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A Hilbert-type integral inequality in the whole plane related to the kernel of exponent function

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Abstract

By using real analysis and weight functions, we obtain a few equivalent statements of a Hilbert-type integral inequality in the whole plane related to the kernel of exponent function with intermediate variables. The constant factor related to the gamma function is proved to be the best possible. We also consider some particular cases and the operator expressions.

MSC: 26D15; 31A10

Keywords: Hilbert-type integral inequality; Weight function; Intermediate variable; Equivalent statement; Operator; Gamma function

1 Introduction

If $0 < \int_0^\infty f^2(x)\,dx < \infty$ and $0 < \int_0^\infty g^2(y)\,dy < \infty$, then we have the following well-known Hilbert integral inequality (see [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi \left(\int_0^\infty f^2(x)\,dx \int_0^\infty g^2(y)\,dy \right)^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible. In 1925, by introducing the pair of conjugate exponents (p, q) ($p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$), Hardy et al. gave an extension of (1) (see [1], Theorem 316). Recently, by means of weight functions, some new extensions of (1) and the Hardy's work were given by Yang [2, 3] and in [4–9]. Most of them are built in the quarter plane of the first quadrant.

In 2007, Yang [10] provided a Hilbert-type integral inequality in the whole plane with the exponent function and intermediate variables as follows:

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(1+e^{x+y})^\lambda}\,dx\,dy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_{-\infty}^\infty e^{-\lambda x} f^2(x)\,dx \int_{-\infty}^\infty e^{-\lambda y} g^2(y)\,dy \right)^{\frac{1}{2}}, \tag{2}$$

where the constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ is the best possible ($\lambda > 0$, $B(u, v)$ is the beta function). He et al. [11–19] proved some new Hilbert-type integral inequalities in the whole plane with the best possible constant factors.

In 2017, Hong [20] gave two equivalent statements between Hilbert-type inequalities with general homogenous kernel and a few parameters. A few authors continue to study this topic (see [21–25]).

In this paper, by using real analysis and weight functions we obtain a few equivalent statements of a Hilbert-type integral inequality in the whole plane related to the exponent function with intermediate variables. The constant factor related to the gamma function is proved to be the best possible. We also consider some particular cases and operator expressions.

2 Some lemmas

For $\gamma, \rho, \sigma > 0$, setting $h(u) := e^{-\rho u^{\gamma}}$ $(u > 0)$, we find

$$\begin{aligned} k_{\rho}^{(\gamma)}(\sigma) &:= \int_{0}^{\infty} h(u)u^{\sigma-1}\,du = \int_{0}^{\infty} e^{-\rho u^{\gamma}} u^{\sigma-1}\,du \quad \bigl(v = \rho u^{\gamma}\bigr) \\ &= \frac{1}{\gamma \rho^{\sigma/\gamma}} \int_{0}^{\infty} e^{-v} v^{(\sigma/\gamma)-1}\,dv = \frac{\Gamma(\sigma/\gamma)}{\gamma \rho^{\sigma/\gamma}} \in \mathbb{R}_{+} = (0,\infty), \end{aligned} \tag{3}$$

where $\Gamma(s) := \int_{0}^{\infty} e^{-v} v^{s-1}\,dv$ $(\operatorname{Re} s > 0)$ is the gamma function (see [26]).

For $\delta \in \{-1, 1\}$, $\alpha, \beta \in (-1, 1)$, we set

$$\begin{aligned} &x_{\alpha} := |x| + \alpha x, \qquad y_{\beta} := |y| + \beta y \quad \bigl(x, y \in \mathbb{R} = (-\infty, \infty)\bigr), \\ &E_{\delta} := \bigl\{t \in \mathbb{R}; |t|^{\delta} \geq 1\bigr\}, \qquad E_{-\delta} = \bigl\{t \in \mathbb{R}; |t|^{\delta} \leq 1\bigr\}. \end{aligned}$$

Lemma 1 *For $c > 0$, $\theta = \alpha, \beta \in (-1, 1)$, we have*

$$\int_{E_{\delta}} t_{\theta}^{-c\delta-1}\,dt = \frac{1}{c}\biggl[\frac{1}{(1+\theta)^{c\delta+1}} + \frac{1}{(1-\theta)^{c\delta+1}}\biggr], \tag{4}$$

$$\int_{E_{-\delta}} t_{\theta}^{c\delta-1}\,dt = \frac{1}{c}\biggl[\frac{1}{(1+\theta)^{-c\delta+1}} + \frac{1}{(1-\theta)^{-c\delta+1}}\biggr]; \tag{5}$$

and for $c \leq 0$, we have

$$\int_{E_{\delta}} t_{\theta}^{-c\delta-1}\,dt = \int_{E_{-\delta}} t_{\theta}^{c\delta-1}\,dt = \infty.$$

Proof Setting $E_{\delta}^{+} := \{t \in \mathbb{R}_{+}; t^{\delta} \geq 1\}$, $E_{\delta}^{-} := \{-t \in \mathbb{R}_{+}; (-t)^{\delta} \geq 1\}$, we find $E_{\delta} = E_{\delta}^{+} \cup E_{\delta}^{-}$ and

$$\begin{aligned} \int_{E_{\delta}} t_{\theta}^{-c\delta-1}\,dt &= \int_{E_{\delta}^{+}} \bigl[(1+\theta)t\bigr]^{-c\delta-1}\,dt + \int_{E_{\delta}^{-}} \bigl[(1-\theta)(-t)\bigr]^{-c\delta-1}\,dt \\ &= \biggl[\frac{1}{(1+\theta)^{c\delta+1}} + \frac{1}{(1-\theta)^{c\delta+1}}\biggr] \int_{E_{\delta}^{+}} t^{-c\delta-1}\,dt. \end{aligned}$$

Setting $t = u^{\frac{1}{\delta}}$, we find

$$\int_{E_{\delta}^{+}} t^{-c\delta-1}\,dt = \frac{1}{|\delta|} \int_{1}^{\infty} u^{\frac{1}{\delta}(-c\delta-1)} u^{\frac{1}{\delta}-1}\,du = \int_{1}^{\infty} u^{-c-1}\,du.$$

Hence, for $c > 0$, (4) follows, and for $c \le 0$, $\int_{E_\delta} t_\theta^{-c\delta-1}\,dt = \infty$. Since, for $c > 0$,

$$\int_{E_{-\delta}} t_\theta^{c\delta-1}\,dt = \int_{E_{(-\delta)}} t_\theta^{-c(-\delta)-1}\,dt = \left[\frac{1}{(1+\theta)^{-c\delta+1}} + \frac{1}{(1-\theta)^{-c\delta+1}}\right]\int_0^1 u^{c-1}\,du,$$

we have (5), and for $c \le 0$, $\int_{E_{-\delta}} t_\theta^{c\delta-1}\,dt = \infty$.

