{V}\left(\frac{S_\lambda}{\overline{B}_\lambda} \le a\right) &= \mathbb{V}\left(\frac{S_{\lambda,N}}{\overline{B}_\lambda} + \frac{S^*_{\lambda,N}}{\overline{B}_\lambda} \le a\right) \\
&\ge \mathbb{V}\left(\frac{S_{\lambda,N}}{\overline{B}_\lambda} \le a - t\right) - \mathbb{V}\left(\frac{|S^*_{\lambda,N}|}{\overline{B}_\lambda} \ge t\right) \\
&\ge \mathbb{V}\left(\frac{S_{\lambda,N}}{\overline{B}_{\lambda,N}} \le a - t\right) - \mathbb{V}\left(|S^*_{\lambda,N}| \ge t\overline{B}_\lambda\right).
\end{aligned}$$

By the same method as before, we have

$$\begin{aligned}
\mathbb{V}&\left(\frac{S_\lambda}{\overline{B}_\lambda} \le a\right) \\
&\ge \widetilde{\mathbb{V}}(\xi \le a - t) - \widetilde{\mathbb{V}}(a - t - \eta < \xi \le a - t + \eta) - \frac{1}{t^2 \overline{B}_\lambda^2} \mathbb{E}\left[|S^*_{\lambda,N}|^2\right] \\
&\quad - \frac{C_h}{\overline{B}_\lambda^2} \sum_{i=1}^\infty c_i^2(\lambda) |\underline{\sigma}^2 \cdot \overline{\sigma}_i^2 - \underline{\sigma}_i^2| - \frac{C_h}{\overline{B}_\lambda^2} \sum_{i=1}^\infty c_i^2(\lambda) \mathbb{E}\left[|X_i|^2 I\left(c_i(\lambda)|X_i| > \frac{\varepsilon}{k}\overline{B}_\lambda\right)\right] \\
&\quad - C_h\left(\frac{1}{\overline{B}_\lambda^2} \sum_{i=1}^\infty c_i^2(\lambda) \mathbb{E}\left[|X_i|^2 I\left(c_i(\lambda)|X_i| > \frac{\varepsilon}{k}\overline{B}_\lambda\right)\right]\right)^{1+\frac{\alpha}{2}} \\
&\quad - C_h \varepsilon^\alpha - C(\sqrt{h} + \sqrt{1+h} - 1).
\end{aligned}$$

Letting $t = (\mathbb{E}[|S^*_{\lambda,N}|^2])^{\frac{1}{3}}$, $N \to \infty$, $\lambda \to \infty$, $\varepsilon \to 0$, and $\eta \to 0$ in turn, we have

$$\liminf_{\lambda \to \infty} \mathbb{V}\left(\frac{S_\lambda}{\overline{B}_\lambda} \le a\right) \ge \widetilde{\mathbb{V}}(\xi \le a) - C(\sqrt{h} + \sqrt{1+h} - 1).$$

By the arbitrariness of $h > 0$, we have

$$\liminf_{\lambda \to \infty} \mathbb{V}\left(\frac{S_\lambda}{\overline{B}_\lambda} \le a\right) \ge \widetilde{\mathbb{V}}(\xi \le a). \tag{5.5}$$

Combining (5.5) with (5.4), we obtain

$$\lim_{\lambda \to \infty} \mathbb{V}\left(\frac{S_\lambda}{\overline{B}_\lambda} \le a\right) = \widetilde{\mathbb{V}}(\xi \le a).$$

Similarly, we can also have

$$\lim_{\lambda \to \infty} \mathbb{V}\left( \frac{S_\lambda}{\overline{B_\lambda}} \geq a \right) = \widetilde{\mathbb{V}}(\xi \geq a).$$

This is equivalent to

$$\lim_{\lambda \to \infty} v\left( \frac{S_\lambda}{\overline{B_\lambda}} \leq a \right) = \widetilde{v}(\xi \leq a). \qquad \square$$

**Competing interests**
The author declares that there are no competing interests.

**Authors' contributions**
The author organized and wrote this paper. The author read and approved the final manuscript.

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