

RESEARCH

Open Access



Jensen–Steffensen inequality for strongly convex functions

M. Klaričić Bakula^{1*} 

*Correspondence: milica@pmfst.hr
¹Faculty of Science, University of Split, Split, Croatia

Abstract

The Jensen inequality for convex functions holds under the assumption that all of the included weights are nonnegative. If we allow some of the weights to be negative, such an inequality is called the Jensen–Steffensen inequality for convex functions. In this paper we prove the Jensen–Steffensen inequality for strongly convex functions.

MSC: 26D15; 39B62; 26B25

Keywords: Strongly convex functions; Jensen inequality; Jensen–Steffensen inequality

1 Introduction

Let $I \subset \mathbb{R}$ be an interval. It is well known that if a function $f : I \rightarrow \mathbb{R}$ is convex, then

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq \sum_{i=1}^n p_i f(x_i) \quad (1.1)$$

for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$, and $p_1, \dots, p_n > 0$ with $p_1 + \dots + p_n = 1$. If f is strictly convex, then (1.1) is strict unless all x_i are equal [7, p. 43]. This classical *Jensen inequality* is one of the most important inequalities in convex analysis, and it has various applications in mathematics, statistics, economics, and engineering sciences.

It is also known that the assumption $p_1, \dots, p_n > 0$ can be relaxed at the expense of restricting x_1, \dots, x_n more severely [9]. Namely, if $\mathbf{p} = (p_1, \dots, p_n)$ is a real n -tuple such that for every $k \in \{1, \dots, n\}$

$$0 \leq p_1 + \dots + p_k \leq p_1 + \dots + p_n = 1, \quad (1.2)$$

then for any monotonic n -tuple $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ (increasing or decreasing) we get

$$\bar{x} = p_1 x_1 + \dots + p_n x_n \in I,$$

and for any function f convex on I (1.1) still holds. Under such assumptions (1.1) is called *the Jensen–Steffensen inequality* for convex functions and (1.2) are called *Steffensen's conditions* due to J. F. Steffensen. Again, for a strictly convex function f , (1.1) remains strict

under certain additional assumptions on \mathbf{x} and \mathbf{p} [1]. It is needless to say that a mathematical mind has to question the limitation $p_1, \dots, p_n > 0$ even if in the usual practice we can cope with it.

Variants of the Jensen inequality are proved for various classes of generalized convex functions, and the class of strongly convex functions is among them. Recall that a function $f : I \rightarrow \mathbb{R}$ is called *strongly convex with modulus* $c > 0$ if

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) - ct(1 - t)(x - y)^2 \tag{1.3}$$

for all $x, y \in I$ and $t \in [0, 1]$ [8]. Strongly convex functions are useful in optimization theory, mathematical economics and approximation theory, and an interested reader can find more about them in an excellent survey paper [5].

As we can easily see, strong convexity is a strengthening of the notion of convexity, and some properties of strongly convex functions are just “stronger versions” of analogous properties of convex functions (for more details, see [5]). One example of such a stronger version is the Jensen inequality for strongly convex functions (see [4] or [5]). If $f : I \rightarrow \mathbb{R}$, $I \subset \mathbb{R}$, is strongly convex with modulus c , then

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq \sum_{i=1}^n p_i f(x_i) - c \sum_{i=1}^n p_i (x_i - \bar{x})^2 \tag{1.4}$$

for all $x_1, \dots, x_n \in I$ and all $p_1, \dots, p_n > 0$ such that $p_1 + \dots + p_n = 1$. If we compare (1.4) with (1.1), we see that (1.4) provides a better upper bound for $f(\bar{x})$ since the term $c \sum_{i=1}^n p_i (x_i - \bar{x})^2$ is always nonnegative. Of course, if $c = 0$, we go right back to convex functions and (1.1).

We must emphasize here that proving a Jensen type inequality for some class of generalized convex functions does not necessarily mean that such inequality holds under Steffensen’s conditions. The goal of this paper is to prove that for the class of strongly convex functions this is not the case.

2 Main result

Strongly convex functions have a very useful characterization: they always have a specific convex representation. This is stated in the following theorem (see [3] or [6]).

