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Hybrid proximal linearized algorithm for the split DC program in infinite-dimensional real Hilbert spaces

Chih-Sheng Chuang[1*] and Pei-Jung Yang[1]

*Correspondence:
cschuang@mail.ncyu.edu.tw;
cschuang1977@gmail.com
[1]Department of Applied Mathematics, National Chiayi University, Chiayi, Taiwan

Abstract
To be the best of our knowledge, the convergence theorem for the DC program and split DC program are proposed in finite-dimensional real Hilbert spaces or Euclidean spaces. In this paper, to study the split DC program, we give a hybrid proximal linearized algorithm and propose related convergence theorems in the settings of finite- and infinite-dimensional real Hilbert spaces, respectively.

MSC: 49J50; 49J53; 49M30; 49M37

Keywords: DC function; Subdifferential; Strongly monotonicity; Critical point

1 Introduction

Let H be a real Hilbert space, and let $f : H \to \mathbb{R}$ be a proper lower semicontinuous and convex function. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by taking $x_1 \in H$ arbitrarily and

$$x_{n+1} = \arg\min_{y\in H}\left\{f(y) + \frac{1}{2\beta_n}\|y - x_n\|^2\right\}, \quad n \in \mathbb{N}. \tag{1.1}$$

Then $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a minimizer of f under suitable conditions, and this is called the proximal point algorithm (PPA). This algorithm is useful, however, only for convex problems, because the idea for this algorithm is based on the monotonicity of subdifferential operators of convex functions. So, it is important to consider the relation between nonconvex functions and proximal point algorithm.

The DC program is the well-known nonconvex problem of the form

$$\text{(DCP)} \quad \text{Find } \bar{x} \in \arg\min_{x\in\mathbb{R}^n}\{f(x) = g(x) - h(x)\},$$

where $g, h : \mathbb{R}^n \to \mathbb{R}$ are proper lower semicontinuous convex functions. Here, the function f is called a DC function, and the functions g and h are called the DC components of f. (In the DC program, the convention $(+\infty) - (+\infty) = +\infty$ is adopted to avoid the ambiguity $(+\infty) - (+\infty)$ that does not present any interest.) It is well known that a necessary condition for $x \in \operatorname{dom}(f) := \{x \in \mathbb{R}^n : (x) < \infty\}$ to be a local minimizer of f is $\partial h(x) \subseteq \partial g(x)$. However, this condition is hard to be reached. So, many researchers focus their attentions on finding points such that $\partial h(x) \cap \partial g(x) \neq \emptyset$, where x is called a critical point of f [1].

It is worth mentioning the richness of the class of DC functions that is a subspace containing the class of lower-$\mathcal{C}^2$ functions. In particular, $\mathcal{DC}(\mathbb{R}^n)$ contains the space $\mathcal{C}^{1,1}$ of functions with locally Lipschitz continuous gradients. Further, $\mathcal{DC}(\mathbb{R}^n)$ is closed under the operations usually considered in optimization. For example, a linear combination, a finite supremum, or the product of two DC functions remain DC. It is also known that the set of DC functions defined on a compact convex set of $\mathbb{R}^n$ is dense in the set of continuous functions on this set.

The interest in the theory of DC functions has much increased in the last years. Some interesting optimality conditions and duality theorems related to the DC program are given. For more details, we refer to [2–9].

In 2003, Sun, Sampaio, and Candido [10] proposed a proximal point algorithm to study problem (DCP).

Algorithm 1.1 (Proximal point algorithm for (DCP) [10]) Let $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$, and let $g,h:\mathbb{R}^k\to\mathbb{R}$ be proper lower semicontinuous and convex functions. Let $\{x_n\}_{n\in\mathbb{N}}$ be generated as follows:

$$\begin{cases} x_1 \in H_1 \text{ is chosen arbitrarily,} \\ \text{Compute } w_n \in \partial h(x_n) \text{ and set } y_n = x_n + \beta_n w_n, \\ x_{n+1} := (I + \beta_n \partial g)^{-1}(y_n), \quad n \in \mathbb{N}. \\ \text{Stop criteria: } x_{n+1} = x_n. \end{cases}$$

In 2016, Souza, Oliveira, and Soubeyran [11] proposed a proximal linearized algorithm to study the DC program.

Algorithm 1.2 (Proximal linearized algorithm [11]) Let $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$, and let $g,h:\mathbb{R}^k\to\mathbb{R}$ be proper lower semicontinuous and convex functions. Let $\{x_n\}_{n\in\mathbb{N}}$ be generated as follows:

$$\begin{cases} x_1 \in H_1 \text{ is chosen arbitrarily,} \\ \text{Compute } w_n \in \partial h(x_n), \\ x_{n+1} := \arg\min_{u\in H_1}\{g(u) + \frac{1}{2\beta_n}\|u-x_n\|^2 - \langle w_n, u-x_n\rangle\}, \quad n \in \mathbb{N}. \\ \text{Stop criteria: } x_{n+1} = x_n. \end{cases}$$

Besides, some algorithms for the DC program are proposed to analyze and solve a variety of highly structured and practical problems (see, for example, [12]).

On the other hand, Chuang [13] introduced the following split DC program (split minimization problems for DC functions):

$$\text{(SDCP)} \quad \text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in \arg\min_{x\in H_1} f_1(x) \text{ and } A\bar{x} \in \arg\min_{y\in H_2} f_2(y),$$

where H_1 and H_2 are real Hilbert spaces, $A: H_1 \to H_2$ is a linear bounded mapping with adjoint A^*, $g_1, h_1 : H_1 \to \mathbb{R}$ and $g_2, h_2 : H_2 \to \mathbb{R}$ are proper lower semicontinuous and convex functions, and $f_1(x) = g_1(x) - h_1(x)$ and $f_2(y) = g_2(y) - h_2(y)$ for all $x \in H_1$ and $y \in H_2$. Further, to study problem (SDCP), Chuang [13] gave the following split proximal linearized algorithm and related convergence theorem in finite-dimensional real Hilbert spaces.

Algorithm 1.3 (Split proximal linearized algorithm) Let $\{x_n\}_{n\in\mathbb{N}}$ be generated as follows:

$$\begin{cases} x_1 \in H_1 \text{ is chosen arbitrarily,} \\ y_n := \arg\min_{v\in H_2}\{g_2(v) + \frac{1}{2\beta_n}\|v - Ax_n\|^2 - \langle\nabla h_2(Ax_n), v - Ax_n\rangle\}, \\ z_n := x_n - r_n A^*(Ax_n - y_n), \\ x_{n+1} := \arg\min_{u\in H_1}\{g_1(u) + \frac{1}{2\beta_n}\|u - z_n\|^2 - \langle\nabla h_1(z_n), u - z_n\rangle\}, \quad n \in \mathbb{N}. \end{cases}$$

Besides, there are also some important algorithms for the related problems in the literature; see, for example, [14–17].

