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Noninstantaneous impulsive inequalities via conformable fractional calculus

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Abstract

We establish some new noninstantaneous impulsive inequalities using the conformable fractional calculus.

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1 Introduction and preliminaries

The subject of fractional differential equations has evolved as an interesting and important field of research in view of numerous applications in physics, mechanics, chemistry, engineering (like traffic, transportation, logistic, etc.), and so forth [1–3]. The tools of fractional calculus play a key role in improving the mathematical modeling of many real-world processes based on classical calculus. For some recent development on the topic, see [4–12] and the references therein.

Various types of fractional derivatives were introduced: Riemann–Liouville, Caputo, Hadamard, Erdélyi–Kober, Grünwald–Letnikov, Marchaud, and Riesz, to just name a few. Commonly, all they are defined as integrals with different singular kernels, that is, they have a nonlocal structure. Due to this fact, there are many inconsistencies of the existing fractional derivatives with classical derivative. Thus they do not obey the familiar product rule, the quotient rule for two functions, and the chain rule. Also, the fractional derivatives do not have a corresponding Rolle's theorem or a corresponding mean value theorem.

On the other hand, a recently introduced definition of the so-called *conformable fractional derivative* involves a limit instead of an integral; see [13, 14]. This local definition enables us to prove many properties analogous to those of integer-order derivatives. The authors in [14] showed that the conformable fractional derivative obeys the product and quotient rules and has results similar to the Rolle theorem and the mean value theorem in classical calculus.

For recent works on conformable derivatives, we refer to [15–19] and the references therein.

Let us recall the definition of the conformable fractional derivative and integral.

Definition 1.1 Let $0 < \alpha \leq 1$. The conformable fractional derivative starting from a point ϕ of a function $f : [\phi, \infty) \to \mathbb{R}$ is defined by

$$ {}_{\phi}D^{\alpha}(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon(t - \phi)^{1-\alpha}) - f(t)}{\epsilon}, \quad t > \phi, \tag{1.1}$$

and ${}_{\phi}D^{\alpha}f(\phi) = \lim_{t\to\phi^+} {}_{\phi}D^{\alpha}f(t)$.

Note that if f is differentiable, then

$$ {}_{\phi}D^{\alpha}f(t) = (t - \phi)^{1-\alpha}\frac{df(t)}{dt}. \tag{1.2}$$

Definition 1.2 Let $0 < \alpha \leq 1$. The conformable fractional integral of a function f : $[\phi, \infty) \to \mathbb{R}$ from a point ϕ is defined by

$$ {}_{\phi}I^{\alpha}(t) = \int_{\phi}^{t} (s - \phi)^{\alpha-1} f(s)\, ds. \tag{1.3}$$

The impulsive differential equations have been used to describe processes that have sudden changes in their states at certain moments. Many mathematical models in physical phenomena that have short-term perturbations at fixed impulse points t_k, $k = 1, 2, 3, \ldots$, caused by external interventions during their evolution appeared in population dynamics, biotechnology processes, chemistry, physics, engineering, and medicine; see [20–22]. In [23, 24], the authors introduced a new class of noninstantaneous impulsive differential equations with initial conditions to describe some certain dynamic changes of evolution processes in the pharmacotherapy. This kind of impulsive differential equations can be distinguished from the usual one as the changing processes containing no ordinary or fractional derivatives of their states work over intervals $(t_k, s_k]$, whereas the usual does at points t_k, $k = 1, 2, 3, \ldots$. There are some papers on existence and stability theory of this kind of impulsive ordinary or fractional differential equations [25–36]. To the best of our knowledge, there is no literature on noninstantaneous impulsive inequalities. The main goal of the paper is to establish some new noninstantaneous impulsive inequalities using the conformable fractional calculus. The main results are presented in Sect. 2. In Sect. 3, the maximum principle and boundedness of solutions for noninstantaneous impulse problems are illustrated.

2 Main results

Assume that the independent variable t is the time defined on the half-line $\mathbb{R}_+ = [0, \infty)$. Let $\{t_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=0}^{\infty}$ be two increasing sequences such that

$$0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < t_3 \leq \cdots < t_i \leq s_i < t_{i+1} \leq \cdots$$

for $i = 1, 2, \ldots$ and $\lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k = \infty$. In addition, we define subsets of $\mathbb{R}_+$ by $U_{s_k} = \bigcup_{k=0}^{\infty}(s_k, t_{k+1}]$, $U_{t_k} = \bigcup_{k=1}^{\infty}(t_k, s_k]$ and $U = U_{s_k} \cup U_{t_k}$. Note that $U \cup \{0\} = \mathbb{R}_+$. Set $PC(U_{s_k}, \mathbb{R}) = \{x : U_{s_k} \to \mathbb{R}; x(t)$ is continuous on U_{s_k}, and $x(s_k^+)$ exists for $k = 0, 1, 2, \ldots\}$, $PC(U_{t_k}, \mathbb{R}) = \{x : U_{t_k} \to \mathbb{R}; x(t)$ is continuous for $t \in U_{t_k}$, and $x(t_k^+)$ exists for $k = 1, 2, 3, \ldots\}$, $PC^{\alpha}_{s_k}(U_{s_k}, \mathbb{R}) = \{x \in PC(U_{s_k}, \mathbb{R}) : {}_{s_k}D^{\alpha}x(t)$ is continuous everywhere for $t \in U_{s_k}$, and

${}_{s_k}D^\alpha x(s_k^+)$ exists for $k = 0,1,2,\ldots\}$, $PC^\alpha_{t_k}(U_{t_k},\mathbb{R}) = \{x \in PC(U_{t_k},\mathbb{R}) : {}_{t_k}I^\alpha x(t)$ is continuous everywhere for all $t \in U_{t_k}$, and ${}_{t_k}I^\alpha x(t_k^+)$ exists for $k = 1,2,3,\ldots\}$, and $PC^\alpha(U,\mathbb{R}) = PC^\alpha_{s_k}(U_{s_k},\mathbb{R}) \cup PC^\alpha_{t_k}(U_{t_k},\mathbb{R})$.

