Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Approximation of the generalized Cauchy–Jensen functional equation in C^*-algebras

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Abstract

In this paper, we prove Hyers–Ulam–Rassias stability of C^*-algebra homomorphisms for the following generalized Cauchy–Jensen equation:

$$\alpha\mu f\left(\frac{x+y}{\alpha}+z\right)=f(\mu x)+f(\mu y)+\alpha f(\mu z),$$

for all $\mu \in \mathbb{S} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and for any fixed positive integer $\alpha \geq 2$, which was introduced by Gao et al. [*J. Math. Inequal.* 3:63–77, 2009], on C^*-algebras by using fixed poind alternative theorem. Moreover, we introduce and investigate Hyers–Ulam–Rassias stability of generalized θ-derivation for such functional equations on C^*-algebras by the same method.

MSC: 39B52; 47H10

Keywords: Cauchy–Jensen functional equations; Hyers–Ulam–Rassias stability; C^*-algebras; Fixed point theorem

1 Introduction and preliminaries

Throughout this paper, let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, the set of real numbers, the set of complex numbers, respectively. The stability problem of functional equations was initiated by Ulam in 1940 [2] arising from concern over the stability of group homomorphisms. This form of asking the question is the object of stability theory. In 1941, Hyers [3] provided a first affirmative partial answer to Ulam's problem for the case of approximately additive mapping in Banach spaces. In 1978, Rassias [4] gave a generalization of Hyers' theorem for linear mapping by considering an unbounded Cauchy difference. A generalization of Rassias' result was developed by Găvruța [5] in 1994 by replacing the unbounded Cauchy difference by a general control function.

In 2006, Baak [6] investigated the Cauchy–Rassis stability of the following Cauchy–Jensen functional equations:

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)=f(x)+2f(z),$$

$$f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)=f(y),$$

or

$$2f\left(\frac{x+y}{2}+z\right)=f(x)+f(y)+2f(z)$$

for all $x, y, z \in X$, in Banach spaces.

The fixed point method was applied to study the stability of functional equations by Baker in 1991 [7] by using the Banach contraction principle. Next, Radu [8] proved a stability of functional equation by the alternative of fixed point which was introduced by Diaz and Margolis [9]. The fixed point method has provided a lot of influence in the development of stability.

In 2008, Park and An [10] proved the Hyers–Ulam–Rassias stability of C^*-algebra homomorphisms and generalized derivations on C^*-algebras by using alternative of fixed point theorem for the Cauchy–Jensen functional equation $2f(\frac{x+y}{2}+z)=f(x)+f(y)+2f(z)$, which was introduced and investigated by Baak [6]

The definition of the generalized Cauchy–Jensen equation was given by Gao et al.[1] in 2009 as follows.

Definition 1.1 ([1]) Let G be an n-divisible abelian group where $n \in \mathbb{N}$ (i.e. $a \mapsto na \mid G \to G$ is a surjection) and X be a normed space with norm $\|\cdot\|_X$. For a mapping $f: G \to X$, the equation

$$nf\left(\frac{x+y}{n}+z\right)=f(x)+f(y)+nf(z)$$

for all $x, y, z \in G$ and for any fixed positive integer $n \geq 2$ is said to be a generalized Cauchy–Jensen equation (GCJE, shortly).

In particular, when $n = 2$, it is called a Cauchy–Jensen equation. Moreover, they gave the following useful properties.

Corollary 1.2 ([1]) *For a mapping $f: G \to X$, the following statements are equivalent.*

(i) *f is additive.*
(ii) $nf(\frac{x+y}{n}+z)=f(x)+f(y)+nf(z)$, *for all $x, y, z \in G$.*
(iii) $\|nf(\frac{x+y}{n}+z)\|_X \geq \|f(x)+f(y)+nf(z)\|_X$, *for all $x, y, z \in G$.*

It is obvious that a vector space is n-divisible abelian group, so Corollary 1.2 works for a vector space G.

All over this paper, $\mathbb{A}$ and $\mathbb{B}$ are C^*-algebras with norm $\|\cdot\|_{\mathbb{A}}$ and $\|\cdot\|_{\mathbb{B}}$, respectively. We recall a fundamental result in fixed point theory. The following is the definition of a generalized metric space which was introduced by Luxemburg in 1958 [11].

Definition 1.3 ([11]) Let X be a set. A function $d: X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:

(i) $d(x, y) = 0$ if and only if $x = y$,
(i) $d(x, y) = d(y, x)$, for all $x, y \in X$,
(iii) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

The following fixed point theorem will play important roles in proving our main results.

Theorem 1.4 ([9]) *Let (X,d) be a complete generalized metric space and $T:X\to X$ be a strictly contractive mapping, that is,*

$$d(Tx,Ty)\le kd(x,y)$$

for all $x,y\in X$ and for some Lipschitz $k<1$. Then, for each given element $x\in X$, either

$$d\bigl(T^nx,T^{n+1}x\bigr)=\infty$$

for all nonnegative integer n or there exists a positive integer n_0 such that
(i) *$d(T^nx,T^{n+1}x)<\infty$ for all $n\ge n_0$,*
(ii) *the sequence $\{T^nx\}$ converges to a fixed point y^* of T,*
(iii) *y^* is the unique fixed point of T in the set $Y=\{y\in X\mid d(T^{n_0}x,y)<\infty\}$,*
(iv) *$d(y,y^*)\le\frac{1}{1-k}d(y,Ty)$, for all $y\in Y$.*

The following lemma is useful for proving our main results.

Lemma 1.5 ([12]) *Let $f:\mathbb{A}\to\mathbb{B}$ be an additive mapping such that $f(\mu x)=\mu f(x)$ for all $x\in\mathbb{A}$ and all $\mu\in\mathbb{S}:=\{\lambda\in\mathbb{C}\mid|\lambda|=1\}$. Then the mapping f is $\mathbb{C}$-linear.*

2 Stability of C^*-algebra homomorphisms

Let f be a mapping of $\mathbb{A}$ into $\mathbb{B}$. We define

$$E_\mu f(x,y,z):=\alpha\mu f\left(\frac{x+y}{\alpha}+z\right)-f(\mu x)-f(\mu y)-\alpha f(\mu z),\tag{2.1}$$

for all $\mu\in\mathbb{S}$, for all $x,y,z\in\mathbb{A}$ and for any fixed positive integer $\alpha\ge 2$.

We prove the Hyers–Ulam–Rassias stability of C^*-algebra homomorphisms for the functional equation $E_\mu f(x,y,z)=0$.