In this paper, motivated by the works mentioned, we first give an hybrid proximal linearized algorithm and then propose a related convergence theorem in finite-dimensional real Hilbert spaces. Next, we propose related convergence theorems in infinite-dimensional real Hilbert space.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We denote the strong and weak convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. For all $x, y, u, v \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \tag{2.1}$$

$$\big\|\lambda x + (1 - \lambda)y\big\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \tag{2.2}$$

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Definition 2.1 Let H be a real Hilbert space, let $B : H \to H$, and let $\beta > 0$. Then,

(i) B is monotone if $\langle x - y, Bx - By\rangle \geq 0$ for all $x, y \in H$.

(ii) B is β-strongly monotone if $\langle x - y, Bx - By\rangle \geq \beta\|x - y\|^2$ for all $x, y \in H$.

Definition 2.2 Let H be a real Hilbert space, and let $B : H \multimap H$ be a set-valued mapping with domain $\mathcal{D}(B) := \{x \in H : B(x) \neq \emptyset\}$. Then,

(i) B is monotone if $\langle u - v, x - y\rangle \geq 0$ for any $u \in B(x)$ and $v \in B(y)$.

(ii) B is maximal monotone if its graph $\{(x, y) : x \in \mathcal{D}(B), y \in B(x)\}$ is not properly contained in the graph of any other monotone mapping.

(iii) B is ρ-strongly monotone ($\rho > 0$) if $\langle x - y, u - v\rangle \geq \rho\|x - y\|^2$ for all $x, y \in H$, $u \in B(x)$, and $v \in B(y)$.

Definition 2.3 Let H be a real Hilbert space, and let $f : H \to \mathbb{R}$. Then,

(i) f is proper if $\mathrm{dom}(f) = \{x \in H : f(x) < \infty\} \neq \emptyset$.

(ii) f is lower semicontinuous if $\{x \in H : f(x) \leq r\}$ is closed for each $r \in \mathbb{R}$.

(iii) f is convex if $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$ for every $x, y \in H$ and $t \in [0, 1]$.

(iv) f is ρ-strongly convex ($\rho > 0$) if

$$f\big(tx + (1 - t)y\big) + \frac{\rho}{2}\cdot t(1 - t)\|x - y\|^2 \leq tf(x) + (1 - t)f(y)$$

for all $x, y \in H$ and $t \in (0, 1)$.

(v) f is Gâteaux differentiable at $x \in H$ if there is $\nabla f(x) \in H$ such that

$$\lim_{t\to 0}\frac{f(x+ty)-f(x)}{t} = \langle y, \nabla f(x)\rangle$$

for each $y \in H$.

(vi) f is Fréchet differentiable at x if there is $\nabla f(x)$ such that

$$\lim_{y\to 0}\frac{f(x+y)-f(x)-\langle \nabla f(x), y\rangle}{\|y\|} = 0.$$

Example 2.1 Let H be a real Hilbert space. Then $g(x) := \|x\|^2$ is a 2-strongly convex function.

Example 2.2 Let $g(x) := \frac{1}{2}\langle Qx, x\rangle - \langle x, b\rangle$, where $Q \in \mathbb{R}^{n\times n}$ is a real symmetric positive definite matrix, and $b \in \mathbb{R}^n$. Then g is a strongly convex function.

Definition 2.4 Let $f : H \to (-\infty, \infty]$ be a proper lower semicontinuous and convex function. Then the subdifferential ∂f of f is defined by

$$\partial f(x) := \{x^* \in H : f(x) + \langle y - x, x^*\rangle \le f(y) \text{ for each } y \in H\}$$

for each $x \in H$.

Lemma 2.1 ([18, 19]) *Let $f : H \to (-\infty, \infty]$ be a proper lower semicontinuous and convex function. Then*:

(i) *∂f is a set-valued maximal monotone mapping*;

(ii) *f is Gâteaux differentiable at $x \in \operatorname{int}(\operatorname{dom}(f))$ if and only if $\partial f(x)$ consists of a single element, that is, $\partial f(x) = \{\nabla f(x)\}$* [18, Prop. 1.1.10];

(iii) *A Fréchet differentiable function f is convex if and only if ∇f is a monotone mapping.*

Lemma 2.2 ([19, Example 22.3(iv)]) *Let $\rho > 0$, let H be a real Hilbert space, and let $f : H \to \mathbb{R}$ be a proper lower semicontinuous and convex function. If f is ρ-strongly convex, then ∂f is ρ-strongly monotone.*

Lemma 2.3 ([19, Prop. 16.26]) *Let H be a real Hilbert space, and let $f : H \to (\infty, \infty]$ be a proper lower semicontinuous and convex function. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ be sequences in H such that $u_n \in \partial f(x_n)$ for all $n \in \mathbb{N}$. Then if $x_n \rightharpoonup x$ and $u_n \to u$, then $u \in \partial f(x)$.*

Lemma 2.4 ([20]) *Let H be a real Hilbert space, let $B : H \multimap H$ be a set-valued maximal monotone mapping, and let $\beta > 0$. The mapping J_β^B defined by $J_\beta^B(x) := (I + \beta B)^{-1}(x)$ for $x \in H$ is a single-valued mapping.*

3 Main results in finite-dimensional real Hilbert space

Let ρ and L be real numbers with $\rho > L > 0$. Let H_1 and H_2 be finite-dimensional real Hilbert spaces, and let $A : H_1 \to H_2$ be a nonzero linear and bounded mapping with adjoint A^*. Let $g_1, h_1 : H_1 \to \mathbb{R}$ be proper lower semicontinuous and convex functions, let $g_2, h_2 : H_2 \to \mathbb{R}$ be proper lower semicontinuous and convex functions, and let $f_1(x) = g_1(x) - h_1(x)$

for $x \in H_1$ and $f_2(y) = g_2(y) - h_2(y)$ for $y \in H_2$. Further, we assume that f_1 and f_2 are bounded from below, h_1 and h_2 are Fréchet differentiable, ∇h_1 and ∇h_2 are L-Lipschitz continuous, and g_1 and g_2 are ρ-strongly convex.