Let the maximums of impulsive points less than or equal to t be defined by

$$s_m = \max\{s_k : s_k \le t, k = 0,1,2,\ldots\} \quad \text{and} \quad t_{\overline{m}} = \max\{t_k : t_k \le t, k = 1,2,3,\ldots\}. \tag{2.1}$$

In addition, we define

$$\begin{aligned} &H_k = e^{d_k \frac{(s_k-t_k)^\alpha}{\alpha}}; \qquad Q_k = e^{\int_{s_{k-1}}^{t_k} p(\xi)(\xi-s_{k-1})^{\alpha-1}\,d\xi}; \\ &G_k = Q_k H_k; \qquad P_k = Q_k H_{k-1}. \end{aligned}$$

Note that

$$H_m G_{m+1} G_{m+2} G_{m+3} \cdots G_{n-1} Q_n = P_{m+1} P_{m+2} \cdots P_{n-1} P_n \tag{2.2}$$

and

$$P_m P_{m+1} P_{m+2} \cdots P_{n-1} P_n H_n = H_{m-1} G_m G_{m+1} \cdots G_{n-1} G_n, \tag{2.3}$$

where $m < n$ are positive integers.

Throughout this paper, we assume that the unknown function $u \in PC^\alpha(U,\mathbb{R})$ is left-continuous at s_k and t_k $(k = 1,2,3,\ldots)$. Now, we are in the position to establish noninstantaneous impulsive differential inequalities.

Theorem 2.1 *Let b_k, c_k, d_k be given constants such that $b_k, c_k \ge 0$ and $d_k > 0$, $k = 1,2,3,\ldots$. Suppose that $p, q \in PC(U_{s_k},\mathbb{R})$ and*

$$\begin{cases} {}_{s_k}D^\alpha u(t) \le p(t)u(t) + q(t), & t \in (s_k, t_{k+1}], k = 0,1,2,\ldots, \\ u(t) \le c_k u(t_k^-) + d_k \int_{t_k}^t (\xi - t_k)^{\alpha-1} u(\xi)\,d\xi + b_k, & t \in (t_k, s_k], k = 1,2,\ldots. \end{cases} \tag{2.4}$$

Then

$$\begin{aligned} u(t) \le{}& e^{\int_{s_m}^t p(\xi)(\xi-s_m)^{\alpha-1}\,d\xi} \Biggl\{ u(s_0) \prod_{0<k\le m} c_k G_k + \sum_{0<k\le m} \Biggl(\prod_{k<j\le m} c_j G_j \Biggr) H_k b_k \\ &+ \sum_{0<k\le m} \Biggl(\prod_{k<j\le m} c_j G_j \Biggr) c_k H_k \int_{s_{k-1}}^{t_k} q(\eta)(\eta - s_{k-1})^{\alpha-1} e^{\int_\eta^{t_k} p(\xi)(\xi-s_{k-1})^{\alpha-1}\,d\xi}\,d\eta \Biggr\} \\ &+ \int_{s_m}^t q(\eta)(\eta - s_m)^{\alpha-1} e^{\int_\eta^t p(\xi)(\xi-s_m)^{\alpha-1}\,d\xi}\,d\eta, \quad t \in (s_k, t_{k+1}], k = 0,1,2,\ldots, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} u(t) \le{}& e^{d_{\overline{m}} \frac{(t-t_{\overline{m}})^\alpha}{\alpha}} \Biggl\{ u(s_0) c_{\overline{m}} Q_{\overline{m}} \prod_{0<k<\overline{m}} c_k G_k + \sum_{0<k\le \overline{m}} \Biggl(\prod_{k<j\le \overline{m}} c_j P_j \Biggr) b_k \\ &+ \sum_{0<k\le \overline{m}} \Biggl(\prod_{k<j\le \overline{m}} c_j P_j \Biggr) c_k \int_{s_{k-1}}^{t_k} q(\eta)(\eta - s_{k-1})^{\alpha-1} e^{\int_\eta^{t_k} p(\xi)(\xi-s_{k-1})^{\alpha-1}\,d\xi}\,d\eta \Biggr\}, \\ & t \in (t_k, s_k], k = 1,2,3,\ldots. \end{aligned} \tag{2.6}$$

Proof For $t \in (s_0, t_1]$, the conformable fractional differential inequality can be written as

$$ {}_{s_0}D^\alpha\Big[u(t)e^{-\int_{s_0}^t p(\xi)(\xi-s_0)^{\alpha-1}\,d\xi}\Big] \le q(t)e^{-\int_{s_0}^t p(\xi)(\xi-s_0)^{\alpha-1}\,d\xi}. $$

By taking the conformable fractional integral of order α from s_0 to $t \in (s_0, t_1]$,

$$ {}_{s_0}I^\alpha\, {}_{s_0}D^\alpha\Big[u(t)e^{-\int_{s_0}^t p(\xi)(\xi-s_0)^{\alpha-1}\,d\xi}\Big] \le {}_{s_0}I^\alpha\Big[q(t)e^{-\int_{s_0}^t p(\xi)(\xi-s_0)^{\alpha-1}\,d\xi}\Big], $$

we obtain

$$ \begin{aligned} u(t) &\le u(s_0)e^{\int_{s_0}^t p(\xi)(\xi-s_0)^{\alpha-1}\,d\xi} \\ &\quad + \int_{s_0}^t q(\eta)(\eta-s_0)^{\alpha-1}e^{\int_\eta^t p(\xi)(\xi-s_0)^{\alpha-1}\,d\xi}\,d\eta, \quad t \in (s_0, t_1], \end{aligned} \tag{2.7} $$

which implies that (2.5) holds for $k = 0$.