Theorem 2.1 *Let $\phi:\mathbb{A}^3\to[0,\infty)$ be a function such that there exists a $k<1$ satisfying*

$$\phi(x,y,z)\le\frac{2+\alpha}{\alpha}k\phi\left(\frac{\alpha}{2+\alpha}x,\frac{\alpha}{2+\alpha}y,\frac{\alpha}{2+\alpha}z\right)\tag{2.2}$$

for all $x,y,z\in\mathbb{A}$. Let f be a mapping of $\mathbb{A}$ into $\mathbb{B}$ satisfying

$$\bigl\|E_\mu f(x,y,z)\bigr\|_{\mathbb{B}}\le\phi(x,y,z),\tag{2.3}$$

$$\bigl\|f(xy)-f(x)f(y)\bigr\|_{\mathbb{B}}\le\phi(x,y,0),\tag{2.4}$$

$$\bigl\|f\bigl(x^*\bigr)-f(x)^*\bigr\|_{\mathbb{B}}\le\phi(x,x,x),\tag{2.5}$$

for all $\mu\in\mathbb{S}$ and for all $x,y,z\in\mathbb{A}$. Then there exists a unique C^*-algebra homomorphism $F:\mathbb{A}\to\mathbb{B}$ such that

$$\bigl\|f(x)-F(x)\bigr\|_{\mathbb{B}}\le\frac{1}{(1-k)(2+\alpha)}\phi(x,x,x)\tag{2.6}$$

for all $x\in\mathbb{A}$.

Proof Consider the set

$$X := \{g \mid \mathbb{A} \to \mathbb{B}\}$$

and introduce the generalized metric on X as follows:

$$d(g,h) = \inf\bigl\{M \in (0,\infty) \mid \|g(x) - h(x)\|_{\mathbb{B}} \le M\phi(x,x,x), \forall x \in \mathbb{A}\bigr\}. \tag{2.7}$$

It is easy to show that (X,d) is complete.

Now, we consider the linear mapping $T : X \to X$ such that

$$Tg(x) := \frac{\alpha}{2+\alpha} g\left(\frac{2+\alpha}{\alpha}x\right)$$

for all $x \in \mathbb{A}$. Next, we will show that T is a strictly contractive self-mapping of X with the Lipschitz constant k. For any $g,h \in X$, let $d(g,h) = K$ for some $K \in \mathbb{R}_+$. Then we have

$$\begin{aligned}
&\|g(x) - h(x)\|_{\mathbb{B}} \le K\phi(x,x,x) \quad \forall x \in \mathbb{A}, \\
&\quad\Rightarrow \quad \left\|g\left(\frac{2+\alpha}{\alpha}x\right) - h\left(\frac{2+\alpha}{\alpha}x\right)\right\|_{\mathbb{B}} \le K\phi\left(\frac{2+\alpha}{\alpha}x, \frac{2+\alpha}{\alpha}x, \frac{2+\alpha}{\alpha}x\right) \quad \forall x \in \mathbb{A}, \\
&\quad\Rightarrow \quad \left\|\frac{\alpha}{2+\alpha}g\left(\frac{2+\alpha}{\alpha}x\right) - \frac{\alpha}{2+\alpha}h\left(\frac{2+\alpha}{\alpha}x\right)\right\|_{\mathbb{B}} \\
&\qquad\qquad \le \frac{\alpha}{2+\alpha}K\phi\left(\frac{2+\alpha}{\alpha}x, \frac{2+\alpha}{\alpha}x, \frac{2+\alpha}{\alpha}x\right) \quad \forall x \in \mathbb{A}.
\end{aligned}$$

By (2.2), we obtain

$$\begin{aligned}
&\|Tg(x) - Th(x)\|_{\mathbb{B}} \\
&\quad \le \frac{\alpha}{2+\alpha}K\frac{2+\alpha}{\alpha}k\phi\left(\frac{\alpha}{2+\alpha}\cdot\frac{2+\alpha}{\alpha}x, \frac{\alpha}{2+\alpha}\cdot\frac{2+\alpha}{\alpha}x, \frac{\alpha}{2+\alpha}\cdot\frac{2+\alpha}{\alpha}x\right) \\
&\quad \Rightarrow \quad \|Tg(x) - Th(x)\|_{\mathbb{B}} \le Kk\phi(x,x,x) \quad \forall x \in \mathbb{A}. \\
&\quad \Rightarrow \quad d(Tg,Th) \le Kk.
\end{aligned}$$

Hence, we obtain

$$d(Tg,Th) \le kd(g,h).$$

Letting $\mu = 1$ and $x = y = z$ in (2.1), we get

$$E_\mu f(x,x,x) = \alpha f\left(\frac{x+x}{\alpha} + x\right) - f(x) - f(x) - \alpha f(x) = \alpha f\left(\frac{2+\alpha}{\alpha}x\right) - (2+\alpha)f(x)$$

for all $x \in \mathbb{A}$. By (2.3), we have

$$\|E_\mu f(x,x,x)\|_{\mathbb{B}} = \left\|\alpha f\left(\frac{2+\alpha}{\alpha}x\right) - (2+\alpha)f(x)\right\|_{\mathbb{B}} \le \phi(x,x,x),$$

which implies that

$$\left\| f(x) - \frac{\alpha}{2+\alpha} f\left(\frac{2+\alpha}{\alpha}x\right) \right\|_{\mathbb{B}} \leq \frac{1}{2+\alpha}\phi(x,x,x)$$

for all $x \in \mathbb{A}$, that is,

$$\left\| f(x) - Tf(x) \right\|_{\mathbb{B}} \leq \frac{1}{2+\alpha}\phi(x,x,x)$$

for all $x \in \mathbb{A}$. It follows from (2.7) that we have

$$d(f, Tf) \leq \frac{1}{2+\alpha}.$$

By Theorem 1.4, there exists a mapping $F : \mathbb{A} \to \mathbb{B}$ such that the following conditions hold.

(1) F is a fixed point of T, that is, $TF(x) = F(x)$ for all $x \in \mathbb{A}$. Then we have

$$F(x) = TF(x) = \frac{\alpha}{2+\alpha} F\left(\frac{2+\alpha}{\alpha}x\right) \quad \Rightarrow \quad F\left(\frac{2+\alpha}{\alpha}x\right) = \frac{2+\alpha}{\alpha}F(x)$$

for all $x \in \mathbb{A}$. Moreover, the mapping F is a unique fixed point of T in the set

$$Y = \{g \in X \mid d(f,g) < \infty\}.$$

From (2.7), there exists $C \in (0,\infty)$ satisfying

$$\left\| f(x) - F(x) \right\|_{\mathbb{B}} \leq C\phi(x,x,x),$$

for all $x \in \mathbb{A}$.