The lemma is proved. □

In the following, We further assume that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\delta \in \{-1,1\}$, $\alpha,\beta \in (-1,1)$, $\gamma,\rho,\sigma > 0$, $\sigma_1 \in \mathbb{R}$, $k_\rho^{(\gamma)}(\sigma)$ is given by (3), and

$$K_{\alpha,\beta}^{(\gamma)}(\sigma) := \frac{2k_\rho^{(\gamma)}(\sigma)}{(1-\alpha^2)^{1/q}(1-\beta^2)^{1/p}}. \tag{6}$$

For $n \in \mathbb{N} = \{1,2,\ldots\}$, $E_{-1} = [-1,1]$, $x \in E_\delta$, we define:

$$\begin{aligned}
&I^{(-)}(x) := \int_{-1}^{0} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} y_\beta^{\sigma+\frac{1}{qn}-1}\,dy, \qquad I^{(+)}(x) := \int_0^1 e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} y_\beta^{\sigma+\frac{1}{qn}-1}\,dy,\\
&I(x) := I^{(-)}(x) + I^{(+)}(x) = \int_{E_{-1}} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} y_\beta^{\sigma+\frac{1}{qn}-1}\,dy.
\end{aligned}$$

For $y_\beta = (\operatorname{sgn}(y)+\beta)y$, where

$$\begin{aligned}
&\operatorname{sgn}(y) := \begin{cases} -1, & y < 0,\\ 0, & y = 0,\\ 1, & y > 0,\end{cases}\\
&x_\alpha^\delta = \bigl(1+\alpha\operatorname{sgn}(x)\bigr)^\delta |x|^\delta \ge \min_{\delta\in\{-1,1\}}\bigl\{\bigl(1\pm|\alpha|\bigr)^\delta\bigr\} \quad (x \in E_\delta),
\end{aligned}$$

and $1-|\alpha| \le (1+|\alpha|)^{-1} \le 1+|\alpha| \le (1-|\alpha|)^{-1}$, we have

$$(1\pm\beta)x_\alpha^\delta \ge m_{\alpha,\beta} := \bigl(1-|\beta|\bigr)\bigl(1-|\alpha|\bigr) > 0 \quad (x \in E_\delta). \tag{7}$$

For fixed $x \in E_\delta$, setting $u = x_\alpha^\delta y_\beta$, we find

$$\begin{aligned}
I^{(-)}(x) &= \frac{x_\alpha^{-\delta(\sigma+\frac{1}{qn})}}{1-\beta}\int_0^{(1-\beta)x_\alpha^\delta} e^{-\rho u^\gamma} u^{\sigma+\frac{1}{qn}-1}\,du \ge \frac{x_\alpha^{-\delta(\sigma+\frac{1}{qn})}}{1-\beta}\int_0^{m_{\alpha,\beta}} e^{-\rho u^\gamma} u^{\sigma+\frac{1}{qn}-1}\,du,\\
I^{(+)}(x) &= \frac{x_\alpha^{-\delta(\sigma+\frac{1}{qn})}}{1+\beta}\int_0^{(1+\beta)x_\alpha^\delta} e^{-\rho u^\gamma} u^{\sigma+\frac{1}{qn}-1}\,du \ge \frac{x_\alpha^{-\delta(\sigma+\frac{1}{qn})}}{1+\beta}\int_0^{m_{\alpha,\beta}} e^{-\rho u^\gamma} u^{\sigma+\frac{1}{qn}-1}\,du,\\
I(x) &= x_\alpha^{-\delta(\sigma+\frac{1}{qn})}\left[\frac{1}{1-\beta}\int_0^{(1-\beta)x_\alpha^\delta} e^{-\rho u^\gamma} u^{\sigma+\frac{1}{qn}-1}\,du + \frac{1}{1+\beta}\int_0^{(1+\beta)x_\alpha^\delta} e^{-\rho u^\gamma} u^{\sigma+\frac{1}{qn}-1}\,du\right]\\
&\ge \frac{2x_\alpha^{-\delta(\sigma+\frac{1}{qn})}}{1+\beta}\int_0^{m_{\alpha,\beta}} e^{-\rho u^\gamma} u^{\sigma+\frac{1}{qn}-1}\,du.
\end{aligned} \tag{8}$$

For $n \in \mathbb{N} = \{1, 2, \ldots\}$, $x \in E_{-\delta}$, we define:

$$J^{(-)}(x) := \int_{-\infty}^{-1} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} y_\beta^{\sigma-\frac{1}{qn}-1}\,dy,$$
$$J^{(+)}(x) := \int_{1}^{\infty} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} y_\beta^{\sigma-\frac{1}{qn}-1}\,dy,$$
$$J(x) := J^{(-)}(x) + J^{(+)}(x) = \int_{E_{-1}} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} y_\beta^{\sigma-\frac{1}{qn}-1}\,dy.$$

Since, for $x \in E_{-\delta}$,

$$x_\alpha^\delta = \bigl(1 + \alpha\operatorname{sgn}(x)\bigr)^\delta |x|^\delta \le \max_{\delta\in\{-1,1\}} \bigl\{\bigl(1 \pm |\alpha|\bigr)^\delta\bigr\} = \bigl(1 - |\alpha|\bigr)^{-1},$$

we have

$$M_{\alpha,\beta} := \bigl(1 + |\beta|\bigr)\bigl(1 - |\alpha|\bigr)^{-1} \ge \bigl(1 \pm |\beta|\bigr)x_\alpha^\delta \quad (x \in E_{-\delta}). \tag{9}$$