For $t \in (t_1, s_1]$, we define the function

$$ z(t) = \int_{t_1}^t (\xi - t_1)^{\alpha-1}u(\xi)\,d\xi. \tag{2.8} $$

Note that $z(t_1) = 0$ and

$$ u(t) \le c_1u\big(t_1^-\big) + d_1z(t) + b_1, \quad t \in (t_1, s_1]. $$

Then, taking the derivative with respect to t, we have

$$ \begin{aligned} z'(t) &= (t-t_1)^{\alpha-1}u(t) \\ &\le (t-t_1)^{\alpha-1}\big[c_1u\big(t_1^-\big) + b_1\big] + d_1(t-t_1)^{\alpha-1}z(t). \end{aligned} $$

Multiplying this inequality by the integrating factor $e^{-d_1\frac{(t-t_1)^\alpha}{\alpha}}$, we get

$$ \frac{d}{dt}\Big[z(t)e^{-d_1\frac{(t-t_1)^\alpha}{\alpha}}\Big] \le \big[c_1u\big(t_1^-\big) + b_1\big](t-t_1)^{\alpha-1}e^{-d_1\frac{(t-t_1)^\alpha}{\alpha}}, $$

which implies that

$$ \begin{aligned} z(t) &\le \big[c_1u\big(t_1^-\big) + b_1\big]e^{d_1\frac{(t-t_1)^\alpha}{\alpha}}\int_{t_1}^t (\eta-t_1)^{\alpha-1}e^{-d_1\frac{(\eta-t_1)^\alpha}{\alpha}}\,d\eta \\ &= \frac{1}{d_1}\big[c_1u\big(t_1^-\big) + b_1\big]\Big[e^{d_1\frac{(t-t_1)^\alpha}{\alpha}} - 1\Big]. \end{aligned} $$

By (2.7) with $t = t_1$ we have

$$ \begin{aligned} u(t) &\le c_1e^{d_1\frac{(t-t_1)^\alpha}{\alpha}}\Bigg[u(s_0)e^{\int_{s_0}^{t_1} p(\xi)(\xi-s_0)^{\alpha-1}\,d\xi} \\ &\quad + \int_{s_0}^{t_1} q(\eta)(\eta-s_0)^{\alpha-1}e^{\int_\eta^{t_1} p(\xi)(\xi-s_0)^{\alpha-1}\,d\xi}\,d\eta\Bigg] + b_1e^{d_1\frac{(t-t_1)^\alpha}{\alpha}}, \quad t \in (t_1, s_1]. \end{aligned} $$

This shows that the bound in (2.6) is true for $k = 1$.

Now, we assume that inequality (2.5) holds for $t \in (s_n, t_{n+1}]$, $n > 0$. By mathematical induction we will show that (2.6) is true for $t \in (t_{n+1}, s_{n+1}]$. Let

$$w(t) = \int_{t_{n+1}}^{t} (\xi - t_{n+1})^{\alpha-1} u(\xi)\,d\xi, \quad t \in (t_{n+1}, s_{n+1}].$$

Then $w(t_{n+1}) = 0$ and $u(t) \le c_{n+1}u(t_{n+1}^-) + d_{n+1}w(t) + b_{n+1}$. Using the above method, we have

$$w'(t) \le (t - t_{n+1})^{\alpha-1}\bigl[c_{n+1}u\bigl(t_{n+1}^-\bigr) + b_{n+1}\bigr] + d_{n+1}(t - t_{n+1})^{\alpha-1}w(t),$$

which leads to

$$w(t) \le \frac{1}{d_{n+1}}\bigl[c_{n+1}u\bigl(t_{n+1}^-\bigr) + b_{n+1}\bigr]\bigl[e^{d_{n+1}\frac{(t-t_{n+1})^{\alpha}}{\alpha}} - 1\bigr].$$

Substituting the bound of $w(t)$ and inequality (2.5) with $t = t_{n+1}$, it follows that

$$\begin{aligned}
u(t) &\le c_{n+1}u\bigl(t_{n+1}^-\bigr) + d_{n+1}w(t) + b_{n+1} \\
&\le c_{n+1}u\bigl(t_{n+1}^-\bigr)e^{d_{n+1}\frac{(t-t_{n+1})^{\alpha}}{\alpha}} + b_{n+1}e^{d_{n+1}\frac{(t-t_{n+1})^{\alpha}}{\alpha}} \\
&\le c_{n+1}\Biggl[e^{\int_{s_n}^{t_{n+1}} p(\xi)(\xi-s_n)^{\alpha-1}\,d\xi}\Biggl\{u(s_0)\prod_{0<k\le n} c_kG_k + \sum_{0<k\le n}\Biggl(\prod_{k<j\le n} c_jG_j\Biggr)H_kb_k \\
&\quad + \sum_{0<k\le n}\Biggl(\prod_{k<j\le n} c_jG_j\Biggr)c_kH_k\int_{s_{k-1}}^{t_k} q(\eta)(\eta - s_{k-1})^{\alpha-1}e^{\int_{\eta}^{t_k} p(\xi)(\xi-s_{k-1})^{\alpha-1}\,d\xi}\,d\eta\Biggr\} \\
&\quad + \int_{s_n}^{t_{n+1}} q(\eta)(\eta - s_n)^{\alpha-1}e^{\int_{\eta}^{t_{n+1}} p(\xi)(\xi-s_n)^{\alpha-1}\,d\xi}\,d\eta\Biggr]e^{d_{n+1}\frac{(t-t_{n+1})^{\alpha}}{\alpha}} \\
&\quad + b_{n+1}e^{d_{n+1}\frac{(t-t_{n+1})^{\alpha}}{\alpha}} \\
&= e^{d_{n+1}\frac{(t-t_{n+1})^{\alpha}}{\alpha}}\Biggl\{u(s_0)c_{n+1}Q_{n+1}\prod_{0<k<n+1} c_kG_k + \sum_{0<k\le n+1}\Biggl(\prod_{k<j\le n+1} c_jP_j\Biggr)b_k \\
&\quad + \sum_{0<k\le n+1}\Biggl(\prod_{k<j\le n+1} c_jP_j\Biggr)c_k\int_{s_{k-1}}^{t_k} q(\eta)(\eta - s_{k-1})^{\alpha-1}e^{\int_{\eta}^{t_k} p(\xi)(\xi-s_{k-1})^{\alpha-1}\,d\xi}\,d\eta\Biggr\}
\end{aligned}$$

by using formula (2.2). Therefore (2.6) is satisfied for $t \in (t_{n+1}, s_{n+1}]$.

