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Monotonicity of the number of positive entries in nonnegative matrix powers

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Abstract

Let A be a nonnegative matrix of order n and $f(A)$ denote the number of positive entries in A. We prove that if $f(A) \leq 3$ or $f(A) \geq n^2 - 2n + 2$, then the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is monotonic for positive integers k.

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1 Introduction

A matrix is *nonnegative (positive)* if all of its entries are nonnegative (positive) real numbers. Nonnegative matrices have many attractive properties and are important in a variety of applications [1, 2]. For two nonnegative matrices A and B of the same size, the notation $A \geq B$ or $B \leq A$ means that $A - B$ is nonnegative.

A *sign pattern* is a matrix whose entries are from the set $\{+, -, 0\}$. In a talk at the 12th ILAS conference (Regina, Canada, June 26–29, 2005), Professor Xingzhi Zhan posed the following problem.

Problem ([4], p. 233) *Characterize those sign patterns of square nonnegative matrices A such that the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is nondecreasing.*

A nonnegative square matrix A is said to be *primitive* if there exists a positive integer k such that A^k is positive. If we denote by $f(A)$ the number of positive entries in A, it seems that the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is increasing for any primitive matrix A. However, Šidák [3] observed that there is a primitive matrix A of order 9 satisfying $f(A) = 18 > f(A^2) = 16$. This is the motivation for us to investigate the nonnegative matrix A such that $\{f(A^k)\}_{k=1}^{\infty}$ is monotonic. It is reasonable to expect that the sequence will be monotonic when $f(A)$ is too small or too large.

Since the value of each positive entry in A does not affect $f(A^k)$ for all positive integers k, it suffices to consider the 0–1 matrix, i.e., the matrix whose entries are either 0 or 1. Denote by E_{ij} the matrix with its entry in the ith row and jth column being 1 and with all other entries being 0. For simplicity we use 0 to denote the zero matrix whose size will be clear from the context.

2 Main results

Let A be a nonnegative square matrix. We will use the fact that if $A^2 \geq A$ ($A^2 \leq A$), then $A^{k+1} \geq A^k$ ($A^{k+1} \leq A^k$) for all positive integers k and thus $\{f(A^k)\}_{k=1}^{\infty}$ is increasing (decreasing).

Theorem 1 *Let A be a* 0–1 *matrix of order n. If $f(A) \leq 2$, then the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is decreasing.*

Proof The case $f(A) = 0$ is trivial.

If $f(A) = 1$, then $A = E_{ij}$, $1 \leq i, j \leq n$. Thus, for $k = 2, 3, \ldots,$

$$A^k = E_{ij}^k = \begin{cases} E_{ii}, & i = j; \\ 0, & i \neq j, \end{cases}$$

which implies that $\{f(A^k)\}_{k=1}^{\infty}$ is decreasing. Next suppose $f(A) = 2$.

Since $\{f(A^k)\}_{k=1}^{\infty}$ is invariant under permutation similarity or transpose of A, it suffices to consider the following cases.

(1) $A = E_{11} + E_{22}$. Then $A^2 = A$.
(2) $A = E_{11} + E_{12}$. Then $A^2 = A$.
(3) $A = E_{11} + E_{23}$. Then $A^2 = E_{11} \leq A$.
(4) $A = E_{12} + E_{13}$. Then $A^2 = 0$.
(5) $A = E_{12} + E_{21}$. Then $A^k = E_{11} + E_{22}$ for all even k, $A^k = A$ for all odd k.
(6) $A = E_{12} + E_{23}$. Then $A^2 = E_{13}$, $A^3 = 0$.
(7) $A = E_{12} + E_{34}$. Then $A^2 = 0$.

It can be seen that in each case $\{f(A^k)\}_{k=1}^{\infty}$ is decreasing. This completes the proof. □

Theorem 2 *Let A be a* 0–1 *matrix of order n. If $f(A) = 3$, then the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is monotonic.*

Proof Under permutation similarity and transpose, it suffices to consider the following cases.