For fixed $x \in E_{-\delta}$, setting $u = x_\alpha^\delta y_\beta$, we find

$$\begin{aligned}
J^{(-)}(x) &= \frac{x_\alpha^{-\delta(\sigma-\frac{1}{qn})}}{1-\beta} \int_{(1-\beta)x_\alpha^\delta}^{\infty} e^{-\rho u^\gamma} u^{\sigma-\frac{1}{qn}-1}\,du \ge \frac{x_\alpha^{-\delta(\sigma-\frac{1}{qn})}}{1-\beta} \int_{M_{\alpha,\beta}}^{\infty} e^{-\rho u^\gamma} u^{\sigma-\frac{1}{qn}-1}\,du,\\
J^{(+)}(x) &= \frac{x_\alpha^{-\delta(\sigma-\frac{1}{qn})}}{1+\beta} \int_{(1+\beta)x_\alpha^\delta}^{\infty} e^{-\rho u^\gamma} u^{\sigma-\frac{1}{qn}-1}\,du \ge \frac{x_\alpha^{-\delta(\sigma-\frac{1}{qn})}}{1+\beta} \int_{M_{\alpha,\beta}}^{\infty} e^{-\rho u^\gamma} u^{\sigma-\frac{1}{qn}-1}\,du,\\
J(x) &= x_\alpha^{-\delta(\sigma-\frac{1}{qn})} \biggl[\frac{1}{1-\beta} \int_{(1-\beta)x_\alpha^\delta}^{\infty} e^{-\rho u^\gamma} u^{\sigma-\frac{1}{qn}-1}\,du + \frac{1}{1+\beta} \int_{(1+\beta)x_\alpha^\delta}^{\infty} e^{-\rho u^\gamma} u^{\sigma-\frac{1}{qn}-1}\,du\biggr]\\
&\ge \frac{2x_\alpha^{-\delta(\sigma-\frac{1}{qn})}}{1+\beta} \int_{M_{\alpha,\beta}}^{\infty} e^{-\rho u^\gamma} u^{\sigma-\frac{1}{qn}-1}\,du.
\end{aligned} \tag{10}$$

In view of (8) and (10), we have the following:

Lemma 2 *We have the following inequalities*:

$$\begin{aligned}
I_1 &:= \int_{E_\delta} I(x) x_\alpha^{\delta(\sigma_1-\frac{1}{pn})-1}\,dx\\
&\ge \frac{2}{1-\beta^2} \int_{E_\delta} x_\alpha^{-\delta(\sigma-\sigma_1+\frac{1}{n})-1}\,dx \int_0^{m_{\alpha,\beta}} e^{-\rho u^\gamma} u^{\sigma+\frac{1}{qn}-1}\,du,
\end{aligned} \tag{11}$$

$$\begin{aligned}
J_1 &:= \int_{E_{-\delta}} J(x) x_\alpha^{\delta(\sigma_1+\frac{1}{pn})-1}\,dx\\
&\ge \frac{2}{1-\beta^2} \int_{E_{-\delta}} x_\alpha^{\delta(\sigma_1-\sigma+\frac{1}{n})-1}\,dx \int_{M_{\alpha,\beta}}^{\infty} e^{-\rho u^\gamma} u^{\sigma-\frac{1}{qn}-1}\,du.
\end{aligned} \tag{12}$$

Lemma 3 *If there exists a constant M such that, for any nonnegative measurable functions $f(x)$ and $g(y)$ in $\mathbb{R}$,*

$$I := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} f(x)g(y)\,dx\,dy \leq M\left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\delta\sigma_1)-1} f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} g^q(y)\,dy\right]^{\frac{1}{q}}, \tag{13}$$

then we have $\sigma_1 = \sigma$.

Proof If $\sigma_1 > \sigma$, then for $n \geq \frac{1}{\sigma_1-\sigma}$ ($n \in \mathbb{N}$), we define the functions:

$$f_n(x) := \begin{cases} x_\alpha^{\delta(\sigma_1-\frac{1}{pn})-1}, & x \in E_\delta, \\ 0, & x \in \mathbb{R}\backslash E_\delta, \end{cases} \qquad g_n(y) := \begin{cases} y_\beta^{\sigma+\frac{1}{qn}-1}, & y \in E_{-1}, \\ 0, & y \in \mathbb{R}\backslash E_{-1}, \end{cases}$$

and by (4) and (5) it follows that

$$\begin{aligned}\tilde{J}_1 &:= \left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\delta\sigma_1)-1} f_n^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} g_n^q(y)\,dy\right]^{\frac{1}{q}} \\ &= \left(\int_{E_\delta} x_\alpha^{-\frac{\delta}{n}-1}\,dx\right)^{\frac{1}{p}}\left(\int_{E_{-1}} y_\beta^{\frac{1}{n}-1}\,dy\right)^{\frac{1}{q}} \\ &= n\left[\frac{1}{(1+\alpha)^{\frac{\delta}{n}+1}} + \frac{1}{(1-\alpha)^{\frac{\delta}{n}+1}}\right]^{\frac{1}{p}}\left[\frac{1}{(1+\beta)^{\frac{-1}{n}+1}} + \frac{1}{(1-\beta)^{\frac{-1}{n}+1}}\right]^{\frac{1}{q}} < \infty.\end{aligned}$$

By (11) and (13) (for $f = f_n$, $g = g_n$) we have

$$\begin{aligned}&\frac{2}{1-\beta^2}\int_{E_\delta} x_\alpha^{-\delta(\sigma-\sigma_1+\frac{1}{n})-1}\,dx \int_0^{m_{\alpha,\beta}} e^{-\rho u^\gamma} u^{\sigma+\frac{1}{qn}-1}\,du \\ &\quad\leq I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} f_n(x)g_n(y)\,dx\,dy \leq M\tilde{J}_1 < \infty.\end{aligned}$$

Since for any $n \geq \frac{1}{\sigma_1-\sigma}$, $\sigma - \sigma_1 + \frac{1}{n} \leq 0$, by Lemma 1 it follows that $\int_{E_\delta} x_\alpha^{-\delta(\sigma-\sigma_1+\frac{1}{n})-1}\,dx = \infty$. In view of $\int_0^{m_{\alpha,\beta}} e^{-\rho u^\gamma} u^{\sigma+\frac{1}{qn}-1}\,du > 0$, we find that $\infty \leq M\tilde{J}_1 < \infty$, which is a contradiction.