Finally, we suppose that estimate (2.6) is fulfilled for $t \in (t_n, s_n]$, where $n > 1$. Next, we will prove that inequality (2.5) holds for $(s_n, t_{n+1}]$. By using the above method, we get the inequality

$$\begin{aligned}
u(t) &\le u(s_n)e^{\int_{s_n}^{t} p(\xi)(\xi-s_n)^{\alpha-1}\,d\xi} \\
&\quad + \int_{s_n}^{t} q(\eta)(\eta - s_n)^{\alpha-1}e^{\int_{\eta}^{t} p(\xi)(\xi-s_n)^{\alpha-1}\,d\xi}\,d\eta, \quad t \in (s_n, t_{n+1}].
\end{aligned}$$

Using (2.6) with $t = s_n$ and applying (2.3), we obtain

$$\begin{aligned}
u(t) &\le e^{\int_{s_n}^{t} p(\xi)(\xi-s_n)^{\alpha-1}\,d\xi}\Bigg[e^{d_n\frac{(s_n-t_n)^\alpha}{\alpha}}\Bigg\{u(s_0)c_nQ_n\prod_{0<k<n}c_kG_k+\sum_{0<k\le n}\Bigg(\prod_{k<j\le n}c_jP_j\Bigg)b_k \\
&\quad+\sum_{0<k\le n}\Bigg(\prod_{k<j\le n}c_jP_j\Bigg)c_k\int_{s_{k-1}}^{t_k}q(\eta)(\eta-s_{k-1})^{\alpha-1}e^{\int_\eta^{t_k}p(\xi)(\xi-s_{k-1})^{\alpha-1}\,d\xi}\,d\eta\Bigg\}\Bigg] \\
&\quad+\int_{s_n}^{t}q(\eta)(\eta-s_n)^{\alpha-1}e^{\int_\eta^{t}p(\xi)(\xi-s_n)^{\alpha-1}\,d\xi}\,d\eta \\
&= e^{\int_{s_n}^{t} p(\xi)(\xi-s_n)^{\alpha-1}\,d\xi}\Bigg[u(s_0)\prod_{0<k\le n}c_kG_k+\sum_{0<k\le n}\Bigg(\prod_{k<j\le n}c_jG_j\Bigg)H_kb_k \\
&\quad+\sum_{0<k\le n}\Bigg(\prod_{k<j\le n}c_jG_j\Bigg)c_kH_k\int_{s_{k-1}}^{t_k}q(\eta)(\eta-s_{k-1})^{\alpha-1}e^{\int_\eta^{t_k}p(\xi)(\xi-s_{k-1})^{\alpha-1}\,d\xi}\,d\eta\Bigg] \\
&\quad+\int_{s_n}^{t}q(\eta)(\eta-s_n)^{\alpha-1}e^{\int_\eta^{t}p(\xi)(\xi-s_n)^{\alpha-1}\,d\xi}\,d\eta.
\end{aligned}$$

Therefore inequality (2.5) is valid on $(s_n, t_{n+1}]$. This completes the proof. □

The following corollary can be obtained by replacing the given functions $p(t)$ and $q(t)$ by constants M and N, respectively.

Corollary 2.1 *Let $b_k, c_k \ge 0$ and $d_k > 0$, $k = 1,2,3,\dots$, be constants. If $M > 0$, $N \in \mathbb{R}$, and*

$$\begin{cases} {}_{s_k}D^\alpha u(t) \le Mu(t) + N, & t \in (s_k, t_{k+1}], k = 0,1,2,\dots, \\ u(t) \le c_k u(t_k^-) + d_k\int_{t_k}^{t}(\xi-t_k)^{\alpha-1}u(\xi)\,d\xi + b_k, & t \in (t_k, s_k], k = 1,2,\dots, \end{cases} \tag{2.9}$$

then

$$\begin{aligned}
u(t) &\le e^{M\frac{(t-s_m)^\alpha}{\alpha}}\Bigg\{u(s_0)\prod_{0<k\le m}c_kG_k^*+\sum_{0<k\le m}\Bigg(\prod_{k<j\le m}c_jG_j^*\Bigg)H_kb_k \\
&\quad+\frac{N}{M}\sum_{0<k\le m}\Bigg(\prod_{k<j\le m}c_jG_j^*\Bigg)c_kH_k\big(e^{M\frac{(t_k-s_{k-1})^\alpha}{\alpha}}-1\big)\Bigg\} \\
&\quad+\frac{N}{M}\big(e^{M\frac{(t-s_m)^\alpha}{\alpha}}-1\big), \quad t \in (s_k, t_{k+1}], k = 0,1,2,\dots,
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
u(t) &\le e^{d_{\overline{m}}\frac{(t-t_{\overline{m}})^\alpha}{\alpha}}\Bigg\{u(s_0)c_{\overline{m}}Q_{\overline{m}}^*\prod_{0<k<\overline{m}}c_kG_k^*+\sum_{0<k\le\overline{m}}\Bigg(\prod_{k<j\le\overline{m}}c_jP_j^*\Bigg)b_k \\
&\quad+\frac{N}{M}\sum_{0<k\le\overline{m}}\Bigg(\prod_{k<j\le\overline{m}}c_jP_j^*\Bigg)c_k\big(e^{M\frac{(t_k-s_{k-1})^\alpha}{\alpha}}-1\big)\Bigg\}, \\
&\quad t \in (t_k, s_k], k = 1,2,3,\dots,
\end{aligned} \tag{2.11}$$

where $Q_k^* = e^{M\frac{(t_k-s_{k-1})^\alpha}{\alpha}}$, $G_k^* = Q_k^*H_k$, and $P_k^* = Q_k^*H_{k-1}$.