Theorem 1 *Let I be an interval in \mathbb{R} . A function $f : I \rightarrow \mathbb{R}$ is strongly convex with modulus c if and only if the function $g = f - c(\cdot)^2$ is convex.*

The Jensen inequality for strongly convex functions can be proved either using Theorem 1 and the Jensen inequality for convex functions or (for I open) directly, using the “support parabola” property [5, Theorem 1]. In this section we prove the Jensen–Steffensen inequality for strongly convex functions using Theorem 1.

In the rest of the paper we use the following notation related to the n -tuples $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{p} = (p_1, \dots, p_n)$, $n \in \mathbb{N}$:

$$\begin{aligned} \bar{x} &= p_1 x_1 + \dots + p_n x_n, \\ P_k &= p_1 + \dots + p_k, \quad k \in \{1, 2, \dots, n\}, \end{aligned}$$

$$\bar{P}_k = p_k + \dots + p_n, \quad k \in \{1, 2, \dots, n\}.$$

Theorem 2 *Let I be an interval in \mathbb{R} . If $f : I \rightarrow \mathbb{R}$ is a strongly convex function with modulus c , then for every monotonic n -tuple $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ and every real n -tuple $\mathbf{p} = (p_1, \dots, p_n)$ such that, for every $i \in \{1, 2, \dots, n\}$,*

$$0 \leq P_i \leq P_n = 1$$

the following inequality holds:

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq \sum_{i=1}^n p_i f(x_i) - c \sum_{i=1}^n p_i (x_i - \bar{x})^2.$$

Proof Suppose that \mathbf{x} is increasing (for \mathbf{x} decreasing the proof is analogous). It can be easily seen that Steffensen’s conditions yield

$$\bar{P}_k \geq 0, \quad k \in \{1, 2, \dots, n\},$$

and

$$x_n - \bar{x} = P_n(x_n - \bar{x}) = \sum_{i=1}^{n-1} P_i(x_{i+1} - x_i) \geq 0,$$

hence we obtain $\bar{x} \leq x_n$. Analogously,

$$\bar{x} - x_1 = P_n(\bar{x} - x_1) = \sum_{i=2}^n \bar{P}_i(x_i - x_{i-1}) \geq 0,$$

and $x_1 \leq \bar{x}$. From that we may conclude $\bar{x} \in [x_1, x_n] \subset I$, which means that $g(\bar{x}) = g(\sum_{i=1}^n p_i x_i)$ is defined.

Using the convex representation $g = f - c(\cdot)^2$ as in Theorem 1 and applying the Jensen–Steffensen inequality for convex functions, we obtain

$$g\left(\sum_{i=1}^n p_i x_i\right) \leq \sum_{i=1}^n p_i g(x_i).$$

Returning back to f , we get

$$\begin{aligned} f\left(\sum_{i=1}^n p_i x_i\right) - c\left(\sum_{i=1}^n p_i x_i\right)^2 &\leq \sum_{i=1}^n p_i (f(x_i) - cx_i^2) \\ &= \sum_{i=1}^n p_i f(x_i) - c \sum_{i=1}^n p_i x_i^2, \end{aligned}$$

or written differently

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq \sum_{i=1}^n p_i f(x_i) - c \left[\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i\right)^2 \right]$$

$$\begin{aligned}
 &= \sum_{i=1}^n p_i f(x_i) - c \left[\sum_{i=1}^n p_i x_i^2 - \bar{x}^2 \right] \\
 &= \sum_{i=1}^n p_i f(x_i) - c \left[\sum_{i=1}^n p_i x_i^2 - 2\bar{x}^2 + \bar{x}^2 \right] \\
 &= \sum_{i=1}^n p_i f(x_i) - c \left[\sum_{i=1}^n p_i x_i^2 - 2\bar{x} \sum_{i=1}^n p_i x_i + \bar{x}^2 \sum_{i=1}^n p_i \right] \\
 &= \sum_{i=1}^n p_i f(x_i) - c \sum_{i=1}^n p_i (x_i - \bar{x})^2. \quad \square
 \end{aligned}$$

3 Alternative reproach

What would happen if we try to prove (1.4) under Steffensen’s conditions directly using the support parabola property? The question is not without sense since in the case of the Jensen inequality for strongly convex functions both ways produce the same inequality as in (1.4) but, generally speaking, any negative weights in \mathbf{p} can at some place interrupt the chain of conclusions in a proof. This is exactly the reason why it is considerably more difficult to prove (1.1) under Steffensen’s conditions. We will see what happens in this case in the next theorem, but first we need the following lemma which basically says that the support parabola in x_0 can be “shifted up” from x_0 to y and still remain “under” $f(x)$ if $x \leq y \leq x_0$.