Choose $\delta \in (0, 0.5)$, let β be a real number, and let $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that

$$0 < \beta, \quad \beta_n < \frac{1}{2\rho - L}.$$

Since $\rho > L > 0$ and $\beta_n > 0$, we have $\beta_n L < \beta_n \rho$, and then

$$0 < \frac{1 + \beta_n L}{1 + 2\beta_n \rho - \beta_n L} < 1.$$

Besides, we know that

$$1 < 1 + 2\beta_n \rho - \beta_n L < 2,$$

which implies that

$$\frac{1}{2} < \frac{1}{1 + 2\beta_n \rho - \beta_n L} < \frac{1 + \beta_n L}{1 + 2\beta_n \rho - \beta_n L} < 1.$$

Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, and let r be a real number with

$$\liminf_{n\to\infty} r_n > 0$$

and

$$0 < r_n, \quad r < \min\left\{ \frac{\sqrt[4]{1-2\delta} \cdot \sqrt{\beta_n(\rho - L)}}{\sqrt{2 + 2\beta_n L} \cdot \|A\|^2}, \frac{\sqrt{\delta}}{(2 + \beta_n L)\|A\|^2} \right\}.$$

Thus we have

$$r_n < \frac{\sqrt{\delta}}{(2 + \beta_n L)\|A\|^2} < \frac{3}{2\|A\|^2}$$

and

$$0 < \frac{4(1 + \beta_n L) \cdot \|A\|^4 \cdot r_n^2}{\sqrt{1 - 2\delta}} < 2\beta_n \rho - 2\beta_n L.$$

So, we have

$$0 < 1 + \beta_n L + \frac{4(1 + \beta_n L) \cdot \|A\|^4 \cdot r_n^2}{\sqrt{1 - 2\delta}} < 1 + 2\beta_n \rho - \beta_n L,$$

and then

$$0 < \frac{1 + \beta_n L}{1 + 2\beta_n \rho - \beta_n L} \cdot \left(1 + \frac{4 \cdot \|A\|^4 \cdot r_n^2}{\sqrt{1 - 2\delta}}\right) < 1.$$

Let Ω_{SDCP} be defined by

$$\Omega_{\mathrm{SDCP}} := \bigl\{x \in H_1 : \nabla h_1(x) \in \partial g_1(x), \nabla h_2(Ax) \in \partial g_2(Ax)\bigr\}.$$

We further assume that $\Omega_{\mathrm{SDCP}} \neq \emptyset$. The following result of Chuang [13] plays an important role in this paper.

Lemma 3.1 ([13]) *Under the assumptions in this section, let*

$$\begin{cases} y := \arg\min_{v \in H_2}\{g_2(v) + \frac{1}{2\beta}\|v - Ax\|^2 - \langle \nabla h_2(Ax), v - Ax\rangle\}, \\ z := x - rA^*(Ax - y), \\ w := \arg\min_{u \in H_1}\{g_1(u) + \frac{1}{2\beta}\|u - z\|^2 - \langle \nabla h_1(z), u - z\rangle\}. \end{cases} \tag{3.1}$$

Then $x \in \Omega_{\mathrm{SDCP}}$ if and only if $x = w$.

Proposition 3.1 ([13]) *If $\rho > L$ and $\Omega_{\mathrm{SDCP}} \neq \emptyset$, then the set Ω_{SDCP} is a singleton.*

In this section, we propose the following algorithm to study the split DC program.

Algorithm 3.1 Let $x_1 \in H_1$ be arbitrary, and let $\{x_n\}_{n\in\mathbb{N}}$ be defined as follows:

$$\begin{cases} y_n := \arg\min_{v \in H_2}\{g_2(v) + \frac{1}{2\beta_n}\|v - Ax_n\|^2 - \langle \nabla h_2(Ax_n), v - Ax_n\rangle\}, \\ z_n := x_n - r_nA^*(Ax_n - y_n), \\ w_n := \arg\min_{u \in H_1}\{g_1(u) + \frac{1}{2\beta_n}\|u - z_n\|^2 - \langle \nabla h_1(z_n), u - z_n\rangle\}, \\ \widehat{y}_n := \arg\min_{v \in H_2}\{g_2(v) + \frac{1}{2\beta_n}\|v - Aw_n\|^2 - \langle \nabla h_2(Aw_n), v - Aw_n\rangle\}, \\ \widehat{z}_n := w_n - r_nA^*(Aw_n - \widehat{y}_n), \\ D_n := z_n - \widehat{z}_n, \\ \alpha_n := \frac{\langle x_n - w_n, D_n\rangle}{\|D_n\|^2}, \\ \widehat{x}_n := x_n - \alpha_n D_n, \\ x_{n+1} := \arg\min_{u \in H_1}\{g_1(u) + \frac{1}{2\beta_n}\|u - \widehat{x}_n\|^2 - \langle \nabla h_1(\widehat{x}_n), u - \widehat{x}_n\rangle\}, \quad n \in \mathbb{N}, \\ \text{stop criteria: } x_n = w_n. \end{cases}$$

Remark 3.1 The stop criteria in Algorithm 3.1 is given by Lemma 3.1.

Theorem 3.1 *Let $\{x_n\}_{n\in\mathbb{N}}$ be generated by Algorithm 3.1. Then $\{x_n\}_{n\in\mathbb{N}}$ converges to $\bar{x}$, where $\Omega_{\mathrm{SDCP}} = \{\bar{x}\}$.*

Proof Take any $w \in \Omega_{\mathrm{SDCP}}$ and $n \in \mathbb{N}$, and let w and n be fixed. First, we know that

$$0 \in \partial g_1(x_{n+1}) + \frac{1}{\beta_n}(x_{n+1} - \widehat{x}_n) - \nabla h_1(\widehat{x}_n). \tag{3.2}$$

By (3.2) and Lemma 2.4 we have

$$x_{n+1} = (I + \beta_n \partial g_1)^{-1}\bigl(\widehat{x}_n + \beta_n \nabla h_1(\widehat{x}_n)\bigr). \tag{3.3}$$

By (3.2) again, there exists $\tau_n \in \partial g_1(x_{n+1})$ such that

$$\nabla h_1(\widehat{x}_n) = \tau_n + \frac{1}{\beta_n}(x_{n+1} - \widehat{x}_n). \tag{3.4}$$

Since $w \in \Omega_{\mathrm{SDCP}}$, we have that $\nabla h_1(w) \in \partial g_1(w)$. By Lemma 2.2, ∂g_1 is ρ-strongly monotone, and this implies that

$$0 \le \langle x_{n+1} - w, \tau_n - \nabla h_1(w)\rangle - \rho\|x_{n+1} - w\|^2. \tag{3.5}$$

By (3.4) and (3.5) we have

$$\begin{aligned}
0 &\le 2\beta_n\langle x_{n+1} - w, \nabla h_1(\widehat{x}_n) - \nabla h_1(w)\rangle - 2\beta_n\rho\|x_{n+1} - w\|^2 \\
&\quad + 2\langle x_{n+1} - w, \widehat{x}_n - x_{n+1}\rangle \\
&\le 2\beta_n L\|x_{n+1} - w\| \cdot \|\widehat{x}_n - w\| - 2\beta_n\rho\|x_{n+1} - w\|^2 \\
&\quad + \|\widehat{x}_n - w\|^2 - \|x_{n+1} - \widehat{x}_n\|^2 - \|x_{n+1} - w\|^2 \\
&\le \beta_n L\big(\|x_{n+1} - w\|^2 + \|\widehat{x}_n - w\|^2\big) - 2\beta_n\rho\|x_{n+1} - w\|^2 \\
&\quad + \|\widehat{x}_n - w\|^2 - \|x_{n+1} - \widehat{x}_n\|^2 - \|x_{n+1} - w\|^2.
\end{aligned} \tag{3.6}$$