(2) The sequence $\{T^n f\}$ converges to F. This implies that we have the equality

$$F(x) = \lim_{n\to\infty} \left(\frac{\alpha}{2+\alpha}\right)^n f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right) \tag{2.8}$$

for all $x \in \mathbb{A}$.

(3) We obtain $d(f,F) \leq \frac{1}{1-k} d(f,Tf)$, which implies that

$$d(f,F) \leq \frac{1}{1-k} d(f,Tf) \leq \frac{1}{(1-k)(2+\alpha)}. \tag{2.9}$$

Therefore, inequality (2.6) holds.

From (2.2), for any $j \in \mathbb{N}$, we have

$$\begin{aligned}
&\left(\frac{\alpha}{2+\alpha}\right)^j \cdot \phi\left(\left(\frac{2+\alpha}{\alpha}\right)^j x, \left(\frac{2+\alpha}{\alpha}\right)^j y, \left(\frac{2+\alpha}{\alpha}\right)^j z\right) \\
&\quad \leq \left(\frac{\alpha}{2+\alpha}\right)^j \cdot \left(\frac{2+\alpha}{\alpha}\right) k\phi\left(\frac{\alpha}{2+\alpha}\left(\frac{2+\alpha}{\alpha}\right)^j x, \frac{\alpha}{2+\alpha}\left(\frac{2+\alpha}{\alpha}\right)^j y, \frac{\alpha}{2+\alpha}\left(\frac{2+\alpha}{\alpha}\right)^j z\right) \\
&\quad = k\left(\frac{\alpha}{2+\alpha}\right)^{j-1} \phi\left(\left(\frac{2+\alpha}{\alpha}\right)^{j-1} x, \left(\frac{2+\alpha}{\alpha}\right)^{j-1} y, \left(\frac{2+\alpha}{\alpha}\right)^{j-1} z\right)
\end{aligned}$$

$$\begin{aligned}
&\leq k\left(\frac{\alpha}{2+\alpha}\right)^{j-1}\left(\frac{2+\alpha}{\alpha}\right)k\phi\left(\frac{\alpha}{2+\alpha}\left(\frac{2+\alpha}{\alpha}\right)^{j-1}x,\right.\\
&\quad\left.\frac{\alpha}{2+\alpha}\left(\frac{2+\alpha}{\alpha}\right)^{j-1}y,\frac{\alpha}{2+\alpha}\left(\frac{2+\alpha}{\alpha}\right)^{j-1}z\right)\\
&= k^2\left(\frac{\alpha}{2+\alpha}\right)^{j-2}\phi\left(\left(\frac{2+\alpha}{\alpha}\right)^{j-2}x,\left(\frac{2+\alpha}{\alpha}\right)^{j-2}y,\left(\frac{2+\alpha}{\alpha}\right)^{j-2}z\right)\\
&\leq\cdots\leq k^j\phi(x,y,z)
\end{aligned}$$

for all $x,y,z\in\mathbb{A}$. Since $0<k<1$, we obtain

$$\lim_{j\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{j}\cdot\phi\left(\left(\frac{2+\alpha}{\alpha}\right)^{j}x,\left(\frac{2+\alpha}{\alpha}\right)^{j}y,\left(\frac{2+\alpha}{\alpha}\right)^{j}z\right)=0 \tag{2.10}$$

for all $x,y,z\in\mathbb{A}$.

It follows from (2.3), (2.8) and (2.10) that

$$\begin{aligned}
&\left\|\alpha F\left(\frac{x+y}{\alpha}+z\right)-F(x)-F(y)-\alpha F(z)\right\|_{\mathbb{B}}\\
&=\left\|\alpha\lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{n}f\left(\left(\frac{2+\alpha}{\alpha}\right)^{n}\left(\frac{x+y}{\alpha}+z\right)\right)-\lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{n}f\left(\left(\frac{2+\alpha}{\alpha}\right)^{n}x\right)\right.\\
&\quad\left.-\lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{n}f\left(\left(\frac{2+\alpha}{\alpha}\right)^{n}y\right)-\alpha\lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{n}f\left(\left(\frac{2+\alpha}{\alpha}\right)^{n}z\right)\right\|_{\mathbb{B}}\\
&=\lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{n}\left\|\alpha f\left(\frac{(\frac{2+\alpha}{\alpha})^nx+(\frac{2+\alpha}{\alpha})^ny}{\alpha}+\left(\frac{2+\alpha}{\alpha}\right)^{n}z\right)-f\left(\left(\frac{2+\alpha}{\alpha}\right)^{n}x\right)\right.\\
&\quad\left.-f\left(\left(\frac{2+\alpha}{\alpha}\right)^{n}y\right)-\alpha f\left(\left(\frac{2+\alpha}{\alpha}\right)^{n}z\right)\right\|_{\mathbb{B}}\\
&\leq\lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{n}\phi\left(\left(\frac{2+\alpha}{\alpha}\right)^{n}x,\left(\frac{2+\alpha}{\alpha}\right)^{n}y,\left(\frac{2+\alpha}{\alpha}\right)^{n}z\right)=0
\end{aligned}$$

for all $x,y,z\in\mathbb{A}$. Hence, we have

$$\alpha F\left(\frac{x+y}{\alpha}+z\right)=F(x)+F(y)+\alpha F(z) \tag{2.11}$$

for all $x,y,z\in\mathbb{A}$. From Corollary 1.2 and (2.11), we see that F is additive, that is,

$$F(x+y)=F(x)+F(y) \tag{2.12}$$

for all $x,y\in\mathbb{A}$. Next, we can show that $F:\mathbb{A}\to\mathbb{B}$ is $\mathbb{C}$-linear. Firstly, we will show that, for any $x\in\mathbb{A}$, $F(\mu x)=\mu F(x)$ for all $\mu\in\mathbb{S}$. For each $\mu\in\mathbb{S}$, substituting x,y,z in (2.1) by $(\frac{2+\alpha}{\alpha})^nx$, we obtain

$$\begin{aligned}
&E_\mu f\left(\left(\frac{2+\alpha}{\alpha}\right)^{n}x,\left(\frac{2+\alpha}{\alpha}\right)^{n}x,\left(\frac{2+\alpha}{\alpha}\right)^{n}x\right)\\
&=\alpha\mu f\left(\frac{(\frac{2+\alpha}{\alpha})^nx+(\frac{2+\alpha}{\alpha})^nx}{\alpha}+\left(\frac{2+\alpha}{\alpha}\right)^{n}x\right)-f\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^{n}x\right)-f\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^{n}x\right)
\end{aligned}$$