(1) $A = E_{11} + E_{22} + E_{33}$. Then $A^2 = A$.
(2) $A = E_{11} + E_{22} + E_{12}$. Then $A^2 = A + E_{12} \geq A$.
(3) $A = E_{11} + E_{22} + E_{13}$. Then $A^2 = A$.
(4) $A = E_{11} + E_{22} + E_{34}$. Then $A^2 = E_{11} + E_{22} \leq A$.
(5) $A = E_{11} + E_{12} + E_{13}$. Then $A^2 = A$.
(6) $A = E_{11} + E_{12} + E_{21}$. Then $A^2 = A + E_{11} + E_{22} \geq A$.
(7) $A = E_{11} + E_{12} + E_{31}$. Then $A^2 = A + E_{32} \geq A$.
(8) $A = E_{11} + E_{12} + E_{23}$. Then $A^k = E_{11} + E_{12} + E_{13}$ for all $k \geq 2$.
(9) $A = E_{11} + E_{12} + E_{32}$. Then $A^2 = E_{11} + E_{12} \leq A$.
(10) $A = E_{11} + E_{12} + E_{34}$. Then $A^2 = E_{11} + E_{12} \leq A$.
(11) $A = E_{11} + E_{23} + E_{24}$. Then $A^2 = E_{11} \leq A$.
(12) $A = E_{11} + E_{23} + E_{32}$. Then $A^k = E_{11} + E_{22} + E_{33}$ for all even k, $A^k = A$ for all odd k.
(13) $A = E_{11} + E_{23} + E_{34}$. Then $A^2 = E_{11} + E_{24}$, $A^k = E_{11}$ for all $k \geq 3$.
(14) $A = E_{11} + E_{23} + E_{45}$. Then $A^2 = E_{11} \leq A$.
(15) $A = E_{12} + E_{13} + E_{14}$. Then $A^2 = 0$.

(16) $A = E_{12} + E_{13} + E_{21}$. Then $A^k = E_{11} + E_{22} + E_{23}$ for all even k, $A^k = A$ for all odd k.
(17) $A = E_{12} + E_{13} + E_{41}$. Then $A^2 = E_{42} + E_{43}$, $A^3 = 0$.
(18) $A = E_{12} + E_{13} + E_{23}$. Then $A^2 = E_{13} \leq A$.
(19) $A = E_{12} + E_{13} + E_{24}$. Then $A^2 = E_{14}$, $A^3 = 0$.
(20) $A = E_{12} + E_{13} + E_{42}$. Then $A^2 = 0$.
(21) $A = E_{12} + E_{13} + E_{45}$. Then $A^2 = 0$.
(22) $A = E_{12} + E_{21} + E_{34}$. Then $A^k = E_{11} + E_{22}$ for all even k, $A^k = E_{12} + E_{21}$ for all odd $k \geq 3$.
(23) $A = E_{12} + E_{23} + E_{31}$. Then

$$A^k = \begin{cases} E_{11} + E_{22} + E_{33}, & k \equiv 0 \text{ (mod3)}; \\ A, & k \equiv 1 \text{ (mod3)}; \\ E_{13} + E_{21} + E_{32}, & k \equiv 2 \text{ (mod3)}. \end{cases}$$

(24) $A = E_{12} + E_{23} + E_{34}$. Then $A^2 = E_{13} + E_{24}$, $A^3 = E_{14}$, $A^4 = 0$.
(25) $A = E_{12} + E_{23} + E_{45}$. Then $A^2 = E_{13}$, $A^3 = 0$.
(26) $A = E_{12} + E_{34} + E_{56}$. Then $A^2 = 0$.

Since in each case $\{f(A^k)\}_{k=1}^{\infty}$ is either increasing or decreasing, this completes the proof. □

Corollary 3 *Let A be a* 0–1 *matrix of order* 2. *Then the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is monotonic.*

Remark When A is of order $n \geq 3$ with $f(A) = 4$, the following example shows that $\{f(A^k)\}_{k=1}^{\infty}$ may not be monotonic. Consider

$$A = E_{12} + E_{13} + E_{21} + E_{31}.$$

Direct computation shows that

$$A^2 = 2E_{11} + E_{22} + E_{23} + E_{32} + E_{33}, \qquad A^3 = 2A.$$

Thus $f(A) = 4 < f(A^2) = 5 > f(A^3) = 4$.

On the one hand, Theorems 1 and 2 show that $\{f(A^k)\}_{k=1}^{\infty}$ is monotonic when $f(A) \leq 3$. On the other hand, $\{f(A^k)\}_{k=1}^{\infty}$ is expected to be also monotonic when $f(A)$ is large enough. Next we discuss the number of positive entries that A has to guarantee the sequence increasing.