If $\sigma_1 < \sigma$, then for $n \geq \frac{1}{\sigma-\sigma_1}$ ($n \in \mathbb{N}$), we define the functions:

$$\tilde{f}_n(x) := \begin{cases} x_\alpha^{\delta(\sigma+\frac{1}{pn})-1}, & x \in E_{-\delta}, \\ 0, & x \in \mathbb{R}\backslash E_{-\delta}, \end{cases} \qquad \tilde{g}_n(y) := \begin{cases} y_\beta^{\sigma-\frac{1}{qn}-1}, & y \in E_1, \\ 0, & y \in \mathbb{R}\backslash E_1, \end{cases}$$

and by (4) and (5) it follows that

$$\begin{aligned}\tilde{J}_2 &:= \left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\delta\sigma_1)-1}\tilde{f}_n^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1}\tilde{g}_n^q(y)\,dy\right]^{\frac{1}{q}} \\ &= \left(\int_{E_{-\delta}} x_\alpha^{\frac{\delta}{n}-1}\,dx\right)^{\frac{1}{p}}\left(\int_{E_1} y_\beta^{\frac{-1}{n}-1}\,dy\right)^{\frac{1}{q}} \\ &= n\left[\frac{1}{(1+\alpha)^{\frac{-\delta}{n}+1}}+\frac{1}{(1-\alpha)^{\frac{-\delta}{n}+1}}\right]^{\frac{1}{p}}\left[\frac{1}{(1+\beta)^{\frac{1}{n}+1}}+\frac{1}{(1-\beta)^{\frac{-}{n}+1}}\right]^{\frac{1}{q}}<\infty.\end{aligned}$$

By (12) and (13) (for $f=\tilde{f}_n$, $g=\tilde{g}_n$) we have

$$\begin{aligned}&\frac{2}{1-\beta^2}\int_{E_{-\delta}} x_\alpha^{\delta(\sigma_1-\sigma+\frac{1}{n})-1}\,dx\int_{M_{\alpha,\beta}}^{\infty} e^{-\rho u^\gamma}u^{\sigma-\frac{1}{qn}-1}\,du \\ &\quad\le J_1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma}\tilde{f}_n(x)\tilde{g}_n(y)\,dx\,dy\le M\tilde{J}_2<\infty.\end{aligned}$$

Since for any $n\ge\frac{1}{\sigma-\sigma_1}$, $\sigma_1-\sigma+\frac{1}{n}\le 0$, by Lemma 1 it follows that $\int_{E_{-\delta}} x_\alpha^{\delta(\sigma_1-\sigma+\frac{1}{n})-1}\,dx=\infty$. In view of $\int_{M_{\alpha,\beta}}^{\infty} e^{-\rho u^\gamma}u^{\sigma-\frac{1}{qn}-1}\,du>0$, we have $\infty\le M\tilde{J}_2<\infty$, which is a contradiction.

Hence we conclude that $\sigma_1=\sigma$.

The lemma is proved. □

Lemma 4 *If there exists a constant M such that, for any nonnegative measurable functions $f(x)$ and $g(y)$ in $\mathbb{R}$,*

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma}f(x)g(y)\,dx\,dy \\ &\quad\le M\left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\delta\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1}g^q(y)\,dy\right]^{\frac{1}{q}},\end{aligned}\tag{14}$$

then we have $K_{\alpha,\beta}^{(\gamma)}(\sigma)\le M$.

Proof For $\sigma_1=\sigma$, by (8) we have

$$\begin{aligned}&I_1=\int_{E_\delta} I(x)x_\alpha^{\delta(\sigma-\frac{1}{pn})-1}\,dx=I_1^{(-)}+I_1^{(+)}, \\ &I_1^{(-)}=\int_{E_\delta} I^{(-)}(x)x_\alpha^{\delta(\sigma-\frac{1}{pn})-1}\,dx,\qquad I_1^{(+)}:=\int_{E_\delta} I^{(+)}(x)x_\alpha^{\delta(\sigma-\frac{1}{pn})-1}\,dx.\end{aligned}$$

In view of the presented results, we find

$$\begin{aligned}I_1^{(-)}&=\frac{1}{1-\beta}\int_{E_\delta} x_\alpha^{-\frac{\delta}{n}-1}\int_0^{(1-\beta)x_\alpha^\delta} e^{-\rho u^\gamma}u^{\sigma+\frac{1}{qn}-1}\,du\,dx \\ &=\frac{1}{1-\beta}\int_{E_\delta} x_\alpha^{-\frac{\delta}{n}-1}\left[\int_0^{\infty} e^{-\rho u^\gamma}u^{\sigma+\frac{1}{qn}-1}\,du-\int_{(1-\beta)x_\alpha^\delta}^{\infty} e^{-\rho u^\gamma}u^{\sigma+\frac{1}{qn}-1}\,du\right]dx\end{aligned}$$

$$= \frac{n}{1-\beta}\left[\frac{1}{(1+\alpha)^{\frac{\delta}{n}+1}} + \frac{1}{(1-\alpha)^{\frac{\delta}{n}+1}}\right] k_\rho^{(\gamma)}\left(\sigma + \frac{1}{qn}\right)$$
$$- \frac{1}{1-\beta}\int_{E_\delta} x_\alpha^{-\frac{\delta}{n}-1}\int_{(1-\beta)x_\alpha^\delta}^{\infty} e^{-\rho u^\gamma} u^{\sigma+\frac{1}{qn}-1}\,du\,dx. \quad (15)$$