$$
\begin{aligned}
&- \alpha f\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^n x\right) \\
&= \alpha\mu f\left(\frac{(2+\alpha)}{\alpha}\cdot\left(\frac{2+\alpha}{\alpha}\right)^n x\right) - (2+\alpha) f\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^n x\right)
\end{aligned}
$$

for all $x \in \mathbb{A}$. By (2.3), we have

$$
\begin{aligned}
&\left\| E_\mu f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&= \left\| \alpha\mu f\left(\frac{2+\alpha}{\alpha}\cdot\left(\frac{2+\alpha}{\alpha}\right)^n x\right) - (2+\alpha) f\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\leq \phi\left(\left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n x\right)
\end{aligned} \tag{2.13}
$$

for all $x \in \mathbb{A}$. From (2.13), in the case $\mu = 1$, we obtain the fact that

$$
\begin{aligned}
&\left\| \alpha f\left(\frac{(2+\alpha)}{\alpha}\cdot\left(\frac{2+\alpha}{\alpha}\right)^n x\right) - (2+\alpha) f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\leq \phi\left(\left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n x\right)
\end{aligned} \tag{2.14}
$$

for all $x \in \mathbb{A}$. It follows from (2.3), (2.13) and (2.14) that

$$
\begin{aligned}
&\left\| (2+\alpha) f\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^n x\right) - (2+\alpha)\mu f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&= \left\| (2+\alpha) f\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^n x\right) - \alpha\mu f\left(\frac{2+\alpha}{\alpha}\cdot\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right. \\
&\quad \left. + \alpha\mu f\left(\frac{2+\alpha}{\alpha}\cdot\left(\frac{2+\alpha}{\alpha}\right)^n x\right) - (2+\alpha)\mu f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\leq \left\| (2+\alpha) f\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^n x\right) - \alpha\mu f\left(\frac{2+\alpha}{\alpha}\cdot\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\quad + \left\| \alpha\mu f\left(\frac{2+\alpha}{\alpha}\cdot\left(\frac{2+\alpha}{\alpha}\right)^n x\right) - (2+\alpha)\mu f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\leq \left\| (2+\alpha) f\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^n x\right) - \alpha\mu f\left(\frac{2+\alpha}{\alpha}\cdot\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\quad + |\mu| \left\| \alpha f\left(\frac{2+\alpha}{\alpha}\cdot\left(\frac{2+\alpha}{\alpha}\right)^n x\right) - (2+\alpha) f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\leq 2\phi\left(\left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n x\right)
\end{aligned}
$$

for all $x \in \mathbb{A}$. This implies that

$$
\begin{aligned}
&\left\| \left(\frac{\alpha}{2+\alpha}\right)^n f\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^n x\right) - \left(\frac{\alpha}{2+\alpha}\right)^n \mu f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\leq \frac{2}{2+\alpha}\left(\frac{\alpha}{2+\alpha}\right)^n \phi\left(\left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n x\right)
\end{aligned}
$$

$$\leq \left(\frac{\alpha}{2+\alpha}\right)^n \phi\left(\left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n x\right)$$

for all $x \in \mathbb{A}$. By (2.10), we have

$$\lim_{n\to\infty}\left\|\left(\frac{\alpha}{2+\alpha}\right)^n f\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^n x\right) - \left(\frac{\alpha}{2+\alpha}\right)^n \mu f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right\|_{\mathbb{B}} = 0,$$

which implies that

$$F(\mu x) = \mu F(x) \tag{2.15}$$

for all $x \in \mathbb{A}$. It follows from (2.12), (2.15) and Lemma 1.5 that $F : \mathbb{A} \to \mathbb{B}$ is $\mathbb{C}$-linear. Next, we will show that F is a C^*-algebra homomorphism. It follows from (2.4) that

$$\begin{aligned}
&\left\|F(xy) - F(x)F(y)\right\|_{\mathbb{B}} \\
&= \left\|\lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{2n} f\left(\left(\frac{2+\alpha}{\alpha}\right)^{2n} xy\right)\right. \\
&\quad \left. - \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^n f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right) \cdot \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^n f\left(\left(\frac{2+\alpha}{\alpha}\right)^n y\right)\right\|_{\mathbb{B}} \\
&= \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{2n}\left\|f\left(\left(\frac{2+\alpha}{\alpha}\right)^{2n} xy\right) - f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right) f\left(\left(\frac{2+\alpha}{\alpha}\right)^n y\right)\right\|_{\mathbb{B}} \\
&\leq \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{2n} \phi\left(\left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n y, 0\right) \\
&\leq \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{n} \phi\left(\left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n y, 0\right) = 0
\end{aligned}$$

for all $x, y \in \mathbb{A}$. Hence

$$F(xy) = F(x)F(y)$$

for all $x, y \in \mathbb{A}$.

Finally, it follows from (2.5) that

$$\begin{aligned}
&\left\|F(x^*) - (F(x))^*\right\|_{\mathbb{B}} \\
&= \left\|\lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^n f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x^*\right) - \left(\lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^n f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right)^*\right\|_{\mathbb{B}} \\
&= \left\|\lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^n f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x^*\right) - \lim_{n\to\infty}\left(\left(\frac{\alpha}{2+\alpha}\right)^n f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right)^*\right\|_{\mathbb{B}} \\
&= \left\|\lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^n f\left(\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)^*\right) - \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^n \left(f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right)^*\right\|_{\mathbb{B}} \\
&= \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^n \left\|f\left(\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)^*\right) - \left(f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)\right)^*\right\|_{\mathbb{B}} \\
&\leq \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^n \phi\left(\left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n x, \left(\frac{2+\alpha}{\alpha}\right)^n x\right) = 0
\end{aligned}$$

for all $x \in \mathbb{A}$, which implies that

$$F(x^*) = (F(x))^*$$

for all $x \in \mathbb{A}$. Therefore, $F : \mathbb{A} \to \mathbb{B}$ is a C^*-algebra homomorphism. □