The *permanent* of a matrix $A = (a_{ij})_{n\times n}$ is defined as

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where S_n is the set of permutations of the integers $1, 2, \ldots, n$. First we have the following important fact.

Lemma 4 *Let A be a* 0–1 *matrix of order n. If* $\operatorname{per} A > 0$, *then the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is increasing.*

Proof Since A is a 0–1 matrix with $\operatorname{per} A > 0$, there exists a permutation matrix P such that $A \geq P$. Now let $A = P + B$, where B is also a 0–1 matrix. Then $A^{k+1} = A \cdot A^k = (P + B)A^k = P \cdot A^k + B \cdot A^k \geq P \cdot A^k$ for all positive integers k. Thus $f(A^{k+1}) \geq f(P \cdot A^k) = f(A^k)$, which implies that $\{f(A^k)\}_{k=1}^{\infty}$ is increasing. □

Theorem 5 *Let A be a 0–1 matrix of order n. If $f(A) \geq n^2 - 2n + 2$, then the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is increasing.*

Proof First if $\operatorname{per} A > 0$, by Lemma 4, $\{f(A^k)\}_{k=1}^{\infty}$ is increasing.

Next suppose $\operatorname{per} A = 0$. Then by the Frobenius–König theorem [4, p. 46], A has an $r \times s$ zero submatrix with $r + s = n + 1$. Since $f(A) \geq n^2 - 2n + 2$, A has at most $2n - 2$ zero entries. Thus $rs \leq 2n - 2$. It can be seen that r and s must be one of the following solutions.

(1) $r = 1, s = n$;

(2) $r = n, s = 1$;

(3) $r = 2, s = n - 1$;

(4) $r = n - 1, s = 2$.

If $r = 1, s = n$ or $r = n, s = 1$, i.e., A has a zero row or a zero column, then A is permutation similar to a matrix of the form

$$\begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix}$$

or its transpose, where B is of order $n - 1$ and C is a column vector. Since A has at most $2n - 2$ zero entries, B has at most $n - 2$ zero entries. Then there exists a permutation matrix Q of order $n - 1$ such that $B \geq Q$. Note that

$$\begin{aligned} \begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix}^{k+1} &= \begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix}^{k} = \begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B^k & B^{k-1}C \\ 0 & 0 \end{bmatrix} \\ &\geq \begin{bmatrix} Q & C \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B^k & B^{k-1}C \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} QB^k & QB^{k-1}C \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} f\left(A^{k+1}\right) &= f\left(\begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix}^{k+1}\right) \geq f\left(\begin{bmatrix} QB^k & QB^{k-1}C \\ 0 & 0 \end{bmatrix}\right) \\ &= f\left(QB^k\right) + f\left(QB^{k-1}C\right) = f\left(B^k\right) + f\left(B^{k-1}C\right) \\ &= f\left(\begin{bmatrix} B^k & B^{k-1}C \\ 0 & 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix}^{k}\right) = f\left(A^k\right) \end{aligned}$$

for all positive integers k, which implies that $\{f(A^k)\}_{k=1}^{\infty}$ is increasing.

If $r = 2$, $s = n - 1$ or $r = n - 1$, $s = 2$, then A is permutation similar to one of the matrices A_1, A_2, A_1^T, A_2^T, where

$$A_1 = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Direct computation shows that $A_1^2 \geq A_1, A_2^2 \geq A_2$. Thus $\{f(A^k)\}_{k=1}^{\infty}$ is increasing. This completes the proof. □

Remark When $f(A) = n^2 - 2n + 1$, the following example shows that $\{f(A^k)\}_{k=1}^{\infty}$ may not be increasing. Consider

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Direct computation shows that $f(A) = n^2 - 2n + 1 > f(A^2) = n^2 - 2n$.

3 Conclusion

This paper considers the number of positive entries $f(A)$ in a nonnegative matrix A and deals with the question of whether the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is monotonic. We prove that if $f(A) \leq 3$ or $f(A) \geq n^2 - 2n + 2$, then the sequence must be monotonic. Some examples show that if $4 \leq f(A) \leq n^2 - 2n + 1$ when $n \geq 3$, then the sequence may not be monotonic.

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