Hence, by (3.6),

$$\|x_{n+1} - w\|^2 \le \frac{1 + \beta_n L}{1 + 2\beta_n\rho - \beta_n L}\|\widehat{x}_n - w\|^2 - \frac{1}{1 + 2\beta_n\rho - \beta_n L}\|x_{n+1} - \widehat{x}_n\|^2. \tag{3.7}$$

Similarly to (3.2), we have

$$0 \in \partial g_2(\widehat{y}_n) + \frac{1}{\beta_n}(\widehat{y}_n - Aw_n) - \nabla h_2(Aw_n) \tag{3.8}$$

and

$$0 \in \partial g_1(w_n) + \frac{1}{\beta_n}(w_n - z_n) - \nabla h_1(z_n). \tag{3.9}$$

Similarly to (3.3), we have

$$y_n = (I + \beta_n\partial g_2)^{-1}\big(Ax_n + \beta_n\nabla h_2(Ax_n)\big) \tag{3.10}$$

and

$$\widehat{y}_n = (I + \beta_n\partial g_2)^{-1}\big(Aw_n + \beta_n\nabla h_2(Aw_n)\big). \tag{3.11}$$

Similarly to (3.7), we have

$$\|w_n - w\|^2 \le \frac{1 + \beta_n L}{1 + 2\beta_n\rho - \beta_n L}\|z_n - w\|^2 - \frac{1}{1 + 2\beta_n\rho - \beta_n L}\|w_n - z_n\|^2, \tag{3.12}$$

$$\|\widehat{y}_n - Aw\|^2 \le \frac{\beta_n L + 1}{1 + 2\beta_n\rho - \beta_n L}\|Aw_n - Aw\|^2 - \frac{\|\widehat{y}_n - Aw_n\|^2}{1 + 2\beta_n\rho - \beta_n L}, \tag{3.13}$$

and

$$\|y_n - Aw\|^2 \leq \frac{\beta_n L + 1}{1 + 2\beta_n \rho - \beta_n L} \|Ax_n - Aw\|^2 - \frac{\|y_n - Ax_n\|^2}{1 + 2\beta_n \rho - \beta_n L}. \tag{3.14}$$

Next, we set

$$\varepsilon_n := r_n \big[A^*(Aw_n - \widehat{y}_n) - A^*(Ax_n - y_n) \big]. \tag{3.15}$$

By (3.10) and (3.11) we have

$$\begin{aligned}
\|\varepsilon_n\| &\leq r_n \|A\| \big(\|Aw_n - Ax_n\| + \|\widehat{y}_n - y_n\| \big) \\
&\leq r_n \|A\| \big(\|Ax_n - Aw_n\| + \|A_n - Aw_n\| + \beta_n L \|Ax_n - Aw_n\| \big) \\
&\leq r_n \|A\|^2 (2 + \beta_n L) \|x_n - w_n\| \\
&\leq \sqrt{\delta} \|x_n - w_n\|.
\end{aligned} \tag{3.16}$$

By (3.15) we have

$$\begin{aligned}
\langle x_n - w_n, D_n \rangle &= \langle x_n - w_n, x_n - w_n + \varepsilon_n \rangle \\
&= \|x_n - w_n\|^2 + \langle x_n - w_n, \varepsilon_n \rangle \\
&\geq \|x_n - w_n\|^2 - \big| \langle x_n - w_n, \varepsilon_n \rangle \big| \\
&\geq (1 - \delta) \|x_n - w_n\|^2
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
\langle x_n - w_n, D_n \rangle &= \langle x_n - w_n, x_n - w_n + \varepsilon_n \rangle \\
&= \|x_n - w_n\|^2 + \langle x_n - w_n, \varepsilon_n \rangle \\
&= \frac{1}{2} \|x_n - w_n\|^2 + \langle x_n - w_n, \varepsilon_n \rangle + \frac{1}{2} \|x_n - w_n\|^2 \\
&\geq \frac{1}{2} \|x_n - w_n\|^2 + \langle x_n - w_n, \varepsilon_n \rangle + \frac{1}{2} \|\varepsilon_n\|^2 \\
&= \frac{1}{2} \|x_n - w_n + \varepsilon_n\|^2 \\
&= \frac{1}{2} \|D_n\|^2.
\end{aligned} \tag{3.18}$$

By (3.18) we know that $\alpha_n \geq \frac{1}{2}$ for each $n \in \mathbb{N}$. Besides, we have

$$\begin{aligned}
\|x_n - w_n + \varepsilon_n\|^2 &= \|x_n - w_n\|^2 + \|\varepsilon_n\|^2 + 2\langle x_n - w_n, \varepsilon_n \rangle \\
&\geq \|x_n - w_n\|^2 + \|\varepsilon_n\|^2 - 2\big| \langle x_n - w_n, \varepsilon_n \rangle \big| \\
&\geq \|x_n - w_n\|^2 + \|\varepsilon_n\|^2 - 2\|x_n - w_n\| \cdot \|\varepsilon_n\| \\
&\geq \|x_n - w_n\|^2 + \|\varepsilon_n\|^2 - 2\delta \|x_n - w_n\|^2 \\
&\geq (1 - 2\delta) \|x_n - w_n\|^2 > 0.
\end{aligned} \tag{3.19}$$

By (3.19) we have

$$\alpha_n^2 \le \left(\frac{\|x_n - w_n\| \cdot \|x_n - w_n + \varepsilon_n\|}{\|x_n - w_n + \varepsilon_n\|^2}\right)^2 \le \frac{\|x_n - w_n\|^2}{(1-2\delta)\|x_n - w_n\|^2} = \frac{1}{1-2\delta}. \tag{3.20}$$

Next, we have

$$\begin{aligned}
\|\widehat{x}_n - w\|^2 &= \|x_n - \alpha_n D_n - w\|^2 \\
&= \|x_n - w\|^2 + \alpha_n^2\|D_n\|^2 - 2\alpha_n\langle x_n - w, D_n\rangle \\
&= \|x_n - w\|^2 + \alpha_n^2\|D_n\|^2 - 2\alpha_n\langle x_n - w_n, D_n\rangle \\
&\quad - 2\alpha_n\langle w_n - w, D_n\rangle \\
&= \|x_n - w\|^2 - \alpha_n^2\|D_n\|^2 - 2\alpha_n\langle w_n - w, D_n\rangle \\
&= \|x_n - w\|^2 - \alpha_n^2\|D_n\|^2 - 2\alpha_n\langle w_n - w, z_n - \widehat{z}_n\rangle \\
&= \|x_n - w\|^2 - \alpha_n^2\|D_n\|^2 - \alpha_n\|w_n - \widehat{z}_n\|^2 - \alpha_n\|z_n - w\|^2 \\
&\quad + \alpha_n\|w_n - z_n\|^2 + \alpha_n\|\widehat{z}_n - w\|^2.
\end{aligned} \tag{3.21}$$