Corollary 2.2 *Let $p \in [0,1)$, $\varepsilon \in [0,\infty)$ and f be a mapping of $\mathbb{A}$ into $\mathbb{B}$ such that*

$$\|E_\mu f(x,y,z)\|_{\mathbb{B}} \le \varepsilon\bigl(\|x\|_{\mathbb{A}}^p + \|y\|_{\mathbb{A}}^p + \|z\|_{\mathbb{A}}^p\bigr), \tag{2.16}$$

$$\|f(xy) - f(x)f(y)\|_{\mathbb{B}} \le \varepsilon\bigl(\|x\|_{\mathbb{A}}^p + \|y\|_{\mathbb{A}}^p\bigr), \tag{2.17}$$

$$\|f(x^*) - f(x)^*\|_{\mathbb{B}} \le 3\varepsilon\|x\|_{\mathbb{A}}^p \tag{2.18}$$

for all $\mu \in \mathbb{S}$ and for all $x,y,z \in \mathbb{A}$. Then there exists a unique C^*-algebra homomorphism $F : \mathbb{A} \to \mathbb{B}$ such that

$$\|f(x) - F(x)\|_{\mathbb{B}} \le \frac{3\varepsilon}{(1 - (\frac{2+\alpha}{\alpha})^{p-1})(2+\alpha)}\|x\|_{\mathbb{A}}^p$$

for all $x \in \mathbb{A}$.

Proof The proof follows from Theorem 2.1 by taking

$$\phi(x,y,z) = \theta\bigl(\|x\|_{\mathbb{A}}^p + \|y\|_{\mathbb{A}}^p + \|z\|_{\mathbb{A}}^p\bigr)$$

for all $x,y,z \in \mathbb{A}$. Then $k = (\frac{2+\alpha}{\alpha})^{p-1}$ and we get the desired results. □

Theorem 2.3 *Let $\phi : \mathbb{A}^3 \to [0,\infty)$ be a function such that there exists a $k < 1$ such that*

$$\phi(x,y,z) \le \left(\frac{\alpha}{2+\alpha}\right)^2 k\phi\left(\frac{2+\alpha}{\alpha}x, \frac{2+\alpha}{\alpha}y, \frac{2+\alpha}{\alpha}z\right) \tag{2.19}$$

for all $x,y,z \in \mathbb{A}$. Let f be a mapping of $\mathbb{A}$ into $\mathbb{B}$ satisfying (2.3), (2.4) and (2.5). Then there exists a unique C^*-algebra homomorphism $F : \mathbb{A} \to \mathbb{B}$ such that

$$\|f(x) - F(x)\|_{\mathbb{B}} \le \frac{\alpha k}{(1-k)(2+\alpha)^2}\phi(x,x,x) \tag{2.20}$$

for all $x \in \mathbb{A}$.

Proof We consider the linear mapping $T : X \to X$ such that

$$Tg(x) := \frac{2+\alpha}{\alpha} g\left(\frac{\alpha}{2+\alpha}x\right) \tag{2.21}$$

for all $x \in \mathbb{A}$. By a similar proof to Theorem 2.1, T is a strictly contractive self-mapping of X with the Lipschitz constant k. Letting $\mu = 1$ and substituting x,y,z in (2.3) by $\frac{\alpha}{2+\alpha}x$, we have

$$\left\|E_\mu f\left(\frac{\alpha}{2+\alpha}x,\frac{\alpha}{2+\alpha}x,\frac{\alpha}{2+\alpha}x\right)\right\|_{\mathbb{B}} = \left\|\alpha f(x)-(2+\alpha)f\left(\frac{\alpha}{2+\alpha}x\right)\right\|_{\mathbb{B}} \leq \phi\left(\frac{\alpha}{2+\alpha}x,\frac{\alpha}{2+\alpha}x,\frac{\alpha}{2+\alpha}x\right) \tag{2.22}$$

for all $x \in \mathbb{A}$. From inequality (2.22) we get

$$\begin{aligned}
&\left\|f(x)-\frac{2+\alpha}{\alpha}f\left(\frac{\alpha}{2+\alpha}x\right)\right\|_{\mathbb{B}} \\
&\quad\leq \frac{1}{\alpha}\phi\left(\frac{\alpha}{2+\alpha}x,\frac{\alpha}{2+\alpha}x,\frac{\alpha}{2+\alpha}x\right) \\
&\quad\leq \frac{1}{\alpha}\cdot\left(\frac{\alpha}{2+\alpha}\right)^2 k\phi\left(\frac{2+\alpha}{\alpha}\cdot\frac{\alpha}{2+\alpha}x,\frac{2+\alpha}{\alpha}\cdot\frac{\alpha}{2+\alpha}x,\frac{2+\alpha}{\alpha}\cdot\frac{\alpha}{2+\alpha}x\right) \\
&\quad= \frac{\alpha k}{(2+\alpha)^2}\cdot\phi(x,x,x)
\end{aligned}$$

for all $x \in \mathbb{A}$, that is,

$$\|Tf(x)-f(x)\|_{\mathbb{B}} \leq \frac{\alpha k}{(2+\alpha)^2}\phi(x,x,x)$$

for all $x \in \mathbb{A}$. Hence, we obtain

$$d(f,Tf) \leq \frac{\alpha k}{(2+\alpha)^2}.$$

By Theorem 1.4, there exists a mapping $F:\mathbb{A}\to\mathbb{B}$ such that the following conditions hold.

(1) F is a fixed point of T, that is, $TF(x)=F(x)$ for all $x \in \mathbb{A}$. Then we have

$$F(x)=TF(x)=\frac{2+\alpha}{\alpha}F\left(\frac{\alpha}{2+\alpha}x\right) \quad\Rightarrow\quad F\left(\frac{\alpha}{2+\alpha}x\right)=\frac{\alpha}{2+\alpha}F(x)$$

for all $x \in \mathbb{A}$. Moreover, the mapping F is a unique fixed point of T in the set

$$Y=\{g\in X \mid d(f,g)<\infty\}.$$

From (2.7), there exists $C\in(0,\infty)$ satisfying

$$\|f(x)-F(x)\|_{\mathbb{B}} \leq C\phi(x,x,x),$$

for all $x \in \mathbb{A}$.

(2) The sequence $\{T^n f\}$ converges to F. This implies that the equality

$$F(x)=\lim_{n\to\infty}\left(\frac{2+\alpha}{\alpha}\right)^n f\left(\left(\frac{\alpha}{2+\alpha}\right)^n x\right) \tag{2.23}$$

for all $x \in \mathbb{A}$.

(3) We obtain $d(f,F) \leq \frac{1}{1-k} d(f,Tf)$, which implies that

$$d(f,F) \leq \frac{1}{1-k} d(f,Tf) \leq \frac{\alpha k}{(1-k)(2+\alpha)^2}.$$

Therefore, inequality (2.20) holds.

