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The obstacle problem for conformal metrics on compact Riemannian manifolds

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Abstract

We prove a priori estimates up to their second order derivatives for solutions to the obstacle problem of curvature equations on Riemannian manifolds (M^n, g) arising from conformal deformation. With the a priori estimates the existence of a $C^{1,1}$ solution to the obstacle problem with Dirichlet boundary value is obtained by approximation.

Keywords: Obstacle problem; A priori estimates; Hessian equations; Viscosity solutions; Riemannian manifolds

1 Introduction

Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 3$ with smooth boundary ∂M, $\bar{M} := M \cup \partial M$. In conformal geometry, it is interesting to find a complete metric $\tilde{g} \in [g]$, the conformal class of g, with which the manifold has prescribed curvature. In general, such conformal deformation can be interpreted by certain partial differential equations. See [8, 13, 22, 25, 26] for more details.

In [8], Guan studied the existence of a complete conformal metric $\tilde{g}$ of negative Ricci curvature on M satisfying

$$f\big(-\lambda\big(\tilde{g}^{-1}\mathrm{Ric}_{\tilde{g}}\big)\big) = \psi \quad \text{in } M, \tag{1.1}$$

where $\mathrm{Ric}_{\tilde{g}}$ is the Ricci tensor of $\tilde{g}$, and $\lambda(\tilde{g}^{-1}\mathrm{Ric}_{\tilde{g}}) = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of $\tilde{g}^{-1}\mathrm{Ric}_{\tilde{g}}$. The transformation formula for the Ricci tensor under conformal deformation $\tilde{g} = e^{2u}g$ is given by

$$\frac{1}{n-2}\mathrm{Ric}_{\tilde{g}} = \frac{1}{n-2}\mathrm{Ric}_g - \nabla^2 u - \left(\frac{\Delta u}{n-2} + |\nabla u|^2\right)g + du \otimes du,$$

where ∇u, $\nabla^2 u$, and Δu denote the gradient, Hessian, and Laplacian of u with respect to the metric g, respectively. When f is homogenous of degree one, it is easy to verify that equation (1.1) is equivalent to the following form:

$$f\left(\lambda\left(g^{-1}\left[\nabla^2 u + \frac{\Delta u}{n-2}g + |\nabla u|^2 g - du \otimes du - \frac{\mathrm{Ric}_g}{n-2}\right]\right)\right) = \frac{\psi(x)}{n-2}e^{2u}. \tag{1.2}$$

In this paper, we study the obstacle problem of equation (1.2). More generally, let

$$T[u] := \nabla^2 u + s\,du \otimes du + \left(\gamma \Delta u - \frac{t}{2}|\nabla u|^2\right)g + \chi,$$

where χ is a smooth $(0,2)$ tensor, $\gamma > 0$ is a constant, and $s, t \in \mathbb{R}$. We consider the following equation:

$$\max\left\{u - h, -\left(f\left(\lambda\left(g^{-1}T[u]\right)\right) - \psi[u]\right)\right\} = 0 \quad \text{in } M \tag{1.3}$$

with the Dirichlet boundary condition

$$u = \varphi \quad \text{on } \partial M, \tag{1.4}$$

where $h \in C^3(\bar{M})$, $\varphi \in C^4(\partial M)$, $h > \varphi$ on ∂M, $\psi[u] = \psi(x,u)$ is a positive function in $C^3(\bar{M} \times \mathbb{R})$.