Let $H(t)$ be the Heaviside function. We define two functions

$$\begin{aligned}\varphi(t) &= \sum_{i=0}^{\infty} H(t-s_i) - H\bigl(t-t_{i+1}^{+}\bigr) \\ &= \begin{cases} 0, & t \in (t_k, s_k], k = 1,2,3,\dots, \\ 1, & t \in (s_k, t_{k+1}], k = 0,1,2,\dots, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\psi(t) &= \sum_{i=1}^{\infty} H(t-t_i) - H\bigl(t-s_i^{+}\bigr) \\ &= \begin{cases} 0, & t \in (s_k, t_{k+1}], k = 0,1,2,\dots, \\ 1, & t \in (t_k, s_k], k = 1,2,3,\dots. \end{cases}\end{aligned}$$

Next, we establish some new noninstantaneous impulsive integral inequalities.

Theorem 2.2 *Let $p \in PC(U_{s_k}, \mathbb{R}_+)$, constants $c_k, b_k \ge 0$, $d_k > 0$, $k = 1,2,3,\dots$, and $A \in \mathbb{R}$. If*

$$\begin{aligned}u(t) \le{}& \left(A + \int_{s_m}^{t} (\xi - s_m)^{\alpha-1} p(\xi) u(\xi)\, d\xi\right)\varphi(t) \\ &+ \left(c_{\overline{m}} u\bigl(t_{\overline{m}}^{-}\bigr) + d_{\overline{m}} \int_{t_{\overline{m}}}^{t} (\xi - t_{\overline{m}})^{\alpha-1} u(\xi)\, d\xi + b_{\overline{m}}\right)\psi(t), \quad t \in \mathbb{R}_+, \end{aligned}\tag{2.12}$$

where s_m and $t_{\overline{m}}$ are defined by (2.1), then we have

$$u(t) \le e^{\int_{s_m}^{t} p(\xi)(\xi - s_m)^{\alpha-1}\, d\xi} \left(A \prod_{0<k\le m} c_k G_k + \sum_{0<k\le m} \left(\prod_{k<j\le m} c_j G_j\right) H_k b_k\right) \tag{2.13}$$

for $t \in (s_k, t_{k+1}]$, $k = 0,1,2,\dots$, and

$$u(t) \le e^{d_{\overline{m}} \frac{(t-t_{\overline{m}})^{\alpha}}{\alpha}} \left(A c_{\overline{m}} Q_{\overline{m}} \prod_{0<k<\overline{m}} c_k G_k + \sum_{0<k\le \overline{m}} \left(\prod_{k<j\le \overline{m}} c_j P_j\right) b_k\right) \tag{2.14}$$

for $t \in (t_k, s_k]$, $k = 1,2,3,\dots$.

Proof To prove inequalities (2.13) and (2.14), for $t \in \mathbb{R}_+$, we define the function

$$\begin{aligned}v(t) ={}& \left(A + \int_{s_m}^{t} (\xi - s_m)^{\alpha-1} p(\xi) u(\xi)\, d\xi\right)\varphi(t) \\ &+ \left(c_{\overline{m}} u\bigl(t_{\overline{m}}^{-}\bigr) + d_{\overline{m}} \int_{t_{\overline{m}}}^{t} (\xi - t_{\overline{m}})^{\alpha-1} u(\xi)\, d\xi + b_{\overline{m}}\right)\psi(t),\end{aligned}$$

which yields $u(t) \le v(t)$ for all $t \in \mathbb{R}_+$ and $v(s_0) = A$. For any $t \in (s_k, t_{k+1}]$, $k = 0,1,2,\dots$, we get

$$v(t) = A + \int_{s_m}^{t} (\xi - s_m)^{\alpha-1} p(\xi) u(\xi)\, d\xi.$$

Also, taking the conformable fractional derivative of order α, we have

$$ {}_{s_m}D^{\alpha}\nu(t) = p(t)u(t) \le p(t)\nu(t). \tag{2.15} $$

For $t \in (t_k, s_k]$, $k = 1, 2, 3, \ldots$, we obtain

$$ \begin{aligned} \nu(t) &= c_{\overline{m}}u\big(t_{\overline{m}}^-\big) + d_{\overline{m}}\int_{t_{\overline{m}}}^t (\xi - t_{\overline{m}})^{\alpha-1}u(\xi)\,d\xi + b_{\overline{m}} \\ &\le c_{\overline{m}}\nu\big(t_{\overline{m}}^-\big) + d_{\overline{m}}\int_{t_{\overline{m}}}^t (\xi - t_{\overline{m}})^{\alpha-1}\nu(\xi)\,d\xi + b_{\overline{m}}. \end{aligned} \tag{2.16} $$

An application of Theorem 2.1 to (2.15) and (2.16) yields

$$ \nu(t) \le e^{\int_{s_m}^t p(\xi)(\xi-s_m)^{\alpha-1}\,d\xi}\Bigg(A\prod_{0<k\le m} c_kG_k + \sum_{0<k\le m}\Bigg(\prod_{k<j\le m} c_jG_j\Bigg)H_kb_k\Bigg) $$

for $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots$, and

$$ \nu(t) \le e^{d_{\overline{m}}\frac{(t-t_{\overline{m}})^{\alpha}}{\alpha}}\Bigg(Ac_{\overline{m}}Q_{\overline{m}}\prod_{0<k<\overline{m}} c_kG_k + \sum_{0<k\le\overline{m}}\Bigg(\prod_{k<j\le\overline{m}} c_jP_j\Bigg)b_k\Bigg) $$

for $t \in (t_k, s_k]$, $k = 1, 2, 3, \ldots$. From $u(t) \le \nu(t)$, $t \in \mathbb{R}_+$, we get the desired results in (2.13) and (2.14). The proof is completed. □