It follows from (2.19) and same argument in Theorem 2.1 that we obtain

$$\lim_{j\to\infty} \left(\frac{2+\alpha}{\alpha}\right)^{2j} \cdot \phi\left(\left(\frac{\alpha}{2+\alpha}\right)^j x, \left(\frac{\alpha}{2+\alpha}\right)^j y, \left(\frac{\alpha}{2+\alpha}\right)^j z\right) = 0 \tag{2.24}$$

for all $x, y, z \in \mathbb{A}$. It follows from (2.3), (2.23), (2.24) that

$$\begin{aligned}
&\left\|\alpha F\left(\frac{x+y}{\alpha} + z\right) - F(x) - F(y) - \alpha F(z)\right\|_{\mathbb{B}} \\
&= \left\|\alpha \lim_{n\to\infty} \left(\frac{2+\alpha}{\alpha}\right)^n f\left(\left(\frac{\alpha}{2+\alpha}\right)^n \left(\frac{x+y}{\alpha} + z\right)\right) - \lim_{n\to\infty} \left(\frac{2+\alpha}{\alpha}\right)^n f\left(\left(\frac{\alpha}{2+\alpha}\right)^n x\right)\right. \\
&\quad \left. - \lim_{n\to\infty} \left(\frac{2+\alpha}{\alpha}\right)^n f\left(\left(\frac{\alpha}{2+\alpha}\right)^n y\right) - \alpha \lim_{n\to\infty} \left(\frac{2+\alpha}{\alpha}\right)^n f\left(\left(\frac{\alpha}{2+\alpha}\right)^n z\right)\right\|_{\mathbb{B}} \\
&= \lim_{n\to\infty} \left(\frac{2+\alpha}{\alpha}\right)^n \left\|\alpha f\left(\frac{(\frac{\alpha}{2+\alpha})^n x + (\frac{\alpha}{2+\alpha})^n y}{\alpha} + \left(\frac{\alpha}{2+\alpha}\right)^n z\right) - f\left(\left(\frac{\alpha}{2+\alpha}\right)^n x\right)\right. \\
&\quad \left. - f\left(\left(\frac{\alpha}{2+\alpha}\right)^n y\right) - \alpha f\left(\left(\frac{\alpha}{2+\alpha}\right)^n z\right)\right\|_{\mathbb{B}} \\
&\leq \lim_{n\to\infty} \left(\frac{2+\alpha}{\alpha}\right)^n \phi\left(\left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n y, \left(\frac{\alpha}{2+\alpha}\right)^n z\right) \\
&\leq \lim_{n\to\infty} \left(\frac{2+\alpha}{\alpha}\right)^{2n} \phi\left(\left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n y, \left(\frac{\alpha}{2+\alpha}\right)^n z\right) = 0
\end{aligned}$$

for all $x, y, z \in \mathbb{A}$. Hence, we have

$$\alpha F\left(\frac{x+y}{\alpha} + z\right) = F(x) + F(y) + \alpha F(z)$$

for all $x, y, z \in \mathbb{A}$. From Corollary 1.2 and the above equation, we see that F is additive for all $x, y \in \mathbb{A}$. Next, we can show that $F : \mathbb{A} \to \mathbb{B}$ is $\mathbb{C}$-linear. Firstly, we will show that, for any $x \in \mathbb{A}$, $F(\mu x) = \mu F(x)$ for all $\mu \in \mathbb{S}$. For each $\mu \in \mathbb{S}$, substituting x, y, z in (2.1) by $(\frac{\alpha}{2+\alpha})^n x$, we obtain

$$\begin{aligned}
&E_\mu f\left(\left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x\right) \\
&= \alpha\mu f\left(\frac{(\frac{\alpha}{2+\alpha})^n x + (\frac{\alpha}{2+\alpha})^n x}{\alpha} + \left(\frac{\alpha}{2+\alpha}\right)^n x\right) - f\left(\mu\left(\frac{\alpha}{2+\alpha}\right)^n x\right) - f\left(\mu\left(\frac{\alpha}{2+\alpha}\right)^n x\right) \\
&\quad - \alpha f\left(\mu\left(\frac{\alpha}{2+\alpha}\right)^n x\right) \\
&= \alpha\mu f\left(\frac{2+\alpha}{\alpha}\left(\frac{\alpha}{2+\alpha}\right)^n x\right) - (2+\alpha) f\left(\mu\left(\frac{\alpha}{2+\alpha}\right)^n x\right)
\end{aligned}$$

for all $x \in \mathbb{A}$. By (2.3), we have

$$\begin{aligned}
&\left\| E_\mu f\left(\left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\quad = \left\|\alpha\mu f\left(\frac{2+\alpha}{\alpha}\left(\frac{\alpha}{2+\alpha}\right)^n x\right) - (2+\alpha)f\left(\mu\left(\frac{\alpha}{2+\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\quad \leq \phi\left(\left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x\right)
\end{aligned} \tag{2.25}$$

for all $x \in \mathbb{A}$. From (2.25), in the case $\mu = 1$, we obtain the fact that

$$\begin{aligned}
&\left\|\alpha f\left(\frac{2+\alpha}{\alpha}\left(\frac{\alpha}{2+\alpha}\right)^n x\right) - (2+\alpha)f\left(\left(\frac{\alpha}{2+\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\quad \leq \phi\left(\left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x\right)
\end{aligned} \tag{2.26}$$

for all $x \in \mathbb{A}$. It follows from (2.3), (2.25) and (2.26) that

$$\begin{aligned}
&\left\|(2+\alpha)f\left(\mu\left(\frac{\alpha}{2+\alpha}\right)^n x\right) - (2+\alpha)\mu f\left(\left(\frac{\alpha}{2+\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\quad = \left\|(2+\alpha)f\left(\mu\left(\frac{\alpha}{2+\alpha}\right)^n x\right) - \alpha\mu f\left(\frac{2+\alpha}{\alpha}\left(\frac{\alpha}{2+\alpha}\right)^n x\right)\right. \\
&\qquad \left. + \alpha\mu f\left(\frac{2+\alpha}{\alpha}\left(\frac{\alpha}{2+\alpha}\right)^n x\right) - (2+\alpha)\mu f\left(\left(\frac{\alpha}{2+\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\quad \leq \left\|(2+\alpha)f\left(\mu\left(\frac{\alpha}{2+\alpha}\right)^n x\right) - \alpha\mu f\left(\frac{2+\alpha}{\alpha}\left(\frac{\alpha}{2+\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\qquad + \left\|\alpha\mu f\left(\frac{2+\alpha}{\alpha}\left(\frac{\alpha}{2+\alpha}\right)^n x\right) - (2+\alpha)\mu f\left(\left(\frac{\alpha}{2+\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\quad = \left\|(2+\alpha)f\left(\mu\left(\frac{\alpha}{2+\alpha}\right)^n x\right) - \alpha\mu f\left(\frac{2+\alpha}{\alpha}\left(\frac{\alpha}{2+\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\qquad + |\mu|\left\|\alpha f\left(\frac{2+\alpha}{\alpha}\left(\frac{\alpha}{2+\alpha}\right)^n x\right) - (2+\alpha)f\left(\left(\frac{\alpha}{2+\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\quad \leq 2\phi\left(\left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x\right)
\end{aligned}$$