Equations as (1.1) and (1.3) are the Hessian equations, which were well studied by many authors such as [2, 7, 9–12, 23, 24]. Generally, $f \in C^2(\Gamma) \cap C^0(\bar{\Gamma})$ is a symmetric function of $\lambda \in \mathbb{R}^n$, defined in an open, convex, and symmetric cone $\Gamma \subsetneqq \mathbb{R}^n$, with vertex at the origin, which contains the positive cone: $\Gamma_n^+ := \{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\}$ and satisfies the following fundamental structure conditions:

$$f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{in } \Gamma, 1 \le i \le n, \tag{1.5}$$

$$f \text{ is a concave function,} \tag{1.6}$$

and

$$f > 0 \quad \text{in } \Gamma, \qquad f = 0 \quad \text{on } \partial\Gamma. \tag{1.7}$$

Here, for convenience, we also assume that

$$f \text{ is homogeneous of degree one.} \tag{1.8}$$

We observe that by the concavity and homogeneity of f,

$$\sum f_i(\lambda) = f(\lambda) + \sum f_i(\lambda)(1 - \lambda_i) \ge f(1, \ldots, 1) > 0 \quad \text{in } \Gamma. \tag{1.9}$$

Important classes of f are the elementary symmetric functions and their quotients, i.e.,

$$f(\lambda) = (\sigma_k)^{\frac{1}{k}}(\lambda) := \left(\sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}\right)^{\frac{1}{k}}, \quad 1 \le k \le n,$$

and

$$f(\lambda) = \left(\frac{\sigma_k}{\sigma_l}\right)^{\frac{1}{k-l}}, \quad 0 \le l < k \le n.$$

Let F be defined by $F(r) = f(\lambda(r))$ for $r = \{r_{ij}\} \in \mathcal{S}^{n\times n}$ with $\lambda(r) \in \Gamma$, where $\mathcal{S}^{n\times n}$ is the set of $n \times n$ symmetric matrices. It is shown in [2] that (1.5) implies F is an elliptic operator and (1.6) ensures that F is concave.

A function $u \in C^2(M)$ is called *admissible* at $x \in M$ if $\lambda(g^{-1}T[u])(x) \in \Gamma$, and we call it *admissible* in M when it is admissible at each x in M. In this paper, we prove the existence of an admissible viscosity solution of (1.3) and (1.4) in $C^{1,1}(\bar{M})$ (see [1, 3] for the definition of viscosity solutions).

Many authors have studied various obstacle problems. In [6], Gerhardt considered a hypersurface bounded from below by an obstacle with prescribed mean curvature in $\mathbb{R}^n$. Lee [17] considered the obstacle problem for the Monge–Ampère equation (i.e., $f = (\sigma_n)^{\frac{1}{n}}$) for the case that $T[u] = D^2u$, $\psi \equiv 1$, and $\varphi \equiv 0$, and proved the $C^{1,1}$ regularity of the viscosity solution in a strictly convex domain in $\mathbb{R}^n$. Xiong and Bao [27] extended the work of Lee to a nonconvex domain in $\mathbb{R}^n$ with general ψ and φ under additional assumptions. Bao, Dong, and Jiao treated a class of obstacle problems in [1] assuming that $T[u] = \nabla^2 u + A(x, u, \nabla u)$, under a certain technical assumption. Because of the term $\gamma\Delta u$ ($\gamma > 0$), here we only need a minimal amount of assumptions. For other works, see [4, 14, 15, 18–21].

Our main result is the following theorem.

Theorem 1.1 *Assume that* (1.5)–(1.8) *and either the following condition*

$$\lim_{z\to+\infty} \psi(x,z) = +\infty, \quad \forall x \in \bar{M}, \tag{1.10}$$

or

$$\frac{2s - nt}{1 + n\gamma} < 2\lambda_1 \tag{1.11}$$

hold, where λ_1 *is the first eigenvalue of the problem*

$$\begin{cases} \Delta u + \lambda(\operatorname{tr}\chi)^+ u = 0 & \text{on } \bar{M}, \\ u = 0 & \text{on } \partial M \end{cases} \tag{1.12}$$