Since $e^{-\rho u^\gamma}u^{2\sigma}$ is continuous in $(0,\infty)$, and $e^{-\rho u^\gamma}u^{2\sigma} \to 0$ $(u \to \infty)$, there exists a positive constant M_1 such that $e^{-\rho u^\gamma}u^{2\sigma} \le M_1$ $(u \in [m_{\alpha,\beta},\infty))$. By (4) it follows that

$$0 < \int_{E_\delta} x_\alpha^{-\frac{\delta}{n}-1}\int_{(1-\beta)x_\alpha^\delta}^{\infty} e^{-\rho u^\gamma} u^{\sigma+\frac{1}{qn}-1}\,du\,dx$$
$$\le M_1\int_{E_\delta} x_\alpha^{-\frac{\delta}{n}-1}\left[\int_{(1-\beta)x_\alpha^\delta}^{\infty} u^{-\sigma+\frac{1}{qn}-1}\,du\right]dx = \frac{M_1\int_{E_\delta} x_\alpha^{-\delta(\sigma+\frac{1}{pn})-1}\,dx}{(1-\beta)^{\sigma-\frac{1}{qn}}}$$
$$= \frac{(\sigma+\frac{1}{pn})^{-1}M_1}{(1-\beta)^{\sigma-\frac{1}{qn}}}\left[\frac{1}{(1+\alpha)^{\delta(\sigma+\frac{1}{pn})+1}} + \frac{1}{(1-\alpha)^{\delta(\sigma+\frac{1}{pn})+1}}\right],$$

so that

$$\frac{1}{1-\beta}\int_{E_\delta} x_\alpha^{-\frac{\delta}{n}-1}\int_{(1-\beta)x_\alpha^\delta}^{\infty} e^{-\rho u^\gamma} u^{\sigma+\frac{1}{qn}-1}\,du\,dx = O(1).$$

By (15) it follows that

$$\frac{1}{n}I_1^{(-)} = \frac{k_\rho^{(\gamma)}(\sigma+\frac{1}{qn})}{1-\beta}\left[\frac{1}{(1+\alpha)^{\frac{\delta}{n}+1}} + \frac{1}{(1-\alpha)^{\frac{\delta}{n}+1}}\right] - \frac{O(1)}{n}. \quad (16)$$

In the same way, we have

$$\frac{1}{n}I_1^{(+)} = \frac{k_\rho^{(\gamma)}(\sigma+\frac{1}{qn})}{1+\beta}\left[\frac{1}{(1+\alpha)^{\frac{\delta}{n}+1}} + \frac{1}{(1-\alpha)^{\frac{\delta}{n}+1}}\right] - \frac{\tilde{O}(1)}{n}. \quad (17)$$

By (14) (for $f = f_n$, $g = g_n$), we have

$$\frac{1}{n}I_1 = \frac{1}{n}\left(I_1^{(-)} + I_1^{(+)}\right) \le \frac{1}{n}M\tilde{J}_1.$$

For $n \to \infty$, by Fatou lemma (see [27]), (16), and (17) we find

$$\frac{2}{1-\beta^2}\frac{2k_\rho^{(\gamma)}(\sigma)}{1-\alpha^2} \le M\left(\frac{2}{1-\alpha^2}\right)^{\frac{1}{p}}\left(\frac{2}{1-\beta^2}\right)^{\frac{1}{q}},$$

so that $K_{\alpha,\beta}^{(\gamma)}(\sigma) = \frac{2k_\rho^{(\gamma)}(\sigma)}{(1-\alpha^2)^{1/q}(1-\beta^2)^{1/p}} \le M$.

The lemma is proved. □

Lemma 5 *We define the following weight functions:*

$$\omega_\delta(\sigma,y) := y_\beta^\sigma\int_{-\infty}^{\infty} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} x_\alpha^{\delta\sigma-1}\,dx \quad (y \in \mathbb{R}), \quad (18)$$

$$\varpi_\delta(\sigma,x) := x_\alpha^{\delta\sigma} \int_{-\infty}^{\infty} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} y_\beta^{\sigma-1}\,dy \quad (x \in \mathbb{R}). \tag{19}$$

Then we have

$$\frac{1-\alpha^2}{2}\omega_\delta(\sigma,y) = \frac{1-\beta^2}{2}\varpi_\delta(\sigma,x) = k_\rho^{(\gamma)}(\sigma) \quad \bigl(x,y \in \mathbb{R}\backslash\{0\}\bigr). \tag{20}$$

Proof For fixed $y \in \mathbb{R}\backslash\{0\}$, setting $u = x_\alpha^\delta y_\beta$, we find

$$\begin{aligned}\omega_\delta(\sigma,y) &= y_\beta^\sigma \int_{-\infty}^{0} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma}\bigl[(1-\alpha)(-x)\bigr]^{\delta\sigma-1}\,dx \\ &\quad + y_\beta^\sigma \int_{0}^{\infty} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma}\bigl[(1+\alpha)x\bigr]^{\delta\sigma-1}\,dx \\ &= \left(\frac{1}{1-\alpha}+\frac{1}{1+\alpha}\right)\int_0^\infty e^{-\rho u^\gamma}u^{\sigma-1}\,du = \frac{2}{1-\alpha^2}k_\rho^{(\gamma)}(\sigma);\end{aligned}$$

for fixed $x \in \mathbb{R}\backslash\{0\}$, setting $u = x_\alpha^\delta y_\beta$, it follows that

$$\begin{aligned}\varpi_\delta(\sigma,x) &= x_\alpha^{\delta\sigma}\int_{-\infty}^{0} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} y_\beta^{\sigma-1}\,dy + x_\alpha^{\delta\sigma}\int_{0}^{\infty} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} y_\beta^{\sigma-1}\,dy \\ &= \frac{2}{1-\beta^2}\int_0^\infty e^{-\rho u^\gamma}u^{\sigma-1}\,du = \frac{2}{1-\beta^2}k_\rho^{(\gamma)}(\sigma).\end{aligned}$$

Hence we have (20).

The lemma is proved. □

3 Main results

Theorem 1 *If M is a constant, then the following statements* (i), (ii), *and* (iii) *are equivalent*:

(i) *For any $f(x) \geq 0$, we have*:

$$\begin{aligned}J &:= \left\{\int_{-\infty}^{\infty} y_\beta^{p\sigma-1}\left[\int_{-\infty}^{\infty} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} f(x)\,dx\right]^p dy\right\}^{\frac{1}{p}} \\ &\leq M\left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\delta\sigma_1)-1} f^p(x)\,dx\right]^{\frac{1}{p}}.\end{aligned} \tag{21}$$

(ii) *For any $f(x), g(y) \geq 0$, we have*:

$$\begin{aligned}I &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} f(x)g(y)\,dx\,dy \\ &\leq M\left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\delta\sigma_1)-1} f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} g^q(y)\,dy\right]^{\frac{1}{q}}.\end{aligned} \tag{22}$$

(iii) $\sigma_1 = \sigma$, *and* $K_{\alpha,\beta}^{(\gamma)}(\sigma) \leq M$.