for all $x \in \mathbb{A}$. This implies that

$$\begin{aligned}
&\left\|\left(\frac{2+\alpha}{\alpha}\right)^n f\left(\mu\left(\frac{\alpha}{2+\alpha}\right)^n x\right) - \left(\frac{2+\alpha}{\alpha}\right)^n \mu f\left(\left(\frac{\alpha}{2+\alpha}\right)^n x\right)\right\|_{\mathbb{B}} \\
&\quad \leq \frac{2}{2+\alpha}\left(\frac{2+\alpha}{\alpha}\right)^n \phi\left(\left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x\right) \\
&\quad \leq \left(\frac{2+\alpha}{\alpha}\right)^n \phi\left(\left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x\right) \\
&\quad \leq \left(\frac{2+\alpha}{\alpha}\right)^{2n} \phi\left(\left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x, \left(\frac{\alpha}{2+\alpha}\right)^n x\right)
\end{aligned}$$

for all $x \in \mathbb{A}$. By (2.24), we have

$$\lim_{n\to\infty}\left\|\left(\frac{2+\alpha}{\alpha}\right)^n f\left(\mu\left(\frac{\alpha}{2+\alpha}\right)^n x\right) - \left(\frac{2+\alpha}{\alpha}\right)^n \mu f\left(\left(\frac{\alpha}{2+\alpha}\right)^n x\right)\right\|_{\mathbb{B}} = 0,$$

which implies that

$$F(\mu x) = \mu F(x)$$

for all $x \in \mathbb{A}$. By Lemma 1.5, we see that F is $\mathbb{C}$-linear. The fact that $F(xy) = F(x)F(y)$ and $F(x^*) = F(x)^*$ for all $x, y \in \mathbb{A}$ can be obtained in a similar method as in the proof of Theorem 2.1. □

Corollary 2.4 *Let $p \in (2,\infty)$, $\varepsilon \in [0,\infty)$ and f be a mapping of $\mathbb{A}$ into $\mathbb{B}$ satisfying* (2.16), (2.17) *and* (2.18). *Then there exists a unique C^*-algebra homomorphism $F : \mathbb{A} \to \mathbb{B}$ such that*

$$\|f(x) - F(x)\|_{\mathbb{B}} \leq \frac{3\alpha\varepsilon}{((\frac{2+\alpha}{\alpha})^{p-2} - 1)(2+\alpha)^2}\|x\|_{\mathbb{A}}^p \tag{2.27}$$

for all $x \in \mathbb{A}$.

Proof The proof follows from Theorem 2.3 and Corollary 2.2 by taking

$$\phi(x,y,z) = \varepsilon\left(\|x\|_{\mathbb{A}}^p + \|y\|_{\mathbb{A}}^p + \|z\|_{\mathbb{A}}^p\right)$$

for all $x, y, z \in \mathbb{A}$. Then $k = (\frac{\alpha}{2+\alpha})^{p-2}$ and we get the desired results. □

Remark 2.5 If $\alpha = 2$, then Theorem 2.1, Corollary 2.2 and Theorem 2.3 we recover Theorem 2.1, Corollary 2.2 and Theorem 2.3 in [10], respectively.

3 Stability of generalized θ-derivations on C^*-algebras

Let f be a mapping of $\mathbb{A}$ into $\mathbb{A}$. We define

$$E_\mu f(x,y,z) := \alpha\mu f\left(\frac{x+y}{\alpha} + z\right) - f(\mu x) - f(\mu y) - \alpha f(\mu z),$$

for all $\mu \in \mathbb{S}$ and all $x, y, z \in \mathbb{A}$ and for any fixed positive integer $\alpha \geq 2$.

Definition 3.1 A generalized θ-derivation $\delta : \mathbb{A} \to \mathbb{A}$ is a $\mathbb{C}$-linear map satisfying

$$\delta(xyz) = \delta(xy)\theta(z) - \theta(x)\delta(y)\theta(z) + \theta(x)\delta(yz).$$

for all $x, y, z \in \mathbb{A}$, where $\theta : \mathbb{A} \to \mathbb{A}$ is a $\mathbb{C}$-linear mapping.

We prove the Hyers–Ulam–Rassias stability of generalized θ-derivation on C^*-algebras for the functional equation $E_\mu f(x,y,z) = 0$.

Theorem 3.1 *Let $\phi : \mathbb{A}^3 \to [0,\infty)$ be a function such that there exists a $k < 1$ satisfying* (2.2). *Let f, h be mappings of $\mathbb{A}$ into itself satisfying*

$$\|E_\mu f(x,y,z)\|_{\mathbb{A}} \le \phi(x,y,z), \tag{3.1}$$

$$\|f(xyz) - f(xy)h(z) + h(x)f(y)h(z) - h(x)f(yz)\|_{\mathbb{A}} \le \phi(x,y,z), \tag{3.2}$$

$$\left\|\mu h\left(\frac{2+\alpha}{2\alpha}(x+y)\right) - \frac{2+\alpha}{2\alpha}\big(h(\mu x) + h(\mu y)\big)\right\|_{\mathbb{A}} \le \phi(x,y,x), \tag{3.3}$$

$$\|f(x^*) - f(x)^*\|_{\mathbb{A}} \le \phi(x,x,x), \tag{3.4}$$

for all $\mu \in \mathbb{S}$ and for all $x,y,z \in \mathbb{A}$. Then there exist unique $\mathbb{C}$-linear mappings $\delta, \theta : \mathbb{A} \to \mathbb{A}$ such that

$$\|f(x) - \delta(x)\|_{\mathbb{A}} \le \frac{1}{(1-k)(2+\alpha)}\phi(x,x,x), \tag{3.5}$$

$$\|h(x) - \theta(x)\|_{\mathbb{A}} \le \frac{\alpha}{(1-k)(2+\alpha)}\phi(x,x,x), \tag{3.6}$$

for all $x \in \mathbb{A}$. Moreover, $\delta : \mathbb{A} \to \mathbb{A}$ is a generalized θ-derivation on $\mathbb{A}$.