Lemma 1 *Let $I \subset \mathbb{R}$ be an open interval, let $f : I \rightarrow \mathbb{R}$ be a strongly convex function with modulus c , and for $x_0 \in I$ let*

$$y = f(x_0) + \lambda(x - x_0) + c(x - x_0)^2 \tag{3.1}$$

be the support parabola for f in x_0 . Then for every $x, y \in I$ such that $x \leq y \leq x_0$

$$f(x) - f(y) \geq \lambda(x - y) + c(x - y)^2, \tag{3.2}$$

and for $x, y \in I$ such that $x_0 \leq x \leq y$

$$f(y) - f(x) \geq \lambda(y - x) + c(y - x)^2. \tag{3.3}$$

Proof Since (3.1) is a support parabola for f in x_0 , it follows that for every $x \in I$

$$f(x) - f(x_0) \geq \lambda(x - x_0) + c(x - x_0)^2. \tag{3.4}$$

Let $x, y \in I$ be such that $x < y < x_0$. The middle element y can be represented as a convex combination of x and z in the following way:

$$y = \frac{x_0 - y}{x_0 - x}x + \frac{y - x}{x_0 - x}x_0.$$

From the definition of strong convexity we have

$$f(y) \leq \frac{x_0 - y}{x_0 - x}f(x) + \frac{y - x}{x_0 - x}f(x_0) - c \frac{x_0 - y}{x_0 - x} \frac{y - x}{x_0 - x} (x - x_0)^2,$$

and since

$$\frac{x_0 - y}{x_0 - x} + \frac{y - x}{x_0 - x} = 1,$$

we can write

$$\begin{aligned} f(y) &= \frac{x_0 - y}{x_0 - x}f(y) + \frac{y - x}{x_0 - x}f(y) \\ &\leq \frac{x_0 - y}{x_0 - x}f(x) + \frac{y - x}{x_0 - x}f(x_0) - c\frac{x_0 - y}{x_0 - x}\frac{y - x}{x_0 - x}(x - x_0)^2. \end{aligned}$$

After a simple calculation we obtain

$$(x_0 - y)(f(x) - f(y)) \geq (x - y)(f(x_0) - f(y)) + c\frac{(x_0 - y)(y - x)}{x_0 - x}(x - x_0)^2$$

and

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(x_0) - f(y)}{x_0 - y} - c(x_0 - x). \tag{3.5}$$

The support parabola property (3.4) gives

$$f(y) - f(x_0) \geq \lambda(y - x_0) + c(y - x_0)^2,$$

and since $y - x_0 < 0$

$$\frac{f(x_0) - f(y)}{x_0 - y} \leq \lambda - c(x_0 - y).$$

Using the above inequality and (3.5), we obtain

$$\begin{aligned} \frac{f(x) - f(y)}{x - y} &\leq \frac{f(x_0) - f(y)}{x_0 - y} - c(x_0 - x) \\ &\leq \lambda - c(x_0 - y) - c(x_0 - x) = \lambda + c(x + y - 2x_0). \end{aligned}$$