Proof (i)=>(ii). By Hölder's inequality (see [28]) we have

$$\begin{aligned} I &= \int_{-\infty}^{\infty}\left[y_\beta^{\sigma-\frac{1}{p}}\int_{-\infty}^{\infty}e^{-\rho(x_\alpha^\delta y_\beta)^\gamma}f(x)\,dx\right]\left(y_\beta^{-\sigma+\frac{1}{p}}g(y)\right)dy \\ &\le J\left[\int_{-\infty}^{\infty}y_\beta^{q(1-\sigma)-1}g^q(y)\,dy\right]^{\frac{1}{q}}. \end{aligned} \tag{23}$$

Then by (21) we have (22).

(ii)=>(iii). By Lemma 1 we have $\sigma_1 = \sigma$. Then by Lemma 2 we have $K_{\alpha,\beta}^{(\gamma)}(\sigma) \le M$.

(iii)=>(i). For $\sigma_1 = \sigma$, by Hölder's inequality with weight (see [28]) and (18) we have

$$\begin{aligned} &\left[\int_{-\infty}^{\infty}e^{-\rho(x_\alpha^\delta y_\beta)^\gamma}f(x)\,dx\right]^p \\ &\quad = \left\{\int_{-\infty}^{\infty}e^{-\rho(x_\alpha^\delta y_\beta)^\gamma}\left[\frac{y_\beta^{(\sigma-1)/p}f(x)}{x_\alpha^{(\delta\sigma-1)/q}}\right]\left[\frac{x_\alpha^{(\delta\sigma-1)/q}}{y_\beta^{(\sigma-1)/p}}\right]dx\right\}^p \\ &\quad \le \int_{-\infty}^{\infty}e^{-\rho(x_\alpha^\delta y_\beta)^\gamma}\frac{y_\beta^{\sigma-1}f^p(x)}{x_\alpha^{(\delta\sigma-1)p/q}}\,dx\left[\int_{-\infty}^{\infty}e^{-\rho(x_\alpha^\delta y_\beta)^\gamma}\frac{x_\alpha^{\delta\sigma-1}}{y_\beta^{(\sigma-1)q/p}}\,dx\right]^{p/q} \\ &\quad = \left[\omega_\delta(\sigma,y)y_\beta^{q(1-\sigma)-1}\right]^{p-1}\int_{-\infty}^{\infty}e^{-\rho(x_\alpha^\delta y_\beta)^\gamma}\frac{y_\beta^{\sigma-1}f^p(x)}{x_\alpha^{(\delta\sigma-1)p/q}}\,dx \\ &\quad = \left(\frac{2k_\rho^{(\gamma)}(\sigma)}{1-\alpha^2}\right)^{p-1}y_\beta^{-p\sigma+1}\int_{-\infty}^{\infty}e^{-\rho(x_\alpha^\delta y_\beta)^\gamma}\frac{y_\beta^{\sigma-1}f^p(x)}{x_\alpha^{(\delta\sigma-1)p/q}}\,dx. \end{aligned} \tag{24}$$

By Fubini's theorem (see [27]), (24), and (19) we have

$$\begin{aligned} J &\le \left(\frac{2k_\rho^{(\gamma)}(\sigma)}{1-\alpha^2}\right)^{\frac{1}{q}}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\rho(x_\alpha^\delta y_\beta)^\gamma}\frac{y_\beta^{\sigma-1}f^p(x)}{x_\alpha^{(\delta\sigma-1)p/q}}\,dx\,dy\right]^{\frac{1}{p}} \\ &= \left(\frac{2k_\rho^{(\gamma)}(\sigma)}{1-\alpha^2}\right)^{\frac{1}{q}}\left[\int_{-\infty}^{\infty}\varpi_\delta(\sigma,x)x_\delta^{p(1-\delta\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}} \\ &= K_{\alpha,\beta}^{(\gamma)}(\sigma)\left[\int_{-\infty}^{\infty}x_\delta^{p(1-\delta\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}}. \end{aligned}$$

For $K_{\alpha,\beta}^{(\gamma)}(\sigma) \le M$, we have (21) (when $\sigma_1 = \sigma$).

Therefore, statements (i), (ii), and (iii) are equivalent.

The theorem is proved. □

Theorem 2 *If M is a constant, then the following statements* (i), (ii), *and* (iii) *are equivalent*:

(i) *For any* $f(x) \ge 0$ *satisfying* $0 < \int_{-\infty}^{\infty}x_\alpha^{p(1-\delta\sigma)-1}f^p(x)\,dx < \infty$, *we have*:

$$\begin{aligned} &\left\{\int_{-\infty}^{\infty}y_\beta^{p\sigma-1}\left[\int_{-\infty}^{\infty}e^{-\rho(x_\alpha^\delta y_\beta)^\gamma}f(x)\,dx\right]^p dy\right\}^{\frac{1}{p}} \\ &\quad < M\left[\int_{-\infty}^{\infty}x_\alpha^{p(1-\delta\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}}. \end{aligned} \tag{25}$$

(ii) *For any* $f(x) \geq 0$ *satisfying* $0 < \int_{-\infty}^{\infty} x_\alpha^{p(1-\delta\sigma)-1} f^p(x)\,dx < \infty$, *and* $g(x) \geq 0$ *satisfying* $0 < \int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} g^q(y)\,dy < \infty$, *we have*:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\rho(x_\alpha^\delta y_\beta)^\gamma} f(x)g(y)\,dx\,dy < M\left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\delta\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} g^q(y)\,dy\right]^{\frac{1}{q}}. \tag{26}$$

(iii) $K_{\alpha,\beta}^{(\gamma)}(\sigma) \leq M$.