($\lambda_1 = +\infty$ *if* $\operatorname{tr}\chi \le 0$). *Then there exists a viscosity solution* $u \in C^{1,1}(\bar{M})$ *to* (1.3) *and* (1.4), *if there exists a subsolution* $\underline{u} \in C^0(\bar{M}) \cap C^1(\bar{M}_\delta)$ *for some* $\delta > 0$ *such that*

$$\begin{cases} f(\lambda(g^{-1}T[\underline{u}])) \ge \psi[\underline{u}], & \text{in } M, \\ \underline{u} = \varphi, & \text{on } \partial M, \\ \underline{u} \le h, & \text{in } M, \end{cases} \tag{1.13}$$

where $M_\delta = \{x \in M : \operatorname{dist}(x, \partial M) \le \delta\}$. *Moreover, we have that* $u \in C^{3,\alpha}(E)$ *for any* $\alpha \in (0,1)$, *and* $f(\lambda(g^{-1}T[u])) = \psi[u]$ *in* E, *where* $E := \{x \in M : u(x) < h(x)\}$.

Remark 1.2 (1.10), as well as (1.11), is used in Lemma 3.2 to derive an upper bound for u. Assumption (1.13) is just applied to derive a lower bound for u on M and $\nabla_\nu u$ on ∂M, where ν is the interior unit normal to ∂M.

Remark 1.3 We can construct some subsolutions of (1.2) satisfying (1.13) as in [15] following ideas from [2] and [7] since

$$|\nabla u|^2 g - du \otimes du$$

is positive definite and that we can obtain a priori upper bound of any admissible function (Lemma 3.2) under additional conditions that there exists a sufficiently large number $R > 0$ such that at each point $x \in \partial M$,

$$(\kappa_1, \dots, \kappa_{n-1}, R) \in \Gamma, \tag{1.14}$$

where $\kappa_1, \dots, \kappa_{n-1}$ are the principal curvatures of ∂M with respect to the interior normal, and that for every $C > 0$ and every compact set K in Γ there is a number $R = R(C, K)$ such that

$$f(R\lambda) \geq C \quad \text{for all } \lambda \in K. \tag{1.15}$$

We use a penalization technique to prove the existence of viscosity solutions to (1.3) and (1.4). We shall consider the following singular perturbation problem:

$$\begin{cases} f(\lambda(g^{-1}T[u])) = \psi[u] + \beta_\varepsilon(u-h) & \text{in } M, \\ u = \varphi & \text{on } \partial M, \end{cases} \tag{1.16}$$

where the penalty function $\beta_\varepsilon \in C^2(\mathbb{R})$ satisfies

$$\begin{aligned} &\beta_\varepsilon, \beta_\varepsilon', \beta_\varepsilon'' \geq 0 \quad \text{on } \mathbb{R}, \beta_\varepsilon(z) = 0, \text{ whenever } z \leq 0; \\ &\beta_\varepsilon(z) \to \infty \quad \text{as } \varepsilon \to 0^+, \text{ whenever } z > 0. \end{aligned} \tag{1.17}$$

An example given in [27] is

$$\beta_\varepsilon(z) = \begin{cases} 0, & z \leq 0, \\ z^3/\varepsilon, & z > 0, \end{cases} \tag{1.18}$$

for $\varepsilon \in (0,1)$. Observe that $\underline{u}$ is also a subsolution to (1.16).

Let

$$\mathcal{U} = \left\{ u_\varepsilon | u_\varepsilon \in C^4(\bar{M}) \text{ is an admissible solution of (1.16) with } u_\varepsilon \geq \underline{u} \text{ on } \bar{M} \right\}.$$

We aim to derive the uniform bound

$$|u_\varepsilon|_{C^2(\bar{M})} \leq C \tag{1.19}$$

for $u_\varepsilon \in \mathcal{U}$, where C is independent of ε. After establishing (1.19), the equation (1.16) becomes uniformly elliptic by (1.7). By Evans–Krylov [5], [16] theorem, we can derive the $C^{2,\alpha}$ estimates (which may depend on ε) of u_ε. Higher estimates can be derived by

Schauder theory. Following the proof as in [8] or [1], we can prove there exists an admissible solution u_ε to (1.16). Then we can conclude by (1.19) that there exists a viscosity solution $u \in C^{1,1}(\bar{M})$ to (1.3) and (1.4), see [1, 27].