On the other hand, we have

$$\begin{aligned}
2\|\widehat{z}_n - w\|^2 &= 2\big\langle \widehat{z}_n - w, w_n - r_n A^*(Aw_n - \widehat{y}_n) - w\big\rangle \\
&= 2\langle \widehat{z}_n - w, w_n - w\rangle - 2r_n\big\langle \widehat{z}_n - w, A^*(Aw_n - \widehat{y}_n)\big\rangle \\
&= 2\langle \widehat{z}_n - w, w_n - w\rangle - 2r_n\langle A\widehat{z}_n - Aw, Aw_n - \widehat{y}_n\rangle \\
&= \|\widehat{z}_n - w\|^2 + \|w_n - w\|^2 - \|\widehat{z}_n - w_n\|^2 - r_n\|A\widehat{z}_n - \widehat{y}_n\|^2 \\
&\quad - r_n\|Aw_n - Aw\|^2 + r_n\|A\widehat{z}_n - Aw_n\|^2 + r_n\|\widehat{y}_n - Aw\|^2,
\end{aligned} \tag{3.22}$$

which implies that

$$\begin{aligned}
\|\widehat{z}_n - w\|^2 &= \|w_n - w\|^2 - \|\widehat{z}_n - w_n\|^2 - r_n\|A\widehat{z}_n - \widehat{y}_n\|^2 \\
&\quad - r_n\|Aw_n - Aw\|^2 + r_n\|A\widehat{z}_n - Aw_n\|^2 + r_n\|\widehat{y}_n - Aw\|^2.
\end{aligned} \tag{3.23}$$

By (3.12), (3.13), (3.21), and (3.23) we have

$$\begin{aligned}
&\|\widehat{x}_n - w\|^2 \\
&\quad = \|x_n - w\|^2 - \alpha_n^2\|D_n\|^2 - 2\alpha_n\|w_n - \widehat{z}_n\|^2 - \alpha_n\|z_n - w\|^2 \\
&\qquad + \alpha_n\|w_n - z_n\|^2 + \alpha_n\|w_n - w\|^2 - \alpha_n r_n\|A\widehat{z}_n - \widehat{y}_n\|^2 \\
&\qquad - \alpha_n r_n\|Aw_n - Aw\|^2 + \alpha_n r_n\|A\widehat{z}_n - Aw_n\|^2 + \alpha_n r_n\|\widehat{y}_n - Aw\|^2 \\
&\quad \le \|x_n - w\|^2 - \alpha_n^2\|D_n\|^2 - \alpha_n\big(2 - r_n\|A\|^2\big)\|\widehat{z}_n - w_n\|^2 - \alpha_n\|z_n - w\|^2 \\
&\qquad + \alpha_n\|w_n - z_n\|^2 + \alpha_n\|w_n - w\|^2 - \alpha_n r_n\|A\widehat{z}_n - \widehat{y}_n\|^2 \\
&\qquad - \alpha_n r_n\|Aw_n - Aw\|^2 + \alpha_n r_n\|\widehat{y}_n - Aw\|^2 \\
&\quad \le \|x_n - w\|^2 - \alpha_n^2\|D_n\|^2 - \alpha_n\big(2 - r_n\|A\|^2\big)\|\widehat{z}_n - w_n\|^2 - \alpha_n\|z_n - w\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_n\|w_n - z_n\|^2 + \alpha_n\|z_n - w\|^2 - \frac{\alpha_n}{1+2\beta_n\rho-\beta_n L}\|w_n - z_n\|^2 \\
&- \alpha_n r_n\|A\widehat{z}_n - \widehat{y}_n\|^2 - \alpha_n r_n\|Aw_n - Aw\|^2 + \alpha_n r_n\|Aw_n - Aw\|^2 \\
\le{}& \|x_n - w\|^2 - \alpha_n^2\|D_n\|^2 - \alpha_n\bigl(2 - r_n\|A\|^2\bigr)\|\widehat{z}_n - w_n\|^2 \\
&+ \alpha_n\|w_n - z_n\|^2 - \frac{\alpha_n}{1+2\beta_n\rho-\beta_n L}\|w_n - z_n\|^2 \\
&- \alpha_n r_n\|A\widehat{z}_n - \widehat{y}_n\|^2.
\end{aligned}
\tag{3.24}
$$

We also have

$$
-2\alpha_n^2\|D_n\|^2 = \alpha_n\|w_n - \widehat{z}_n\|^2 + \alpha_n\|x_n - z_n\|^2 - \alpha_n\|w_n - z_n\|^2 - \alpha_n\|x_n - \widehat{z}_n\|^2. \tag{3.25}
$$

By (3.24) and (3.25) we have

$$
\begin{aligned}
&\|\widehat{x}_n - w\|^2 \\
&\quad\le \|x_n - w\|^2 - \alpha_n\left(\frac{3}{2} - r_n\|A\|^2\right)\|\widehat{z}_n - w_n\|^2 - \alpha_n r_n\|A\widehat{z}_n - \widehat{y}_n\|^2 \\
&\qquad - \frac{\alpha_n}{1+2\beta_n\rho-\beta_n L}\|w_n - z_n\|^2 - \frac{1}{2}\cdot\alpha_n\|x_n - \widehat{z}_n\|^2 + \frac{1}{2}\cdot\alpha_n\|x_n - z_n\|^2 \\
&\qquad + \frac{1}{2}\cdot\alpha_n\|w_n - z_n\|^2.
\end{aligned}
\tag{3.26}
$$

By (3.14) we have

$$
\begin{aligned}
\|x_n - z_n\| &= \bigl\|r_n A^*(Ax_n - y_n)\bigr\| \\
&\le r_n\|A\|\bigl(\|Ax_n - Aw\| + \|y_n - Aw\|\bigr) \\
&\le 2r_n\|A\|\cdot\|Ax_n - Aw\| \\
&\le 2r_n\|A\|^2\|x_n - w\|.
\end{aligned}
\tag{3.27}
$$

By (3.7), (3.26), and (3.27) we have

$$
\begin{aligned}
\|x_{n+1} - w\|^2 &\le \frac{1+\beta_n L}{1+2\beta_n\rho-\beta_n L}\|\widehat{x}_n - w\|^2 - \frac{1}{1+2\beta_n\rho-\beta_n L}\|x_{n+1} - \widehat{x}_n\|^2 \\
&\le \frac{1+\beta_n L}{1+2\beta_n\rho-\beta_n L}\left(\|x_n - w\|^2 - \alpha_n\left(\frac{3}{2} - r_n\|A\|^2\right)\|\widehat{z}_n - w_n\|^2\right. \\
&\quad - \alpha_n r_n\|A\widehat{z}_n - \widehat{y}_n\|^2 - \alpha_n\cdot\left(\frac{1}{1+2\beta_n\rho-\beta_n L} - \frac{1}{2}\right)\|w_n - z_n\|^2 \\
&\quad \left. - \frac{1}{2}\cdot\alpha_n\|x_n - \widehat{z}_n\|^2 + \frac{1}{2}\cdot\alpha_n\|x_n - z_n\|^2\right) \\
&\quad - \frac{1}{1+2\beta_n\rho-\beta_n L}\|x_{n+1} - \widehat{x}_n\|^2 \\
&\le \frac{1+\beta_n L}{1+2\beta_n\rho-\beta_n L}\bigl(1 + 2\alpha_n r_n^2\|A\|^4\bigr)\|x_n - w\|^2 \\
&\le \|x_n - w\|^2.
\end{aligned}
\tag{3.28}
$$