Moreover, if statement (iii) *holds, then the constant factor* $M = K_{\alpha,\beta}^{(\gamma)}(\sigma)$ *in* (25) *and* (26) *is the best possible.*

In particular, (1) *for* $\delta = 1$, $M = K_{\alpha,\beta}^{(\gamma)}(\sigma)$, *we have the following equivalent inequalities with nonhomogeneous kernel*:

$$\left\{\int_{-\infty}^{\infty} y_\beta^{p\sigma-1}\left[\int_{-\infty}^{\infty} e^{-\rho(x_\alpha y_\beta)^\gamma} f(x)\,dx\right]^p dy\right\}^{\frac{1}{p}} < K_{\alpha,\beta}^{(\gamma)}(\sigma)\left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}, \tag{27}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\rho(x_\alpha y_\beta)^\gamma} f(x)g(y)\,dx\,dy < K_{\alpha,\beta}^{(\gamma)}(\sigma)\left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} g^q(y)\,dy\right]^{\frac{1}{q}}, \tag{28}$$

where $K_{\alpha,\beta}^{(\gamma)}(\sigma)$ *is the best possible constant factor*;

(2) *for* $\delta = -1$, $M = K_{\alpha,\beta}^{(\gamma)}(\sigma)$, *we have the following equivalent inequalities with homogeneous kernel of degree* 0:

$$\left\{\int_{-\infty}^{\infty} y_\beta^{p\sigma-1}\left[\int_{-\infty}^{\infty} e^{-\rho(y_\beta/x_\alpha)^\gamma} f(x)\,dx\right]^p dy\right\}^{\frac{1}{p}} < K_{\alpha,\beta}^{(\gamma)}(\sigma)\left[\int_{-\infty}^{\infty} x_\alpha^{p(1+\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}, \tag{29}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\rho(y_\beta/x_\alpha)^\gamma} f(x)g(y)\,dx\,dy < K_{\alpha,\beta}^{(\gamma)}(\sigma)\left[\int_{-\infty}^{\infty} x_\alpha^{p(1+\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} g^q(y)\,dy\right]^{\frac{1}{q}}, \tag{30}$$

where $K_{\alpha,\beta}^{(\gamma)}(\sigma)$ *is the best possible constant factor.*

Proof For $\sigma_1 = \sigma$, under and the assumption of statement (i), if (24) takes the form of equality for $y \in \mathbb{R}\backslash\{0\}$, then there exist constants A and B such that they are not both zero and (see [28])

$$A\frac{y_\beta^{\sigma-1}}{x_\alpha^{(\delta\sigma-1)p/q}} f^p(x) = B\frac{x_\alpha^{\delta\sigma-1}}{y_\beta^{(\sigma-1)q/p}} \quad \text{a.e. in } \mathbb{R}.$$

We suppose that $A \neq 0$ (otherwise, $B = A = 0$). Then it follows that

$$x_\alpha^{p(1-\delta\sigma)-1} f^p(x) = y_\beta^{q(\sigma-1)} \frac{B}{Ax_\alpha} \quad \text{a.e. in } \mathbb{R}.$$

Since $\int_{-\infty}^{\infty} x_\alpha^{-1}\,dx = \infty$, this contradicts the fact that $0 < \int_{-\infty}^{\infty} x_\alpha^{p(1-\delta\sigma)-1} f^p(x)\,dx < \infty$. Hence (24) takes the form of strict inequality, and so does (21). Hence (25) and (26) are valid.

In view of Theorem 1, we still can conclude that statements (i), (ii), and (iii) in Theorem 2 are equivalent.

When statement (iii) holds, namely, $K_{\alpha,\beta}^{(\gamma)}(\sigma) \leq M$, if there exists a constant $M(\leq K_{\alpha,\beta}^{(\gamma)}(\sigma))$ such that (26) is valid, then $M = K_{\alpha,\beta}^{(\gamma)}(\sigma)$, and we can conclude that the constant factor $M = K_{\alpha,\beta}^{(\gamma)}(\sigma)$ in (26) is the best possible.

The constant factor $M = K_{\alpha,\beta}^{(\gamma)}(\sigma)$ in (25) is still the best possible. Otherwise, by (23) (for $\sigma_1 = \sigma$), we would get a contradiction that the constant factor $M = K_{\alpha,\beta}^{(\gamma)}(\sigma)$ in (26) is not the best possible.

The theorem is proved. □

4 Operator expressions

We set the following functions: $\varphi(x) := x_\alpha^{p(1-\delta\sigma)-1}$ $(x \in \mathbb{R})$, and $\psi(y) := y_\beta^{q(1-\sigma)-1}$, where from $\psi^{1-p}(y) := y_\beta^{p\sigma-1}$ $(y \in \mathbb{R})$. Define the following real normed linear spaces:

$$L_{p,\varphi}(\mathbb{R}) := \left\{f; \|f\|_{p,\varphi} := \left(\int_{-\infty}^{\infty} \varphi(x)\left|f(x)\right|^p dx\right)^{\frac{1}{p}} < \infty\right\},$$

$$L_{q,\psi}(\mathbb{R}) := \left\{g; \|g\|_{q,\psi} = \left(\int_{-\infty}^{\infty} \psi(y)\left|g(y)\right|^q dy\right)^{\frac{1}{q}} < \infty\right\},$$

$$L_{p,\psi^{1-p}}(\mathbb{R}) := \left\{h; \|h\|_{p,\psi^{1-p}} = \left(\int_{-\infty}^{\infty} \psi^{1-p}(y)\left|h(y)\right|^p dy\right)^{\frac{1}{p}} < \infty\right\}.$$

In view of Theorem 2, for $f \in L_{p,\varphi}(\mathbb{R})$, setting

$$h_1(y) := \int_{-\infty}^{\infty} e^{-\rho(y_\beta/x_\alpha)^\gamma} f(x)\,dx \quad (y \in \mathbb{R}),$$

by (25) we have

$$\|h_1\|_{p,\psi^{1-p}} = \left(\int_{-\infty}^{\infty} \psi^{1-p}(y)\left|h_1(y)\right|^p dy\right)^{\frac{1}{p}} \leq M\|f\|_{p,\varphi} < \infty. \tag{31}$$

Definition 1 Define the Hilbert-type integral operator $T : L_{p,\varphi}(\mathbb{R}) \to L_{p,\psi^{1-p}}(\mathbb{R})$ as follows: For any $f \in L_{p,\varphi}(\mathbb{R})$, there exists a unique representation $Tf = h_1 \in L_{p,\psi^{1-p}}(\mathbb{R})$, satisfying for any $y \in \mathbb{R}$, $Tf(y) = h_1(y)$.

In view of (31), it follows that $\|Tf\|_{p,\psi^{1-p}} = \|h_1\|_{p,\psi^{1-p}} \leq M\|f\|_{p,\varphi}$, and then the operator T is bounded and satisfies

$$\|T\| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbb{R})} \frac{\|Tf\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq M.$$

If we define the formal inner product of Tf and g as

$$(Tf,g) := \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-\rho(y_\beta/x_\alpha)^\gamma} f(x)\,dx\right]g(y)\,dy,$$

then we can rewrite Theorem 2 as follows.