Thus, our main work is focused on the a priori estimates for admissible solutions up to their second order derivatives. In Sect. 2, we achieve the estimates for second order derivatives. Finally, we end this paper with gradient and C^0 estimates in Sect. 3.

2 Estimates for second order derivatives

In this section, we prove a priori estimates of second order derivatives for admissible solutions. From now on, we drop the subscript ε when there is no possible confusion.

Theorem 2.1 *Assume that f satisfies (1.5)–(1.8) and $u \in C^4(\bar{M})$ is an admissible solution to (1.16). Then*

$$\sup_M \left|\nabla^2 u\right| \le C\Big(1 + \sup_{\partial M}\left|\nabla^2 u\right|\Big), \tag{2.1}$$

where C depends on $|u|_{C^1(\bar{M})}$ and other known data.

Proof Set

$$W(x) = \max_{\xi\in T_xM, |\xi|=1} \left(\nabla_{\xi\xi} u + s|\nabla_\xi u|^2\right)e^{\phi}, \quad x \in \bar{M},$$

where ϕ is a function to be determined. Assume that W is achieved at an interior point $x_0 \in M$ and a unit direction $\xi \in T_{x_0}M$. Choose a smooth orthonormal local frame $e_1, \dots, e_n$ about x_0 such that $\xi = e_1$, $\nabla_i e_j(x_0) = 0$ and that $T_{ij}(x_0)$ is diagonal. We write $G = \nabla_{11}u + s|\nabla_1 u|^2$. Assume $G(x_0) > 0$ (otherwise we are done).

At the point x_0, where the function $\log G + \phi$ (defined near x_0) attains its maximum, we have

$$\frac{\nabla_i G}{G} + \nabla_i \phi = 0, \quad i = 1, \dots, n, \tag{2.2}$$

and

$$\frac{\nabla_{ii} G}{G} - \left(\frac{\nabla_i G}{G}\right)^2 + \nabla_{ii}\phi \le 0. \tag{2.3}$$

By (2.3) we have

$$F^{ii}\left(\nabla_{ii}G + G\nabla_{ii}\phi - G|\nabla_i\phi|^2\right) \le 0 \tag{2.4}$$

and

$$\Delta G + G\Delta\phi - G|\nabla\phi|^2 \le 0. \tag{2.5}$$

Since $\gamma > 0$, we obtain

$$F^{ii}\left(\nabla_{ii}G + \gamma\Delta G + G\nabla_{ii}\phi + \gamma G\Delta\phi - G|\nabla_i\phi|^2 - \gamma G|\nabla\phi|^2\right) \le 0. \tag{2.6}$$

By calculation, we get

$$\nabla_i G = \nabla_{i11} u + 2s \nabla_1 u \nabla_{i1} u, \tag{2.7}$$

and

$$\nabla_{ii} G = \nabla_{ii11} u + 2s\big(|\nabla_{i1} u|^2 + \nabla_1 u \nabla_{ii1} u\big). \tag{2.8}$$

Recall the formula for interchanging order of covariant derivatives

$$\nabla_{ijk} v - \nabla_{kij} v = R^l_{kij} \nabla_l v, \tag{2.9}$$

and

$$\begin{aligned} \nabla_{ijkl} v - \nabla_{klij} v = {} & R^m_{ljk} \nabla_{im} v + \nabla_i R^m_{ljk} \nabla_m v + R^m_{lik} \nabla_{jm} v \\ & + R^m_{jik} \nabla_{lm} v + R^m_{jil} \nabla_{km} v + \nabla_k R^m_{jil} \nabla_m v. \end{aligned} \tag{2.10}$$