Since $x - y < 0$ we get

$$f(x) - f(y) \geq \lambda(x - y) + c(x - y)(x + y - 2x_0),$$

and because of $x + y - 2x_0 < x - y$, we end up with

$$f(x) - f(y) \geq \lambda(x - y) + c(x - y)^2.$$

If $x_0 < x < y$, in an analogous way we can prove

$$f(y) - f(x) \geq \lambda(y - x) + c(y - x)^2.$$

Note that the above inequalities still hold in the trivial way if $x = y$. □

Remark 1 (3.2) and (3.3) can be also proved using the convex representation $g = f - c(\cdot)^2$. We start from the support parabola property in $x_0 \in I$

$$f(x) - f(x_0) \geq \lambda(x - x_0) + c(x - x_0)^2.$$

Then

$$g(x) - g(x_0) + cx^2 - cx_0^2 \geq \lambda(x - x_0) + c(x - x_0)^2,$$

that is,

$$\begin{aligned} g(x) - g(x_0) &\geq \lambda(x - x_0) + c(x - x_0)^2 - cx^2 + cx_0^2 \\ &= (\lambda - 2cx_0)(x - x_0) = \lambda'(x - x_0), \end{aligned}$$

hence g has a support line in x_0 for $\lambda' = \lambda - 2cx_0$. Since g is convex, we know that for every $x_0 \leq x \leq y$ [7]

$$g(y) - g(x) \geq \lambda'(y - x) = (\lambda - 2cx_0)(y - x).$$

Returning to f , we obtain

$$f(y) - cy^2 - f(x) + cx^2 \geq (\lambda - 2cx_0)(y - x),$$

hence

$$\begin{aligned} f(y) - f(x) &\geq (\lambda - 2cx_0)(y - x) + cy^2 - cx^2 \\ &= \lambda(y - x) + c(y - x)(x + y - 2x_0) \\ &\geq \lambda(y - x) + c(y - x)(x + y - 2x) \\ &= \lambda(y - x) + c(y - x)^2. \end{aligned}$$

Consequently,

$$f(y) - f(x) \geq \lambda(y - x) + c(y - x)^2, \quad x_0 \leq x \leq y.$$

Analogously, we can prove

$$f(x) - f(y) \geq \lambda(x - y) + c(x - y)^2, \quad x \leq y \leq x_0.$$

Theorem 3 *Let $I \subset \mathbb{R}$ be an open interval. If $f : I \rightarrow \mathbb{R}$ is a strongly convex function with modulus c , then for every monotonic n -tuple $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ and every real n -tuple $\mathbf{p} = (p_1, \dots, p_n)$ such that for every $i \in \{1, 2, \dots, n\}$*

$$0 \leq P_i \leq P_n = 1,$$

there exists $k \in \{1, \dots, n - 1\}$ such that $\bar{x} \in [x_k, x_{k+1}]$ for \mathbf{x} increasing or $\bar{x} \in [x_{k+1}, x_k]$ for \mathbf{x} decreasing, and

$$\begin{aligned} & \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \\ & \geq c \left[\sum_{i=1}^{k-1} P_i(x_i - x_{i+1})^2 + P_k(x_k - \bar{x})^2 + \bar{P}_{k+1}(x_{k+1} - \bar{x})^2 + \sum_{i=k+2}^n \bar{P}_i(x_i - x_{i-1})^2 \right] \\ & \geq 0. \end{aligned}$$

Proof Suppose that \mathbf{x} is increasing (for \mathbf{x} decreasing the proof is analogous).

First observe that as in Theorem 2 we know that $\bar{x} \in [x_1, x_n] \subset I$, and we may conclude that there exists some $k \in \{1, \dots, n - 1\}$ such that $\bar{x} \in [x_k, x_{k+1}]$.

From (3.4), choosing $x_0 = \bar{x}$, we get

$$f(x) - f(\bar{x}) \geq \lambda(x - \bar{x}) + c(x - \bar{x})^2$$

for some $\lambda \in \mathbb{R}$ and every $x \in I$.

Next we use the Abel transformation to obtain the identities (similar can be found in [1])

$$\begin{aligned} 0 &= \sum_{i=1}^n p_i x_i - \bar{x} \\ &= \sum_{i=1}^{k-1} P_i(x_i - x_{i+1}) + P_k(x_k - \bar{x}) \\ &\quad + \bar{P}_{k+1}(x_{k+1} - \bar{x}) + \sum_{i=k+2}^n \bar{P}_i(x_i - x_{i-1}) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} & \sum_{i=1}^n p_i f(x_i) - f(\bar{x}) \\ &= \sum_{i=1}^{k-1} P_i(f(x_i) - f(x_{i+1})) + P_k(f(x_k) - f(\bar{x})) \\ &\quad + \bar{P}_{k+1}(f(x_{k+1}) - f(\bar{x})) + \sum_{i=k+2}^n \bar{P}_i(f(x_i) - f(x_{i-1})), \end{aligned} \tag{3.7}$$

where in the case $k = 1$ we assume $\sum_{i=1}^{k-1}$ to be 0, while in the case $k = n - 1$ we assume $\sum_{i=k+2}^n$ to be 0.