Theorem 2.3 *Let $p \in PC(U_{s_k}, \mathbb{R}_+)$, let h be a positive fractional integrable function of order α, and let $c_k, b_k \ge 0$ and $d_k > 0$, $k = 1, 2, 3, \ldots$, be constants. If*

$$ \begin{aligned} u(t) \le{}& h(t) + \left(\int_{s_m}^t (\xi - s_m)^{\alpha-1}p(\xi)u(\xi)\,d\xi\right)\varphi(t) \\ &+ \left(c_{\overline{m}}u\big(t_{\overline{m}}^-\big) + d_{\overline{m}}\int_{t_{\overline{m}}}^t (\xi - t_{\overline{m}})^{\alpha-1}u(\xi)\,d\xi + b_{\overline{m}}\right)\psi(t), \quad t \in \mathbb{R}_+, \end{aligned} \tag{2.17} $$

where s_m and $t_{\overline{m}}$ are defined by (2.1), then we have

$$ \begin{aligned} u(t) \le{}& h(t) + e^{\int_{s_m}^t p(\xi)(\xi-s_m)^{\alpha-1}\,d\xi}\Bigg\{\sum_{0<k\le m}\Bigg(\prod_{k<j\le m} c_jG_j\Bigg)H_k\big(b_k + c_kh\big(t_k^-\big) + d_kK_k\big) \\ &+ \sum_{0<k\le m}\Bigg(\prod_{k<j\le m} c_jG_j\Bigg)c_kH_k\int_{s_{k-1}}^{t_k} p(\eta)h(\eta)(\eta - s_{k-1})^{\alpha-1}e^{\int_{\eta}^{t_k} p(\xi)(\xi-s_{k-1})^{\alpha-1}\,d\xi}\,d\eta\Bigg\} \\ &+ \int_{s_m}^t p(\eta)h(\eta)(\eta - s_m)^{\alpha-1}e^{\int_{\eta}^t p(\xi)(\xi-s_m)^{\alpha-1}\,d\xi}\,d\eta, \\ &t \in (s_k, t_{k+1}], k = 0, 1, 2, \ldots, \end{aligned} \tag{2.18} $$

and

$$u(t) \le h(t) + e^{d_{\overline{m}}\frac{(t-t_{\overline{m}})^\alpha}{\alpha}} \Biggl\{ \sum_{0<k\le\overline{m}} \biggl(\prod_{k<j\le\overline{m}} c_j P_j \biggr) \bigl(b_k + c_k h\bigl(t_k^-\bigr) + d_k K_k\bigr)$$
$$+ \sum_{0<k\le\overline{m}} \biggl(\prod_{k<j\le\overline{m}} c_j P_j \biggr) c_k \int_{s_{k-1}}^{t_k} p(\eta)h(\eta)(\eta - s_{k-1})^{\alpha-1} e^{\int_\eta^{t_k} p(\xi)(\xi-s_{k-1})^{\alpha-1}\,d\xi}\, d\eta \Biggr\},$$
$$t \in (t_k, s_k], k = 1,2,3,\ldots, \tag{2.19}$$

where the constants K_k, $k = 1,2,3,\ldots$, are defined by $K_k = \int_{t_k}^{s_k} (\xi - t_k)^{\alpha-1} h(\xi)\,d\xi$.

Proof For $t \in \mathbb{R}_+$, setting

$$y(t) = \biggl(\int_{s_m}^t (\xi - s_m)^{\alpha-1} p(\xi) u(\xi)\,d\xi \biggr) \varphi(t)$$
$$+ \biggl(c_{\overline{m}} u\bigl(t_{\overline{m}}^-\bigr) + d_{\overline{m}} \int_{t_{\overline{m}}}^t (\xi - t_{\overline{m}})^{\alpha-1} u(\xi)\,d\xi + b_{\overline{m}} \biggr) \psi(t),$$

we have

$${}_{s_m}D^\alpha y(t) = p(t)u(t), \qquad y(s_0) = 0,$$

for $t \in (s_k, t_{k+1}]$, $k = 0,1,2,\ldots$, and

$$y(t) = c_{\overline{m}} u\bigl(t_{\overline{m}}^-\bigr) + d_{\overline{m}} \int_{t_{\overline{m}}}^t (\xi - t_{\overline{m}})^{\alpha-1} u(\xi)\,d\xi + b_{\overline{m}}$$

for $t \in (t_k, s_k]$, $k = 1,2,3,\ldots$. Since $u(t) \le h(t) + y(t)$, $t \in \mathbb{R}_+$, this reduces to

$${}_{s_m}D^\alpha y(t) \le p(t)y(t) + p(t)h(t), \qquad y(s_0) = 0, \tag{2.20}$$

and

$$y(t) \le c_{\overline{m}} y\bigl(t_{\overline{m}}^-\bigr) + d_{\overline{m}} \int_{t_{\overline{m}}}^t (\xi - t_{\overline{m}})^{\alpha-1} y(\xi)\,d\xi$$
$$+ \bigl(b_{\overline{m}} + c_{\overline{m}} h\bigl(t_{\overline{m}}^-\bigr) + d_{\overline{m}} K_{\overline{m}}\bigr). \tag{2.21}$$

Now Theorem 2.1, together with the inequality $u(t) \le h(t) + y(t)$, yields estimates (2.18) and (2.19), completing the proof. □

Next, we obtain the following corollary by putting constant values $h(t) \equiv B > 0$ and $p(t) \equiv M > 0$.