It follows from (2.10)

$$\nabla_{ii} G \geq \nabla_{11ii} u + 2s\big(|\nabla_{i1} u|^2 + \nabla_1 u \nabla_{1ii} u\big) - C(1+G), \tag{2.11}$$

and

$$\begin{aligned} \nabla_{ii} G + \gamma \Delta G \geq {} & \nabla_{11ii} u + 2s\big(|\nabla_{i1} u|^2 + \nabla_1 u \nabla_{1ii} u\big) + \gamma \nabla_{11}(\Delta u) \\ & + 2s\gamma\big(|\nabla_{i1} u|^2 + \nabla_1 u \nabla_1(\Delta u)\big) - C(1+G). \end{aligned} \tag{2.12}$$

Differentiating equation (1.16) once at x_0, we obtain for $1 \leq k \leq n$,

$$\nabla_k F = F^{ii} \nabla_k T_{ii} = \psi_{x_k} + \psi_z \nabla_k u + \nabla_k \beta_\varepsilon(u-h). \tag{2.13}$$

It is easy to see that

$$\begin{aligned} F^{ii} \nabla_1(\nabla_{ii} u + \gamma \Delta u) &= F^{ii} \nabla_1 \left(T_{ii}[u] - s|\nabla_i u|^2 + \frac{t}{2}|\nabla u|^2 - \chi_{ii} \right) \\ &\geq \nabla_1 F - 2sF^{ii} \nabla_i u \nabla_{1i} u + t \nabla_k u \nabla_{1k} u \sum_i F^{ii} - \sum_i F^{ii} \end{aligned} \tag{2.14}$$

and that

$$\begin{aligned} F^{ii} \nabla_{11}(\nabla_{ii} u + \gamma \Delta u) &= F^{ii} \nabla_{11} \left(T_{ii}[u] - s|\nabla_i u|^2 + \frac{t}{2}|\nabla u|^2 - \chi_{ii} \right) \\ &\geq F^{ii} \nabla_{11} T_{ii}[u] - 2sF^{ii}\big(\nabla_i u \nabla_{11i} u + |\nabla_{1i} u|^2\big) \\ &\quad + t \sum_k \big(\nabla_k u \nabla_{11k} u + |\nabla_{1k} u|^2\big) \sum_i F^{ii} - C \sum_i F^{ii}. \end{aligned} \tag{2.15}$$

With (2.9) we see

$$
\begin{aligned}
2s\nabla_i u\nabla_{11i}u &\le 2s\nabla_i u(\nabla_i G - 2s\nabla_1 u\nabla_{i1}u) + C \\
&\le -4s^2\nabla_i u\nabla_1 u\nabla_{i1}u + C\bigl(1 + G|\nabla\phi|\bigr),
\end{aligned} \tag{2.16}
$$

and similarly

$$
t\nabla_k u\nabla_{11k}u \ge -2st\nabla_k u\nabla_1 u\nabla_{k1}u - C\bigl(1 + G|\nabla\phi|\bigr). \tag{2.17}
$$

With (2.12), (2.14)–(2.17), and the concavity of F, we derive

$$
\begin{aligned}
F^{ii}(\nabla_{ii}G + \gamma\Delta G) &\ge \nabla_{11}F + 2s\nabla_1 u\nabla_1 F + (2s\gamma + t)\sum_k |\nabla_{1k}u|^2 \sum F^{ii} \\
&\quad - C\biggl(G + G|\nabla\phi| + \sum_{j,k}|\nabla_{jk}u|\biggr) \\
&\ge \nabla_{11}F + 2s\nabla_1 u\nabla_1 F - C\bigl(G^2 + G|\nabla\phi|\bigr).
\end{aligned} \tag{2.18}
$$

By (1.9) and $\beta_\varepsilon'' > 0$ it follows from (2.6) and (2.18) that

$$
\begin{aligned}
&F^{ii}\bigl(\nabla_{ii}\phi - |\nabla_i\phi|^2\bigr) + \gamma\bigl(\Delta\phi - |\nabla\phi|^2\bigr)\sum F^{ii} \\
&\quad \le C\bigl(G + |\nabla\phi|\bigr)\sum F^{ii} + \biggl(\frac{C}{G} - 1\biggr)\beta_\varepsilon'(u - h).
\end{aligned} \tag{2.19}
$$