From (3.7), using (3.2), (3.3), and then (3.6), we get

$$\begin{aligned} & \sum_{i=1}^n p_i f(x_i) - f(\bar{x}) \\ & \geq \sum_{i=1}^{k-1} P_i(\lambda(x_i - x_{i+1}) + c(x_i - x_{i+1})^2) + P_k(\lambda(x_k - \bar{x}) + c(x_k - \bar{x})^2) \end{aligned}$$

$$\begin{aligned}
 & + \bar{P}_{k+1}(\lambda(x_{k+1} - \bar{x}) + c(x_{k+1} - \bar{x})^2) + \sum_{i=k+2}^n \bar{P}_i(\lambda(x_i - x_{i-1}) + c(x_i - x_{i-1})^2) \\
 & = \lambda \left[\sum_{i=1}^{k-1} P_i(x_i - x_{i+1}) + P_k(x_k - \bar{x}) + \bar{P}_{k+1}(x_{k+1} - \bar{x}) + \sum_{i=k+2}^n \bar{P}_i(x_i - x_{i-1}) \right] \\
 & \quad + c \left[\sum_{i=1}^{k-1} P_i(x_i - x_{i+1})^2 + P_k(x_k - \bar{x})^2 + \bar{P}_{k+1}(x_{k+1} - \bar{x})^2 + \sum_{i=k+2}^n \bar{P}_i(x_i - x_{i-1})^2 \right] \\
 & = c \left[\sum_{i=1}^{k-1} P_i(x_i - x_{i+1})^2 + P_k(x_k - \bar{x})^2 + \bar{P}_{k+1}(x_{k+1} - \bar{x})^2 + \sum_{i=k+2}^n \bar{P}_i(x_i - x_{i-1})^2 \right]. \quad \square
 \end{aligned}$$

It was hopeful to think that this way we can end up with

$$\sum_{i=1}^n p_i f(x_i) - f(\bar{x}) \geq c \sum_{i=1}^n p_i (x_i - \bar{x})^2$$

since this is exactly what happens in the analogous proofs (direct and indirect) for convex functions. It would be possible if

$$\begin{aligned}
 & \sum_{i=1}^{k-1} P_i(x_i - x_{i+1})^2 + P_k(x_k - \bar{x})^2 + \bar{P}_{k+1}(x_{k+1} - \bar{x})^2 + \sum_{i=k+2}^n \bar{P}_i(x_i - x_{i-1})^2 \\
 & \geq \sum_{i=1}^n p_i (x_i - \bar{x})^2, \tag{3.8}
 \end{aligned}$$

but sadly this is not generally true.

Example 1 Let $\mathbf{x} = (1, 2, 3, 4)$, $\mathbf{p} = (1, -1, 0, 1)$. Then

$$\begin{aligned}
 & P_1 = 1, \quad P_2 = 0, \quad P_3 = 0, \quad P_4 = 1, \\
 & \bar{P}_1 = 1, \quad \bar{P}_2 = 0, \quad \bar{P}_3 = 1, \quad \bar{P}_4 = 1, \\
 & \bar{x} = 3 \in [2, 3], \quad k = 2 \text{ (or } k = 3), \\
 & \sum_{i=1}^1 P_i(x_i - x_{i+1})^2 + P_2(x_2 - \bar{x})^2 + \bar{P}_3(x_3 - \bar{x})^2 + \sum_{i=4}^4 \bar{P}_i(x_i - x_{i-1})^2 \\
 & \quad = (1 - 2)^2 + 0 + (3 - 3)^2 + (4 - 3)^2 = 2, \\
 & \sum_{i=1}^4 p_i(x_i - \bar{x})^2 = (1 - 3)^2 - (2 - 3)^2 + 0 + (4 - 3)^2 = 4 > 2.
 \end{aligned}$$

In fact, the following theorem holds.