By (3.28), $\lim_{n\to\infty}\|x_n - w\|$ exists, $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence, and

$$\begin{cases} \lim_{n\to\infty} \frac{1+\beta_n L}{1+2\beta_n\rho-\beta_n L}\cdot \alpha_n(\frac{3}{2} - r_n\|A\|^2)\|\widehat{z}_n - w_n\|^2 = 0, \\ \lim_{n\to\infty} \frac{1+\beta_n L}{1+2\beta_n\rho-\beta_n L}\cdot \alpha_n r_n\|A\widehat{z}_n - \widehat{y}_n\|^2 = 0, \\ \lim_{n\to\infty} \frac{1+\beta_n L}{1+2\beta_n\rho-\beta_n L}\cdot \frac{\alpha_n}{1+2\beta_n\rho-\beta_n L}\|w_n - z_n\|^2 = 0, \\ \lim_{n\to\infty} \frac{1+\beta_n L}{1+2\beta_n\rho-\beta_n L}\cdot \frac{1}{2}\cdot \alpha_n\|x_n - \widehat{z}_n\|^2 = 0. \end{cases} \tag{3.29}$$

By the assumptions we have

$$\lim_{n\to\infty}\|\widehat{z}_n - w_n\| = \lim_{n\to\infty}\|A\widehat{z}_n - \widehat{y}_n\| = \lim_{n\to\infty}\|w_n - z_n\| = \lim_{n\to\infty}\|x_n - \widehat{z}_n\| = 0. \tag{3.30}$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_k}\to\bar{x}\in H_1$. Thus, $w_{n_k}\to\bar{x}$, $z_{n_k}\to\bar{x}$, $Aw_{n_k}\to A\bar{x}$, and $\widehat{y}_{n_k}\to A\bar{x}$. By (3.8), (3.9), and Lemma 2.3 we get that $\bar{x}\in\Omega_{\mathrm{SDCP}}$. By Proposition 3.1, $\Omega_{\mathrm{SDCP}} = \{\bar{x}\}$. Further, $\lim_{n\to\infty}\|x_n - \bar{x}\| = \lim_{k\to\infty}\|x_{n_k} - \bar{x}\| = 0$. Therefore the proof is completed. □

4 Main results in infinite-dimensional real Hilbert space

Let H_1 and H_2 be infinite-dimensional real Hilbert spaces. Let δ, ρ, L, A, A^*, g_1, h_1, g_2, h_2, f_1, f_2, $\{r_n\}_{n\in\mathbb{N}}$, and $\{\beta_n\}_{n\in\mathbb{N}}$ be the same as in Sect. 3.

Definition 4.1 Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T: C\to H$. Let $\mathrm{Fix}(T) := \{x\in C: Tx = x\}$. Then:

(i) T is a nonexpansive mapping if $\|Tx - Ty\|\le\|x-y\|$ for all $x,y\in C$;

(ii) T is a firmly nonexpansive mapping if $\|Tx - Ty\|^2\le\langle x-y, Tx-Ty\rangle$ for all $x,y\in C$, that is, $\|Tx-Ty\|^2\le\|x-y\|^2 - \|(I-T)x-(I-T)y\|^2$ for all $x,y\in C$.

Lemma 4.1 ([21]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T: C\to H$ be a nonexpansive mapping, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in C. If $x_n\rightharpoonup w$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then $Tw = w$.*

Definition 4.2 Let $\beta > 0$, let H be a real Hilbert space, and let $g: H\to\mathbb{R}$ be a proper lower-semicontinuous and convex function. Then the proximal operator of g of order β is defined by

$$\mathrm{prox}_{\beta,g}(x) := \operatorname*{argmin}_{v\in H}\left\{g(v) + \frac{1}{2\beta}\|v-x\|^2\right\}$$

for each $x\in H$. In fact, we know that $\mathrm{prox}_{\beta,g}(x) = (I+\beta\partial g)^{-1}(x) = J_\beta^{\partial g}(x)$ and $T(x) := \mathrm{prox}_{\beta,g}(x)$ is a firmly nonexpansive mapping.

Lemma 4.2 ([22, Lemma 2.3]) *Let H be a real Hilbert space, and let $g: H\to\mathbb{R}$ be a proper lower-semicontinuous and convex function. For $\beta_2\ge\beta_1>0$, we have*

$$\mathrm{prox}_{\beta_2,g}(x) = \mathrm{prox}_{\beta_1,g}\left(\frac{\beta_1}{\beta_2}x + (1-\frac{\beta_1}{\beta_2})\,\mathrm{prox}_{\beta_2,g}(x)\right).$$

The following result plays an important role when we study our convergence theorem in an infinite-dimensional real Hilbert space.

Lemma 4.3 *Let H be a real Hilbert space, let $g, h : H \to \mathbb{R}$ be proper lower-semicontinuous and convex functions, and suppose that h is Fréchet differentiable. Then for all $x \in H$ and $0 < \beta_1 \leq \beta_2$, we have*

$$\left\| x - \operatorname{prox}_{\beta_1,g}(x + \beta_1 \nabla h(x)) \right\| \leq 2 \left\| x - \operatorname{prox}_{\beta_2,g}(x + \beta_2 \nabla h(x)) \right\|.$$

Proof By Lemma 4.2 we have

$$\operatorname{prox}_{\beta_2,g}\bigl(x + \beta_2 \nabla h(x)\bigr) = \operatorname{prox}_{\beta_1,g}\left(\frac{\beta_1}{\beta_2}\bigl(x + \beta_2 \nabla h(x)\bigr) + \left(1 - \frac{\beta_1}{\beta_2}\right) \operatorname{prox}_{\beta_2,g}\bigl(x + \beta_2 \nabla h(x)\bigr) \right).$$

Thus,

$$\begin{aligned}
&\left\| \operatorname{prox}_{\beta_1,g}\bigl(x + \beta_1 \nabla h(x)\bigr) - \operatorname{prox}_{\beta_2,g}\bigl(x + \beta_2 \nabla h(x)\bigr) \right\| \\
&\quad \leq \left\| x + \beta_1 \nabla h(x) - \left(\frac{\beta_1}{\beta_2}\bigl(x + \beta_2 \nabla h(x)\bigr) + \left(1 - \frac{\beta_1}{\beta_2}\right) \operatorname{prox}_{\beta_2,g}\bigl(x + \beta_2 \nabla h(x)\bigr) \right) \right\| \\
&\quad = \left(1 - \frac{\beta_1}{\beta_2}\right) \left\| x - \operatorname{prox}_{\beta_2,g}\bigl(x + \beta_2 \nabla h(x)\bigr) \right\| \\
&\quad \leq \left\| x - \operatorname{prox}_{\beta_2,g}\bigl(x + \beta_2 \nabla h(x)\bigr) \right\|,
\end{aligned}$$

and then

$$\begin{aligned}
&\left\| x - \operatorname{prox}_{\beta_1,g}\bigl(x + \beta_1 \nabla h(x)\bigr) \right\| \\
&\quad \leq \left\| x - \operatorname{prox}_{\beta_2,g}\bigl(x + \beta_2 \nabla h(x)\bigr) \right\| + \left\| \operatorname{prox}_{\beta_2,g}\bigl(x + \beta_2 \nabla h(x)\bigr) - \operatorname{prox}_{\beta_1,g}\bigl(x + \beta_1 \nabla h(x)\bigr) \right\| \\
&\quad \leq 2 \left\| x - \operatorname{prox}_{\beta_2,g}\bigl(x + \beta_2 \nabla h(x)\bigr) \right\|.
\end{aligned}$$