Let

$$
\phi := \eta(w) = \biggl(1 - \frac{w}{2a}\biggr)^{-1/2}, \quad w = \frac{|\nabla u|^2}{2},
$$

where $a > \sup_M w$ is a constant to be determined. We have

$$
1 \le \eta < \sqrt{2}, \quad \eta' = \frac{\eta^3}{4a}, \quad \eta'' = \frac{3\eta'^2}{\eta}
$$

and

$$
\nabla_{ii}\phi - |\nabla_i\phi|^2 = \eta'\nabla_{ii}w + \bigl(\eta'' - \eta'^2\bigr)|\nabla_i w|^2 \ge \eta'\nabla_{ii}w. \tag{2.20}
$$

Next, by (2.14)

$$
\begin{aligned}
&F^{ii}(\nabla_{ii}w + \gamma\Delta w) \\
&\quad = F^{ii}\biggl(\sum_l |\nabla_{il}u|^2 + \gamma\sum_{k,l}|\nabla_{kl}u|^2\biggr) + F^{ii}\nabla_l u\bigl(\nabla_{iil}u + \gamma\Delta(\nabla_l u)\bigr) \\
&\quad \ge F^{ii}\nabla_l u\biggl(\nabla_{lii}u + \gamma\sum_k \nabla_{lkk}u\biggr) + \bigl(\gamma G^2 - CG\bigr)\sum F^{ii} \\
&\quad \ge -C\beta_\varepsilon'(u - h) + \bigl(\gamma G^2 - CG\bigr)\sum F^{ii}.
\end{aligned} \tag{2.21}
$$

Combining (2.19), (2.20), (2.21), and $|\nabla\phi| \le C\eta' G$, we have

$$\eta'\left(\gamma G^2 - CG\right)\sum F^{ii} \le C\left(G + \eta' G\right)\sum F^{ii} + \left(\frac{C}{G} - 1 + C\eta'\right)\beta_\varepsilon'(u-h). \tag{2.22}$$

We could assume that $G \ge 2C$. When $a > 2C$, the coefficient of $\beta_\varepsilon'(u-h)$ is negative. Then we can derive $G \le \frac{4aC}{\gamma}$. □

To derive the boundary estimates for $\nabla^2 u$, we note that $\operatorname{tr}(sdu \otimes du - \frac{t}{2}|\nabla u|^2 g + \chi) \le C$ on $\bar{M}$, where C is independent of ε, though it may depend on $|u|_{C^1(\bar{M})}$. As in [1, 4], let H be the solution to

$$\begin{cases} (1+n\gamma)\Delta H + C = 0 & \text{in } M, \\ H = \varphi & \text{on } \partial M. \end{cases}$$

Then we have $u \le H$ in M by the maximum principle and $\beta_\varepsilon(u-h) \equiv 0$ in $M_\delta = \{x \in M : \operatorname{dist}(x, \partial M) \le \delta\}$, where δ is sufficiently small. Thus,

$$\begin{cases} f(\lambda(g^{-1}T[u])) = \psi[u] & \text{in } M_\delta, \\ u = \varphi & \text{on } \partial M. \end{cases} \tag{2.23}$$

By the same arguments of Sect. 4 in [8], we obtain that

$$\sup_{\partial M}\left|\nabla^2 u\right| \le C, \tag{2.24}$$

where C depends on $|u|_{C^1(\bar{M})}$ and other known data.

Combining (2.1) and (2.24), we therefore get the full estimates for second order derivatives.

3 Gradient estimates, maximum principle, and existence

For the gradient estimates, we have the following theorem.