Theorem 4 *Let $f, \mathbf{p}, \mathbf{x}$, and k be as in Theorem 3. Then*

$$\sum_{i=1}^{k-1} P_i(x_i - x_{i+1})^2 + P_k(x_k - \bar{x})^2 + \bar{P}_{k+1}(x_{k+1} - \bar{x})^2 + \sum_{i=k+2}^n \bar{P}_i(x_i - x_{i-1})^2$$

$$\leq \sum_{i=1}^n p_i(x_i - \bar{x})^2.$$

Proof For the sake of simplicity, we introduce the following notation:

$$I_k = \sum_{i=1}^{k-1} P_i(x_i - x_{i+1})^2 + P_k(x_k - \bar{x})^2 + \bar{P}_{k+1}(x_{k+1} - \bar{x})^2 + \sum_{i=k+2}^n \bar{P}_i(x_i - x_{i-1})^2,$$

$$\bar{x}^2 = \sum_{i=1}^n p_i x_i^2.$$

Suppose that \mathbf{x} is increasing (for \mathbf{x} decreasing the proof is analogous). First note that for k as in Theorem 3 we have

$$x_i \leq \bar{x}, \quad i = 1, 2, \dots, k,$$

$$\bar{x} \leq x_i, \quad i = k + 1, \dots, n.$$

Using this notation, we get

$$I_k = \sum_{i=1}^{k-1} P_i(x_i^2 - x_{i+1}^2) + P_k(x_k^2 - \bar{x}^2) + \bar{P}_{k+1}(x_{k+1}^2 - \bar{x}^2) + \sum_{i=k+2}^n \bar{P}_i(x_i^2 - x_{i-1}^2)$$

$$- 2 \sum_{i=1}^{k-1} P_i x_i x_{i+1} + 2 \sum_{i=1}^{k-1} P_i x_i^2 + P_k(-2x_k \bar{x} + \bar{x}^2) + P_k \bar{x}^2 + \bar{P}_{k+1} \bar{x}^2$$

$$+ \bar{P}_{k+1}(-2x_{k+1} \bar{x} + \bar{x}^2) - 2 \sum_{i=k+2}^n \bar{P}_i x_i x_{i-1} + 2 \sum_{i=k+2}^n \bar{P}_i x_{i-1}^2.$$

Applying (3.6) on \mathbf{p} and $\mathbf{x}^2 = (x_1^2, \dots, x_n^2)$, we obtain

$$\sum_{i=1}^{k-1} P_i(x_i^2 - x_{i+1}^2) + P_k(x_k^2 - \bar{x}^2) + \bar{P}_{k+1}(x_{k+1}^2 - \bar{x}^2) + \sum_{i=k+2}^n \bar{P}_i(x_i^2 - x_{i-1}^2) = 0,$$

hence

$$I_k = 2 \sum_{i=1}^{k-1} P_i x_{i+1}(x_{i+1} - x_i) + 2 \sum_{i=k+2}^n \bar{P}_i x_{i-1}(x_{i-1} - x_i) + P_k \bar{x}^2 + \bar{P}_{k+1} \bar{x}^2$$

$$+ P_k(-2x_k \bar{x} + \bar{x}^2) + \bar{P}_{k+1}(-2x_{k+1} \bar{x} + \bar{x}^2)$$

$$= 2 \sum_{i=1}^{k-1} P_i x_{i+1}(x_{i+1} - x_i) + 2 \sum_{i=k+2}^n \bar{P}_i x_{i-1}(x_{i-1} - x_i) + \bar{x}^2$$

$$+ P_k(-2x_k \bar{x} + \bar{x}^2) + \bar{P}_{k+1}(-2x_{k+1} \bar{x} + \bar{x}^2).$$