Therefore the proof is completed. □

Lemma 4.4 *Let $\beta > 0$, let H be a real Hilbert space, and let $g : H \to \mathbb{R}$ be a proper lower semicontinuous and ρ-strongly convex function. Then $T(x) := \operatorname{prox}_{\beta,g}(x)$ is a contraction mapping. In fact, $\|Tx - Ty\| \leq \frac{1}{1+\beta\rho}\|x - y\|$.*

Lemma 4.5 *Let $\beta > 0$, let H be a real Hilbert space, and let $g, h : H \to \mathbb{R}$ be proper lower semicontinuous and convex functions. Further, we assume that h is Fréchet differentiable, ∇h is L-Lipschitz continuous, and g is ρ-strongly convex. Let $T : H \to H$ be defined by $Tx := \operatorname{prox}_{\beta,g}(x + \beta \nabla h(x))$ for each $x \in H$. Then the following are satisfied.*

(i) *If $\rho > L > 0$, then T is a contraction mapping.*

(ii) *If $\rho = L > 0$, then T is a nonexpansive mapping.*

Proof For $x, y \in H$, we have

$$\begin{aligned}
\|Tx - Ty\| &\leq \frac{1}{1 + \beta\rho} \left\| \bigl(x + \beta \nabla h(x)\bigr) - \bigl(y + \beta \nabla h(y)\bigr) \right\| \\
&\leq \frac{1}{1 + \beta\rho} \bigl(\|x - y\| + \beta \left\| \nabla h(x) - \nabla h(y)\bigr) \right\| \bigr)
\end{aligned}$$

$$\leq \frac{1}{1+\beta\rho}\big(\|x-y\| + \beta L\|x-y\|\big)$$
$$= \frac{1+\beta L}{1+\beta\rho}\|x-y\|.$$

Thus the proof is completed. □

Theorem 4.1 *In Theorem 3.1, let H_1 and H_2 be an infinite-dimensional real Hilbert space and assume that* $\liminf_{n\to\infty}\beta_n > 0$. *Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by Algorithm 3.1 converges weakly to the unique solution $\bar{x}$ of problem* (SDCP).

Proof By Proposition 3.1 we know that $\Omega_{\mathrm{SDCP}} = \{\bar{x}\}$. Since $\liminf_{n\to\infty}\beta_n > 0$, we may assume that there exists a real number β^* such that $\beta_n > \beta^* > 0$. By (3.11) we have

$$\widehat{y}_n = (I+\beta_n\partial g_2)^{-1}\big(Aw_n + \beta_n\nabla h_2(Aw_n)\big) = \operatorname{prox}_{\beta_n,g_2}\big(Aw_n + \beta_n\nabla h_2(Aw_n)\big). \tag{4.1}$$

Similarly, we have

$$w_n = (I+\beta_n\partial g_1)^{-1}\big(z_n + \beta_n\nabla h_1(z_n)\big) = \operatorname{prox}_{\beta_n,g_1}\big(z_n + \beta_n\nabla h_1(z_n)\big). \tag{4.2}$$

By (3.30) we know that

$$\lim_{n\to\infty}\|Aw_n - \widehat{y}_n\| = \lim_{n\to\infty}\|w_n - z_n\| = \lim_{n\to\infty}\|x_n - w_n\| = 0. \tag{4.3}$$

By (4.2) and (4.3) we have

$$\lim_{n\to\infty}\big\|z_n - \operatorname{prox}_{\beta_n,g_1}\big(z_n + \beta_n\nabla h_1(z_n)\big)\big\| = 0 \tag{4.4}$$

and

$$\lim_{n\to\infty}\big\|Aw_n - \operatorname{prox}_{\beta_n,g_2}\big(Aw_n + \beta_n\nabla h_2(Aw_n)\big)\big\| = 0. \tag{4.5}$$

By (4.4), (4.5), and Lemma 4.3 we have

$$\lim_{n\to\infty}\big\|z_n - \operatorname{prox}_{\beta^*,g_1}\big(z_n + \beta^*\nabla h_1(z_n)\big)\big\| = 0 \tag{4.6}$$

and

$$\lim_{n\to\infty}\big\|Aw_n - \operatorname{prox}_{\beta^*,g_2}\big(Aw_n + \beta^*\nabla h_2(Aw_n)\big)\big\| = 0. \tag{4.7}$$

Besides, we have to show that $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence. Since H_1 is infinite dimensional, there exist $\bar{x}\in H_1$ and a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup x^* \in H_1$. By (4.3) we know that $z_{n_k} \rightharpoonup x^*$ and $w_{n_k} \rightharpoonup x^*$. Hence, by (4.6), Lemma 4.1, and Lemma 4.5 we have that $x^* = \operatorname{prox}_{\beta^*,g_1}(x^* + \beta^*\nabla h_1(x^*))$, which implies that $\nabla h_1(x^*) \in \partial g_1(x^*)$. Since A is linear, we have $Aw_{n_k} \rightharpoonup Ax^*$. Hence, by (4.7), Lemma 4.1 and Lemma 4.5, we have $Ax^* = \operatorname{prox}_{\beta^*,g_2}(Ax^* + \beta^*\nabla h_2(Ax^*))$, which implies that $\nabla h_2(Ax^*) \in \partial g_2(Ax^*)$. So, $x^* \in \Omega_{\mathrm{SDCP}}$, and thus $\lim_{n\to\infty}\|x_n - x^*\|$ exists. So, by Opial's condition, we get $x_n \rightharpoonup x^*$. Therefore the proof is completed. □

Remark 4.1 To the best of our knowledge, the convergence theorems for the DC program and split DC program are proposed in finite-dimensional Hilbert spaces. Here, Theorem 4.1 is a convergence theorem for the split DC program in infinite-dimensional real Hilbert spaces.

Following the same argument as in the proof of Theorem 4.1, we get the following convergence theorem in infinite-dimensional real Hilbert spaces.