Theorem 3.1 *Assume that* (1.5)–(1.8) *hold. Let* $u \in C^3(\bar{M})$ *be an admissible solution to* (1.16). *Then*

$$\sup_M |\nabla u| \le C\Big(1 + \sup_{\partial M}|\nabla u|\Big), \tag{3.1}$$

where C *depends on* $|u|_{C^0(\bar{M})}$ *and other known data.*

Proof Suppose that we^ϕ, where $w = \frac{|\nabla u|^2}{2}$ and $\phi = \phi(u)$ to be determined satisfying that $\phi'(u) > 0$, achieves a maximum at an interior point $x_0 \in M$. As before, we choose a smooth orthonormal local frame $e_1, \dots, e_n$ about x_0 such that $\nabla_{e_i} e_j = 0$ at x_0 and $\{T_{ij}(x_0)\}$ is diagonal. Differentiating we^ϕ at x_0 twice, we have

$$\nabla_i w + w\nabla_i\phi = 0 \tag{3.2}$$

and

$$\nabla_{ii} w - w(\nabla_i \phi)^2 + w\nabla_{ii}\phi \le 0. \tag{3.3}$$

Differentiating w, we see

$$\nabla_i w = \sum_k \nabla_k u \nabla_{ik} u, \qquad \nabla_{ii} w = \sum_k (\nabla_{ik} u)^2 + \sum_k \nabla_k u \nabla_{iik} u.$$

Using (3.2) it follows from (3.3) that

$$F^{ii}\left(\delta_{kl} - \frac{\nabla_k u \nabla_l u}{2w}\right)\nabla_{ik} u \nabla_{il} u + F^{ii}\nabla_k u \nabla_{iik} u - wF^{ii}\left(\frac{(\nabla_i \phi)^2}{2} - \nabla_{ii}\phi\right) \le 0 \tag{3.4}$$

and

$$\sum_{i,k,l}\left(\delta_{kl} - \frac{\nabla_k u \nabla_l u}{2w}\right)\nabla_{ik} u \nabla_{il} u + \nabla_k u \Delta(\nabla_k u) - \frac{w}{2}|\nabla \phi|^2 + w\Delta\phi \le 0. \tag{3.5}$$

Note that the first term in (3.4) and (3.5) is nonnegative. Multiply $\gamma \sum F^{ii}$ to (3.5) and add what we got to (3.4). Thus, by (2.9) we obtain

$$\begin{aligned} &F^{ii}\nabla_k u(\nabla_{kii} u + \gamma \nabla_k \Delta u) - \frac{w}{2}F^{ii}\left(|\nabla_i \phi|^2 + \gamma|\nabla\phi|^2\right) \\ &\quad + wF^{ii}(\nabla_{ii}\phi + \gamma\Delta\phi) \le C|\nabla u|^2 \sum F^{ii}. \end{aligned} \tag{3.6}$$

Now we compute the first term in (3.6). Firstly, we have

$$\nabla_i \phi = \phi' \nabla_i u, \qquad \nabla_{ii}\phi = \phi'\nabla_{ii} u + \phi''(\nabla_i u)^2.$$

Using (3.2), we easily get that

$$\begin{aligned} &F^{ii}\nabla_k u(\nabla_{kii} u + \gamma\nabla_k \Delta u) \\ &\quad = F^{ii}\nabla_k u \nabla_k\left(T_{ii} - s|\nabla_i u|^2 + \frac{t}{2}|\nabla u|^2 - \chi_{ii}\right) \\ &\quad = \nabla_k u \nabla_k(\psi + \beta_\varepsilon) + w\phi' F^{ii}\left(2s|\nabla_i u|^2 - t|\nabla u|^2\right) - F^{ii}\nabla_k u \nabla_k \chi_{ii}. \end{aligned} \tag{3.7}$$