Corollary 2.2 *Let constants $c_k, b_k \ge 0$ and $d_k > 0$, $k = 1,2,3,\ldots$. If*

$$u(t) \le B + \biggl(M \int_{s_m}^t (\xi - s_m)^{\alpha-1} u(\xi)\,d\xi \biggr) \varphi(t)$$
$$+ \biggl(c_{\overline{m}} u\bigl(t_{\overline{m}}^-\bigr) + d_{\overline{m}} \int_{t_{\overline{m}}}^t (\xi - t_{\overline{m}})^{\alpha-1} u(\xi)\,d\xi + b_{\overline{m}} \biggr) \psi(t), \quad t \in \mathbb{R}_+, \tag{2.22}$$

where s_m and $t_{\overline{m}}$ are defined by (2.1), then we have

$$\begin{aligned} u(t) \le{}& Be^{M\frac{(t-s_m)^\alpha}{\alpha}} + e^{M\frac{(t-s_m)^\alpha}{\alpha}} \Biggl\{ \sum_{0<k\le m} \Biggl(\prod_{k<j\le m} c_j G_j^* \Biggr) H_k Z_k \\ &+ B \sum_{0<k\le m} \Biggl(\prod_{k<j\le m} c_j G_j^* \Biggr) c_k H_k \bigl(e^{M\frac{(t_k-s_{k-1})^\alpha}{\alpha}} - 1 \bigr) \Biggr\} \\ & t \in (s_k, t_{k+1}], k = 0,1,2,\ldots, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} u(t) \le{}& B + e^{d_{\overline{m}}\frac{(t-t_{\overline{m}})^\alpha}{\alpha}} \Biggl\{ \sum_{0<k\le \overline{m}} \Biggl(\prod_{k<j\le \overline{m}} c_j P_j^* \Biggr) Z_k \\ &+ B \sum_{0<k\le \overline{m}} \Biggl(\prod_{k<j\le \overline{m}} c_j P_j^* \Biggr) c_k \bigl(e^{M\frac{(t_k-s_{k-1})^\alpha}{\alpha}} - 1 \bigr) \Biggr\}, \\ & t \in (t_k, s_k], k = 1,2,3,\ldots, \end{aligned} \tag{2.24}$$

where G_j^* and P_j^* are defined as in Corollary 2.1, and $Z_k = b_k + Bc_k + Bd_k(s_k - t_k)^\alpha/\alpha$.

3 Applications

In this section, we establish two applications of noninstantaneous impulsive differential and integral inequalities. Let $J = [0,T]$ with $t_{n+1} = T$ and $\overline{J} = [0,\overline{T}]$ with $s_{n+1} = \overline{T}$ for some $n \ge 1$. The first purpose is accomplished by considering two problems that have the end points at t_{n+1} and s_{n+1}, respectively. Now, we consider

$$\begin{cases} {}_{s_k}D^\alpha u(t) - Mu(t) + a(t) \le 0, & t \in (s_k, t_{k+1}], k = 0,1,2,\ldots,n, \\ u(t) \le c_k u(t_k^-) + d_k \int_{t_k}^t (\xi - t_k)^{\alpha-1} u(\xi)\, d\xi, & t \in (t_k, s_k], k = 1,2,3,\ldots,n, \\ u(0) = u(T) + \lambda, \end{cases} \tag{3.1}$$

and

$$\begin{cases} {}_{s_k}D^\alpha v(t) - Mv(t) + a(t) \le 0, & t \in (s_k, t_{k+1}], k = 0,1,2,\ldots,n, \\ v(t) \le c_k v(t_k^-) + d_k \int_{t_k}^t (\xi - t_k)^{\alpha-1} v(\xi)\, d\xi, & t \in (t_k, s_k], k = 1,2,3,\ldots,n+1, \\ v(0) = v(\overline{T}) + \lambda, \end{cases} \tag{3.2}$$

where $M > 0$, $a(t) \in C[\mathbb{R}_+, \mathbb{R}_+]$, $c_k \ge 0$, and $d_k > 0$. Let us state the following conditions:

(H_1) $e^{M\frac{(T-s_n)^\alpha}{\alpha}} \prod_{k=1}^n c_k G^* < 1$,

(H_2) $\lambda \le e^{M\frac{(T-s_n)^\alpha}{\alpha}} \int_{s_n}^T a(\eta)(\eta - s_n)^{\alpha-1} e^{-M\frac{(\eta-s_n)^\alpha}{\alpha}}\, d\eta$,

(H_3) $\prod_{k=1}^{n+1} c_k G_k^* < 1$,

(H_4) $\lambda \le e^{d_{n+1}\frac{(\overline{T}-t_{n+1})^\alpha}{\alpha}} \sum_{k=1}^{n+1} (\prod_{k<j\le n+1} c_j P_j^*) c_k D_k$, where D_k is defined by

$$D_k = e^{M\frac{(t_k-s_{k-1})^\alpha}{\alpha}} \int_{s_{k-1}}^{t_k} a(\eta)(\eta - s_{k-1})^{\alpha-1} e^{-M\frac{(\eta-s_{k-1})^\alpha}{\alpha}}\, d\eta, \quad k = 1,2,\ldots,n+1.$$

Corollary 3.1 *Let u and v be unknown functions satisfying* (3.1) *and* (3.2), *respectively. If* (H_1)–(H_2) *hold, then $u(t) \le 0$ for $t \in J$. If* (H_3)–(H_4) *hold, then $v(t) \le 0$ for $t \in \overline{J}$.*

Proof Applying Theorem 2.1 to the first two inequalities in problem (3.1), we have

$$\begin{aligned} u(t) &\le u(0)e^{M\frac{(t-s_m)^\alpha}{\alpha}} \prod_{0<k\le m} c_k G_k^* - e^{M\frac{(t-s_m)^\alpha}{\alpha}} \sum_{0<k\le m} \Bigl(\prod_{k<j\le m} c_j G_j^*\Bigr) c_k H_k D_k \\ &\quad - e^{M\frac{(t-s_m)^\alpha}{\alpha}} \int_{s_m}^{t} a(\eta)(\eta - s_m)^{\alpha-1} e^{-M\frac{(\eta-s_m)^\alpha}{\alpha}}\, d\eta, \\ &\quad t \in (s_k, t_{k+1}], k = 0,1,2,\dots,n. \end{aligned}$$