Theorem 4.2 *Let H_1 and H_2 be infinite-dimensional real Hilbert spaces. Let A, A^*, g_1, h_1, g_2, h_2, f_1, and f_2 be the same as in Sect. 3. Let $\rho \geq L > 0$. Let $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence in $[a,b] \subseteq (0,\infty)$. Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, \frac{1}{\|A\|^2})$ such that $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < \frac{1}{\|A\|^2}$. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by Algorithm 1.3 converges weakly to some $\bar{x} \in \Omega_{\mathrm{SDCP}}$.*

5 Application to DC program

Let ρ, L, δ, $\{\beta_n\}_{n\in\mathbb{N}}$ be the same as in Sect. 3. Let H be an infinite-dimensional Hilbert space, and let $g, h : H \to \mathbb{R}$ be proper lower semicontinuous and convex functions. Besides, we also assume that h is Fréchet differentiable, ∇h is L-Lipschitz continuous, and g is ρ-strongly convex. Let $f(x) = g(x) - h(x)$ for all $x \in H$ and assume that f is bounded from below.

Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ with $\liminf_{n\to\infty} r_n > 0$ and

$$0 < r_n < \min\left\{\frac{\sqrt[4]{1-2\delta}\cdot\sqrt{\beta_n(\rho-L)}}{\sqrt{2+2\beta_n L}}, \frac{\sqrt{\delta}}{(2+\beta_n L)}\right\}.$$

Let Ω_{DCP} be defined by

$$\Omega_{\mathrm{DCP}} := \bigl\{x \in H : \nabla h(x) \in \partial g(x)\bigr\},$$

and assume that $\Omega_{\mathrm{DCP}} \neq \emptyset$.

The following algorithm and convergence theorem are given by Algorithm 3.1 and Theorem 4.1, respectively.

Algorithm 5.1 Let $x_1 \in H$ be arbitrary, and let $\{x_n\}_{n\in\mathbb{N}}$ be generated as follows:

$$\begin{cases}
y_n := \arg\min_{v\in H}\{g(v) + \frac{1}{2\beta_n}\|v - x_n\|^2 - \langle\nabla h(x_n), v - x_n\rangle\},\\
z_n := x_n - r_n(x_n - y_n),\\
w_n := \arg\min_{u\in H_1}\{g(u) + \frac{1}{2\beta_n}\|u - z_n\|^2 - \langle\nabla h(z_n), u - z_n\rangle\},\\
\widehat{y}_n := \arg\min_{v\in H}\{g(v) + \frac{1}{2\beta_n}\|v - w_n\|^2 - \langle\nabla h(w_n), v - w_n\rangle\},\\
\widehat{z}_n := w_n - r_n(w_n - \widehat{y}_n),\\
D_n := z_n - \widehat{z}_n,\\
\alpha_n := \frac{\langle x_n - w_n, D_n\rangle}{\|D_n\|^2},\\
\widehat{x}_n := x_n - \alpha_n D_n,\\
x_{n+1} := \arg\min_{u\in H}\{g(u) + \frac{1}{2\beta_n}\|u - \widehat{x}_n\|^2 - \langle\nabla h(\widehat{x}_n), u - \widehat{x}_n\rangle\}, \quad n \in \mathbb{N},\\
\text{stop criteria: } x_n = w_n.
\end{cases}$$

Theorem 5.1 *Assume that* $\liminf_{n\to\infty} \beta_n > 0$. *Then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *generated by Algorithm* 5.1 *converges weakly to the unique solution* $\bar{x}$ *of problem* (SDCP).

The following algorithm is a particular case of Algorithm 1.3.

Algorithm 5.2 ([13]) Let $x_1 \in H$ be arbitrary, and let $\{x_n\}_{n\in\mathbb{N}}$ be generated as follows:

$$\begin{cases} y_n := \arg\min_{v\in H}\{g(v) + \frac{1}{2\beta_n}\|v - x_n\|^2 - \langle \nabla h(x_n), v - x_n\rangle\}, \\ z_n := (1 - r_n)x_n + r_n y_n, \\ x_{n+1} := \arg\min_{u\in H}\{g(u) + \frac{1}{2\beta_n}\|u - z_n\|^2 - \langle \nabla h(z_n), u - z_n\rangle\}, \quad n \in \mathbb{N}. \end{cases}$$

By Theorem 4.2 we get the following result, which it is a generalization of [13, Thm. 4.1].

Theorem 5.2 *Let* $\rho \geq L > 0$. *Let* $\{\beta_n\}_{n\in\mathbb{N}}$ *be a sequence in* $[a, b] \subseteq (0, \infty)$. *Let* $\{r_n\}_{n\in\mathbb{N}}$ *be a sequence in* $(0, 1)$ *such that* $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 1$. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be generated by Algorithm* 5.2. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges weakly to some* $\bar{x} \in \Omega_{\mathrm{DCP}}$.

Next, we can get the following algorithm and convergence theorem by Algorithm 5.2 and Theorem 5.2, respectively. Further, Theorem 5.3 is a generalization of [13, Thm. 4.2].

Algorithm 5.3 ([13]) Let $x_1 \in H$ be arbitrary, and let $\{x_n\}_{n\in\mathbb{N}}$ be generated as follows:

$$\begin{cases} z_n := \arg\min_{u\in H}\{g(u) + \frac{1}{2\beta_n}\|u - x_n\|^2 - \langle \nabla h(x_n), u - x_n\rangle\}, \\ y_n := \arg\min_{v\in H}\{g(v) + \frac{1}{2\beta_n}\|v - z_n\|^2 - \langle \nabla h(z_n), v - z_n\rangle\}, \\ x_{n+1} := (1 - r_n)z_n + r_n y_n, \quad n \in \mathbb{N}. \end{cases}$$

Theorem 5.3 *Let* $\rho \geq L > 0$. *Let* $\{\beta_n\}_{n\in\mathbb{N}}$ *be a sequence in* $[a, b] \subseteq (0, \infty)$. *Let* $\{r_n\}_{n\in\mathbb{N}}$ *be a sequence in* $(0, 1)$ *such that* $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 1$. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be generated by Algorithm* 5.3. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges weakly to some* $\bar{x} \in \Omega_{\mathrm{DCP}}$.

If $r_n = 0$ for all $n \in \mathbb{N}$, then we have the following result.

Theorem 5.4 *Let* $\rho \geq L > 0$. *Let* $\{\beta_n\}_{n\in\mathbb{N}}$ *be a sequence in* $[a, b] \subseteq (0, \infty)$. *Let* $x_1 \in H$ *be arbitrary, and let* $\{x_n\}_{n\in\mathbb{N}}$ *be generated by*

$$x_{n+1} := \arg\min_{u\in H}\left\{g(u) + \frac{1}{2\beta_n}\|u - x_n\|^2 - \left\langle \nabla h(x_n), u - x_n\right\rangle\right\}, \quad n \in \mathbb{N}.$$

Then $\{x_n\}_{n\in\mathbb{N}}$ *converges weakly to some* $\bar{x} \in \Omega_{\mathrm{DCP}}$.

Proof Following similar argument as in the proof of Theorem 4.1, we get the statement of Theorem 5.4. □

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Competing interests
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Authors' contributions

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