By the homogeneity of F, we also get

$$\begin{aligned} &F^{ii}(\nabla_{ii}\phi + \gamma\Delta\phi) \\ &\quad = \phi'' F^{ii}\left(|\nabla_i u|^2 + \gamma|\nabla u|^2\right) + \phi' F^{ii}\left(T_{ii} - s|\nabla_i u|^2 + \frac{t}{2}|\nabla u|^2 - \chi_{ii}\right) \\ &\quad = \phi'' F^{ii}\left(|\nabla_i u|^2 + \gamma|\nabla u|^2\right) + \phi'\left(F - sF^{ii}|\nabla_i u|^2 + \frac{t}{2}F^{ii}|\nabla u|^2 - F^{ii}\chi_{ii}\right). \end{aligned} \tag{3.8}$$

According to (3.7) and (3.8), it follows from (3.6)

$$\begin{aligned}
&\gamma|\nabla u|^2\left(\phi'' - \frac{1}{2}(\phi')^2 - \frac{t}{2\gamma}\phi'\right)\sum F^{ii} + \left(\phi'' - \frac{1}{2}(\phi')^2 + s\phi'\right)F^{ii}(\nabla_i u)^2 \\
&\quad \le -\phi'(\psi + \beta_\varepsilon - F^{ii}\chi_{ii}) + C\sum F^{ii} - \frac{\nabla_k u \nabla_k(\psi + \beta_\varepsilon)}{w} \\
&\quad \le -\left(\phi'\beta_\varepsilon + \frac{\nabla_k u \nabla_k(u-h)\beta_\varepsilon'}{w}\right) + C\sum F^{ii} + C.
\end{aligned} \tag{3.9}$$

Let

$$\phi(u) = v^{-a}, \qquad v = 1 - u + \sup_M u.$$

We have

$$\phi'(u) = av^{-a-1}, \qquad \phi''(u) = \frac{(a+1)\phi'}{v},$$

and

$$\phi'' - \frac{1}{2}(\phi')^2 = \phi'\left(\frac{a+1}{v} - \frac{av^{-a}}{2v}\right) \ge \frac{\phi' a}{2v} > 0$$

since $v^{-a} \le 1$. When $|\nabla u(x_0)|$ is sufficiently large, we see $\nabla_k u \nabla_k(u-h) > 0$. Hence we have that the first term on the right-hand side of (3.9) is negative as $\beta_\varepsilon, \beta_\varepsilon' > 0$. From (3.9) and (1.9) when a is sufficiently large, we then obtain that

$$\frac{\phi' a\gamma|\nabla u|^2}{4v} \le C, \tag{3.10}$$

from which we conclude that (3.1) holds. □

In order to prove (1.19), it remains to bound $\sup_M |u| + \sup_{\partial M} |\nabla u|$. We quote two lemmas in [8], the ingredients of whose proofs are the maximum principle.

Lemma 3.2 *If either* (1.10) *or* (1.11) *holds, then any admissible solution* u *of* (1.16) *admits the a priori bound*

$$\sup_M u \le c_0. \tag{3.11}$$

Lemma 3.3 *If* u *is admissible such that* $\operatorname{tr} T[u] \ge 0$ *and* $|u|_{C^0(M)} \le \mu$, *then*

$$\sup_{\partial M} \nabla_\nu u \le c_1(\mu), \tag{3.12}$$

where ν *is the interior unit normal to* ∂M.

Now with the above two lemmas and the fact $\nabla_\nu u \ge \nabla_\nu \underline{u}$ on ∂M when $u \in \mathcal{U}$, we then have the following.

Theorem 3.4 *Suppose that* (1.5)–(1.8), *and either* (1.10) *or* (1.11) *hold. Then, for* $u \in \mathcal{U}$, (1.19) *holds*.

Therefore, the uniform estimates (1.19) ensure that there exist a subsequence $\{u_{\varepsilon_k}\}$ of $\{u_\varepsilon\}$ and a function $u \in C^{1,1}(\bar{M})$ such that $u_{\varepsilon_k} \to u$ in M as $\varepsilon_k \to 0$. It is easy to verify that u satisfies (1.3) and (1.4) and $u \in C^{3,\alpha}(E)$ for any $\alpha \in (0,1)$. Consequently, Theorem 1.1 is established.

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Competing interests
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