Taking into account that \mathbf{x} is increasing and

$$P_i, \bar{P}_i \geq 0, \quad i = 1, 2, \dots, n,$$

$$\begin{aligned}
 x_i &\leq \bar{x}, \quad i = 1, 2, \dots, k, \\
 \bar{x} &\leq x_i, \quad i = k + 1, \dots, n,
 \end{aligned}$$

we obtain

$$\begin{aligned}
 I_k &\leq 2\bar{x} \sum_{i=1}^{k-1} P_i(x_{i+1} - x_i) + 2\bar{x} \sum_{i=k+2}^n \bar{P}_i(x_{i-1} - x_i) + \bar{\mathbf{x}}^2 \\
 &\quad - 2P_k x_k \bar{x} - 2\bar{P}_{k+1} x_{k+1} \bar{x} + \bar{x}^2.
 \end{aligned}$$

Applying again (3.6) on \mathbf{p} and \mathbf{x} , we get

$$\sum_{i=1}^{k-1} P_i(x_{i+1} - x_i) + \sum_{i=k+2}^n \bar{P}_i(x_{i-1} - x_i) = P_k(x_k - \bar{x}) + \bar{P}_{k+1}(x_{k+1} - \bar{x}),$$

hence

$$\begin{aligned}
 I_k &\leq 2\bar{x}[P_k(x_k - \bar{x}) + \bar{P}_{k+1}(x_{k+1} - \bar{x})] + \bar{\mathbf{x}}^2 - 2P_k x_k \bar{x} - 2\bar{P}_{k+1} x_{k+1} \bar{x} + \bar{x}^2 \\
 &= -2P_k \bar{x}^2 - 2\bar{P}_{k+1} \bar{x}^2 + \bar{x}^2 + \bar{\mathbf{x}}^2 = -2\bar{x}^2 + \bar{x}^2 + \bar{\mathbf{x}}^2 = \bar{\mathbf{x}}^2 - \bar{x}^2,
 \end{aligned}$$

or written differently

$$I_k \leq \sum_{i=1}^n p_i x_i^2 - \bar{x}^2 = \sum_{i=1}^n p_i (x_i - \bar{x})^2. \quad \square$$

We have just proven that the Jensen–Steffensen inequality for strongly convex functions behaves differently than the Jensen inequality for strongly convex functions: applying the same proof techniques, we end up with two different bounds, and surprisingly the indirect proof gives the better one.

4 Integral version

The integral version of the Jensen–Steffensen inequality for convex functions was proved by Boas in 1970 [2].

Theorem 5 *Let $x : [\alpha, \beta] \rightarrow (a, b)$ be a continuous and monotonic function, where $-\infty < \alpha < \beta < +\infty$ and $-\infty \leq a < b \leq +\infty$, and let $f : (a, b) \rightarrow \mathbb{R}$ be a convex function. If $\lambda : [\alpha, \beta] \rightarrow \mathbb{R}$ is either continuous or of bounded variation satisfying*

$$\begin{aligned}
 (\forall t \in [\alpha, \beta]) \quad \lambda(\alpha) &\leq \lambda(t) \leq \lambda(\beta), \\
 \lambda(\beta) - \lambda(\alpha) &> 0,
 \end{aligned}$$

then

$$f\left(\frac{\int_{\alpha}^{\beta} x(t) \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)}\right) \leq \frac{\int_{\alpha}^{\beta} f(x(t)) \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)}.$$

Since the indirect proof as in Theorem 2 produced a better bound, we will use the same technique to prove the integral version of the Jensen–Steffensen inequality for strongly convex functions.

Theorem 6 *Let $x : [\alpha, \beta] \rightarrow (a, b)$ be a continuous and monotonic function, where $-\infty < \alpha < \beta < +\infty$ and $-\infty \leq a < b \leq +\infty$, and let $f : (a, b) \rightarrow \mathbb{R}$ be a strongly convex function with modulus c . If $\lambda : [\alpha, \beta] \rightarrow \mathbb{R}$ is either continuous or of bounded variation satisfying*

$$(\forall t \in [\alpha, \beta]) \quad \lambda(\alpha) \leq \lambda(t) \leq \lambda(\beta),$$

$$\lambda(\beta) - \lambda(\alpha) > 0,$$

then

$$f(\mu) \leq \frac{\int_{\alpha}^{\beta} f(x(t)) \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} - c \frac{\int_{\alpha}^{\beta} (x(t) - \mu)^2 \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)},$$

where

$$\mu = \frac{\int_{\alpha}^{\beta} x(t) \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)}.$$