Theorem 3 *If M is a constant, then the following statements* (i), (ii), *and* (iii) *are equivalent*:

(i) *For any $f(x) \geq 0, f \in L_{p,\varphi}(\mathbb{R})$, $\|f\|_{p,\varphi} > 0$, we have*:

$$\|Tf\|_{p,\psi^{1-p}} < M\|f\|_{p,\varphi}. \tag{32}$$

(ii) *For $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbb{R})$, $g \in L_{q,\psi}(\mathbb{R})$, $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$, we have*:

$$(Tf,g) < M\|f\|_{p,\varphi}\|g\|_{q,\psi}. \tag{33}$$

(iii) $K_{\alpha,\beta}^{(\gamma)}(\sigma) \leq M$.

Moreover, if statement (iii) *holds, then the constant factor $M = K_{\alpha,\beta}^{(\gamma)}(\sigma)$ in* (32) *and* (33) *is the best possible, namely, $\|T\| = K_{\alpha,\beta}^{(\gamma)}(\sigma)$.*

Remark 1 (1) In particular, for $\alpha = \beta = 0$ in (27) and (28), we have the following equivalent inequalities:

$$\begin{aligned}&\left\{\int_{-\infty}^{\infty}|y|^{p\sigma-1}\left[\int_{-\infty}^{\infty}e^{-\rho|xy|^\gamma}f(x)\,dx\right]^p dy\right\}^{\frac{1}{p}}\\&\quad< \frac{2\Gamma(\sigma/\gamma)}{\gamma\rho^{\sigma/\gamma}}\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}},\end{aligned} \tag{34}$$

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\rho|xy|^\gamma}f(x)g(y)\,dx\,dy\\&\quad< \frac{2\Gamma(\sigma/\gamma)}{\gamma\rho^{\sigma/\gamma}}\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^q(y)\,dy\right]^{\frac{1}{q}},\end{aligned} \tag{35}$$

where $\frac{2\Gamma(\sigma/\gamma)}{\gamma\rho^{\sigma/\gamma}}$ is the best possible constant factor.

If $f(-x) = f(x)$, $g(-y) = g(y)$ $(x, y \in \mathbb{R}_+)$, then we have the following equivalent inequalities:

$$\begin{aligned}&\left\{\int_0^{\infty}y^{p\sigma-1}\left[\int_0^{\infty}e^{-\rho(xy)^\gamma}f(x)\,dx\right]^p dy\right\}^{\frac{1}{p}}\\&\quad< \frac{\Gamma(\sigma/\gamma)}{\gamma\rho^{\sigma/\gamma}}\left[\int_0^{\infty}x^{p(1-\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}},\end{aligned} \tag{36}$$

$$\begin{aligned}&\int_0^{\infty}\int_0^{\infty}e^{-\rho(xy)^\gamma}f(x)g(y)\,dx\,dy\\&\quad< \frac{\Gamma(\sigma/\gamma)}{\gamma\rho^{\sigma/\gamma}}\left[\int_0^{\infty}x^{p(1-\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_0^{\infty}y^{q(1-\sigma)-1}g^q(y)\,dy\right]^{\frac{1}{q}},\end{aligned} \tag{37}$$

where $\frac{\Gamma(\sigma/\gamma)}{\gamma\rho^{\sigma/\gamma}}$ is the best possible constant factor.

(2) For $\alpha = \beta = 0$ in (29) and (30), we have the following equivalent inequalities:

$$\left\{\int_{-\infty}^{\infty} |y|^{p\sigma-1}\left[\int_{-\infty}^{\infty} e^{-\rho|y/x|^{\gamma}} f(x)\,dx\right]^p dy\right\}^{\frac{1}{p}} < \frac{2\Gamma(\sigma/\gamma)}{\gamma\rho^{\sigma/\gamma}}\left[\int_{-\infty}^{\infty} |x|^{p(1+\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}, \tag{38}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\rho|y/x|^{\gamma}} f(x)g(y)\,dx\,dy < \frac{2\Gamma(\sigma/\gamma)}{\gamma\rho^{\sigma/\gamma}}\left[\int_{-\infty}^{\infty} |x|^{p(1+\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)\,dy\right]^{\frac{1}{q}}, \tag{39}$$

where $\frac{2\Gamma(\sigma/\gamma)}{\gamma\rho^{\sigma/\gamma}}$ is the best possible constant factor.

If $f(-x) = f(x)$, $g(-y) = g(y)$ $(x, y \in \mathbb{R}_+)$, then we have the following equivalent inequalities:

$$\left\{\int_{0}^{\infty} y^{p\sigma-1}\left[\int_{0}^{\infty} e^{-\rho(y/x)^{\gamma}} f(x)\,dx\right]^p dy\right\}^{\frac{1}{p}} < \frac{\Gamma(\sigma/\gamma)}{\gamma\rho^{\sigma/\gamma}}\left[\int_{0}^{\infty} x^{p(1+\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}, \tag{40}$$

$$\int_{0}^{\infty}\int_{0}^{\infty} e^{-\rho(y/x)^{\gamma}} f(x)g(y)\,dx\,dy < \frac{\Gamma(\sigma/\gamma)}{\gamma\rho^{\sigma/\gamma}}\left[\int_{0}^{\infty} x^{p(1+\sigma)-1} f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{0}^{\infty} y^{q(1-\sigma)-1} g^q(y)\,dy\right]^{\frac{1}{q}}, \tag{41}$$

where $\frac{\Gamma(\sigma/\gamma)}{\gamma\rho^{\sigma/\gamma}}$ is the best possible constant factor.

5 Conclusions

In this paper, by using real analysis and weight functions we obtain a few equivalent statements of a Hilbert-type integral inequality in the whole plane related to the kernel of exponent function with the intermediate variables (Theorem 1). The constant factor related to the gamma function is proved to be the best possible in Theorem 2. We also consider some particular cases and the operator expressions in Remark 1 and Theorem 3. The lemmas and theorems provide an extensive account of this type of inequalities.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

BY carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. YZ and MH participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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