Since $a(t) \ge 0$ for all $t \in \mathbb{R}_+$ and all constants are positive, it is sufficient to show that $u(0) \le 0$. At the end point $t = T$, we obtain

$$\begin{aligned} u(T) &\le u(0)e^{M\frac{(T-s_n)^\alpha}{\alpha}} \prod_{k=1}^{n} c_k G_k^* - e^{M\frac{(T-s_n)^\alpha}{\alpha}} \sum_{k=1}^{n} \Bigl(\prod_{k<j\le n} c_j G_j^*\Bigr) c_k H_k D_k \\ &\quad - e^{M\frac{(T-s_n)^\alpha}{\alpha}} \int_{s_n}^{T} a(\eta)(\eta - s_n)^{\alpha-1} e^{-M\frac{(\eta-s_n)^\alpha}{\alpha}}\, d\eta. \end{aligned}$$

By conditions (H_1)–(H_2) we have

$$\begin{aligned} u(0)\Biggl(1 - e^{M\frac{(T-s_n)^\alpha}{\alpha}} \prod_{k=1}^{n} c_k G_k^*\Biggr) &\le \lambda - e^{M\frac{(T-s_n)^\alpha}{\alpha}} \sum_{k=1}^{n} \Bigl(\prod_{k<j\le n} c_j G_j^*\Bigr) c_k H_k D_k \\ &\quad - e^{M\frac{(T-s_n)^\alpha}{\alpha}} \int_{s_n}^{T} a(\eta)(\eta - s_n)^{\alpha-1} e^{-M\frac{(\eta-s_n)^\alpha}{\alpha}}\, d\eta \\ &\le 0, \end{aligned}$$

which yields $u(0) \le 0$. Therefore $u(t) \le 0$ for $t \in [0, T]$.

Next, we will show that $v(t) \le 0$ for $t \in \overline{J}$. The application of Theorem 2.1 for the first two inequalities in problem (3.2) leads to

$$\begin{aligned} v(t) &\le e^{d_{\overline{m}}\frac{(t-t_{\overline{m}})^\alpha}{\alpha}} \Biggl\{ v(0) c_{\overline{m}} Q_{\overline{m}}^* \prod_{0<k<\overline{m}} c_k G_k^* - \sum_{0<k\le \overline{m}} \Bigl(\prod_{k<j\le \overline{m}} c_j P_j^*\Bigr) c_k D_k \Biggr\}, \\ &\quad t \in (t_k, s_k], k = 1,2,3,\dots,n+1. \end{aligned}$$

Substituting the end point at $t = \overline{T}$, we have

$$v(\overline{T}) \le v(0) \prod_{k=1}^{n+1} c_k G_k^* - e^{d_{n+1}\frac{(\overline{T}-t_{n+1})^\alpha}{\alpha}} \sum_{k=1}^{n+1} \Bigl(\prod_{k<j\le n+1} c_j P_j^*\Bigr) c_k D_k,$$

which implies

$$\begin{aligned} v(0)\Biggl(1 - \prod_{k=1}^{n+1} c_k G_k^*\Biggr) &\le \lambda - e^{d_{n+1}\frac{(\overline{T}-t_{n+1})^\alpha}{\alpha}} \sum_{k=1}^{n+1} \Bigl(\prod_{k<j\le n+1} c_j P_j^*\Bigr) c_k D_k \\ &\le 0, \end{aligned}$$

by conditions (H_3)–(H_4). This means that $v(0) \leq 0$. In the same way, we can conclude that $v(t) \leq 0$ for $t \in \overline{J}$. The proof is completed. □

Finally, we apply the noninstantaneous impulsive inequality to the initial value problem of the form

$$\begin{cases} {}_{s_k}D^{\alpha} u(t) = f(t, u(t)), & t \in (s_k, t_{k+1}], k = 0, 1, 2, \dots, \\ u(t) = c_k u(t_k^-) + d_k \int_{t_k}^{t} (\xi - t_k)^{\alpha-1} u(\xi)\, d\xi, & t \in (t_k, s_k], k = 1, 2, \dots, \\ u(0) = u_0, \end{cases} \tag{3.3}$$

where $0 < \alpha \leq 1$, $c_k \geq 0$, $d_k > 0$, $u_0 \in \mathbb{R}$, and the given function $f \in PC(U_{s_k} \times \mathbb{R}, \mathbb{R})$ satisfies

(H_5) $|f(t, u)| \leq M|u|$, $M > 0$, for all $t \in U_{s_k}$.

Corollary 3.2 *If* (H_5) *holds, then the solution* $u(t)$ *of problem* (3.3) *is estimated as*

$$\left|u(t)\right| \leq e^{M\frac{(t-s_m)^{\alpha}}{\alpha}} |u_0| \prod_{0<k\leq m} c_k G_k^*, \quad t \in (s_k, t_{k+1}], k = 0, 1, 2, \dots, \tag{3.4}$$

and

$$\left|u(t)\right| \leq e^{d_{\overline{m}}\frac{(t-t_{\overline{m}})^{\alpha}}{\alpha}} |u_0| c_{\overline{m}} Q_{\overline{m}}^* \prod_{0<k<\overline{m}} c_k G_k^*, \quad t \in (t_k, s_k], k = 1, 2, 3, \dots. \tag{3.5}$$

Proof Taking the conformable fractional integral of order α to the first equation of problem (3.3), we obtain

$$u(t) = u(s_k) + \int_{s_k}^{t} (\xi - s_k)^{\alpha-1} f\big(\xi, u(\xi)\big)\, d\xi, \quad t \in (s_k, t_{k+1}], k = 0, 1, 2, \dots.$$

From condition (H_5) it follows that

$$\begin{aligned} \left|u(t)\right| &\leq \left|u(s_k)\right| + \int_{s_k}^{t} (\xi - s_k)^{\alpha-1} \left|f\big(\xi, u(\xi)\big)\right| d\xi, \\ &\leq \left|u(s_k)\right| + M \int_{s_k}^{t} (\xi - s_k)^{\alpha-1} \left|u(\xi)\right| d\xi. \end{aligned}$$

Since $u(s_0) = u_0$, by Theorem 2.2 inequalities (3.4)–(3.5) hold, and the proof is completed. □

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