Proof Using the convex representation $g = f - c(\cdot)^2$ as in Theorem 1 and applying the integral Jensen–Steffensen inequality for convex functions, we obtain

$$g(\mu) = g\left(\frac{\int_{\alpha}^{\beta} x(t) \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)}\right) \leq \frac{\int_{\alpha}^{\beta} g(x(t)) \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)}.$$

Going back to f we get

$$f(\mu) - c\mu^2 \leq \frac{\int_{\alpha}^{\beta} (f(x(t)) - cx(t)^2) \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} = \frac{\int_{\alpha}^{\beta} f(x(t)) \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} - c \frac{\int_{\alpha}^{\beta} x(t)^2 \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)},$$

or written differently

$$\begin{aligned} f(\mu) &\leq \frac{\int_{\alpha}^{\beta} f(x(t)) \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} - c \frac{\int_{\alpha}^{\beta} x(t)^2 \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} + c\mu^2 \\ &= \frac{\int_{\alpha}^{\beta} f(x(t)) \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} - c \left[\frac{\int_{\alpha}^{\beta} x(t)^2 \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} - \mu^2 \right] \\ &= \frac{\int_{\alpha}^{\beta} f(x(t)) \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} - c \left[\frac{\int_{\alpha}^{\beta} x(t)^2 \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} - 2\mu^2 + \mu^2 \right] \\ &= \frac{\int_{\alpha}^{\beta} f(x(t)) \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} - c \left[\frac{\int_{\alpha}^{\beta} x(t)^2 \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} - 2\mu \frac{\int_{\alpha}^{\beta} x(t) \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} + \mu^2 \frac{\int_{\alpha}^{\beta} d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} \right] \\ &= \frac{\int_{\alpha}^{\beta} f(x(t)) \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} - c \frac{\int_{\alpha}^{\beta} (x(t) - \mu)^2 \, d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)}. \end{aligned}$$

□

Funding

University of Split, Faculty of Science, Split, Croatia.

Availability of data and materials

Not applicable.

Competing interests

The author declares that there are no competing interests.

Authors' contributions

Author read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 31 August 2018 Accepted: 2 November 2018 Published online: 12 November 2018

References

1. Abramovich, S., Klaričić Bakula, M., Matić, M., Pečarić, J.: A variant of Jensen–Steffensen's inequality and quasi-arithmetic means. *J. Math. Anal. Appl.* **307**(1), 370–386 (2005)
2. Boas, R.P. Jr.: The Jensen–Steffensen inequality. *Publ. Elektroteh. Fak. Univ. Beogr., Ser. Mat. Fiz.* **302–319**, 1–8 (1970)
3. Hiriart-Urruty, J.-B., Lemaréchal, C.: *Fundamentals of Convex Analysis*. Abridged Version of Convex Analysis and Minimization Algorithms I and II. Grundlehren Text Editions. Springer, Berlin (2001)
4. Merentes, N., Nikodem, K.: Remarks on strongly convex functions. *Aequ. Math.* **80**(1–2), 193–199 (2010)
5. Nikodem, K.: On strongly convex functions and related classes of functions. In: *Handbook of Functional Equations*. Springer Optim. Appl., vol. 95, pp. 365–405. Springer, New York (2014)
6. Nikodem, K., Páles, Z.: Characterizations of inner product spaces by strongly convex functions. *Banach J. Math. Anal.* **5**(1), 83–87 (2011)
7. Pečarić, J.E., Proschan, F., Tong, Y.L.: *Convex Functions, Partial Orderings, and Statistical Applications*. Mathematics in Science and Engineering, vol. 187. Academic Press, Boston (1992). ISBN 0-12-549250-2
8. Polyak, B.T.: Existence theorems and convergence of minimizing sequences in extremum problems with restrictions. *Sov. Math. Dokl.* **7**, 72–75 (1966)
9. Steffensen, J.F.: On certain inequalities and methods of approximation. *J. Inst. Actuar.* **51**, 274–297 (1919)

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)