Proof Let (X,d) be the generalized metric space as in the proof of Theorem 2.1. We consider the linear mapping $T : X \to X$ such that

$$Tg(x) := \frac{\alpha}{2+\alpha} g\left(\frac{2+\alpha}{\alpha}x\right)$$

for all $x \in \mathbb{A}$ and for all $g \in X$. Letting $\mu = 1$ and $y = x$ in (3.3), we get

$$\left\|h\left(\frac{2+\alpha}{\alpha}x\right) - \frac{2+\alpha}{\alpha}h(x)\right\|_{\mathbb{A}} \le \phi(x,x,x)$$

for all $x \in \mathbb{A}$, so we have

$$\left\|h(x) - \frac{\alpha}{2+\alpha}h\left(\frac{2+\alpha}{\alpha}x\right)\right\|_{\mathbb{A}} \le \frac{\alpha}{2+\alpha}\phi(x,x,x)$$

for all $x \in \mathbb{A}$. Hence, we obtain

$$d(h, Th) \le \frac{\alpha}{2+\alpha}.$$

It follows from the proof of Theorem 2.1 that

$$d(f, Tf) \le \frac{1}{2+\alpha}.$$

By the same reasoning as the proof of Theorem 2.1, there exist a unique involutive $\mathbb{C}$-linear mapping $\delta : \mathbb{A} \to \mathbb{A}$ and a mapping $\theta : \mathbb{A} \to \mathbb{A}$ satisfying (3.5) and (3.6), respectively. The mappings δ and θ are given by

$$\delta(x) = \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^n f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)$$

and

$$\theta(x) = \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^n h\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right)$$

for all $x \in \mathbb{A}$, respectively. It follows from (3.2) that

$$\begin{aligned}
&\left\|\delta(xyz) - \delta(xy)\theta(z) + \theta(x)\delta(y)\theta(z) - \theta(x)\delta(yz)\right\|_{\mathbb{A}} \\
&\quad = \left\| \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{3n} f\left(\left(\frac{2+\alpha}{\alpha}\right)^{3n} xyz\right)\right. \\
&\qquad - \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{2n} f\left(\left(\frac{2+\alpha}{\alpha}\right)^{2n} xy\right)\cdot \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{n} h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} z\right) \\
&\qquad + \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{n} h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right)\cdot \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{n} f\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} y\right) \\
&\qquad \cdot \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{n} h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} z\right) \\
&\qquad \left. - \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{n} h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right)\cdot \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{2n} f\left(\left(\frac{2+\alpha}{\alpha}\right)^{2n} yz\right)\right\|_{\mathbb{A}} \\
&\quad = \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{3n} \left\| f\left(\left(\frac{2+\alpha}{\alpha}\right)^{3n} xyz\right) - f\left(\left(\frac{2+\alpha}{\alpha}\right)^{2n} xy\right)\cdot h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} z\right)\right. \\
&\qquad + h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right)\cdot f\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} y\right)\cdot h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} z\right) - h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right) \\
&\qquad \left.\cdot f\left(\left(\frac{2+\alpha}{\alpha}\right)^{2n} yz\right)\right\|_{\mathbb{A}} \\
&\quad \le \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{3n} \phi\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x, \left(\frac{2+\alpha}{\alpha}\right)^{n} y, \left(\frac{2+\alpha}{\alpha}\right)^{n} z\right) \\
&\quad \le \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^{n} \phi\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x, \left(\frac{2+\alpha}{\alpha}\right)^{n} y, \left(\frac{2+\alpha}{\alpha}\right)^{n} z\right) = 0
\end{aligned}$$

for all $x, y, z \in \mathbb{A}$. Hence

$$\delta(xyz) = \delta(xy)\theta(z) - \theta(x)\delta(y)\theta(z) + \theta(x)\delta(yz)$$

for all $x, y, z \in \mathbb{A}$. Next, we can show that $\theta : \mathbb{A} \to \mathbb{A}$ is $\mathbb{C}$-linear. Firstly, we will show that, for any $x \in \mathbb{A}$, $\mu(\theta x) = \theta(\mu x)$ for all $\mu \in \mathbb{S}$. For each $\mu \in \mathbb{S}$, substituting x, y, z in (3.3) by $(\frac{2+\alpha}{\alpha})^n x$, we obtain

$$\begin{aligned}
&\left\| \mu h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n+1} x\right) - \frac{2+\alpha}{\alpha} h\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right)\right\|_{\mathbb{A}} \\
&\quad \le \phi\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x, \left(\frac{2+\alpha}{\alpha}\right)^{n} x, \left(\frac{2+\alpha}{\alpha}\right)^{n} x\right) \qquad (3.7)
\end{aligned}$$

for all $x \in \mathbb{A}$. For $\mu = 1$, we also have

$$\begin{aligned}
&\left\| h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n+1} x\right) - \frac{2+\alpha}{\alpha} h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right)\right\|_{\mathbb{A}} \\
&\quad \le \phi\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x, \left(\frac{2+\alpha}{\alpha}\right)^{n} x, \left(\frac{2+\alpha}{\alpha}\right)^{n} x\right)
\end{aligned} \tag{3.8}$$

for all $x \in \mathbb{A}$. It follows from (3.7) and (3.8) that

$$\begin{aligned}
&\left\| \frac{2+\alpha}{\alpha} h\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right) - \frac{2+\alpha}{\alpha} \mu h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right)\right\|_{\mathbb{A}} \\
&\quad = \left\| \frac{2+\alpha}{\alpha} h\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right) - \mu h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n+1} x\right)\right. \\
&\qquad \left. + \mu h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n+1} x\right) - \frac{2+\alpha}{\alpha} \mu h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right)\right\|_{\mathbb{A}} \\
&\quad \le \left\| \frac{2+\alpha}{\alpha} h\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right) - \mu h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n+1} x\right)\right\|_{\mathbb{A}} \\
&\qquad + \left\| \mu h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n+1} x\right) - \frac{2+\alpha}{\alpha} \mu h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right)\right\|_{\mathbb{A}} \\
&\quad = \left\| \frac{2+\alpha}{\alpha} h\left(\mu\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right) - \mu h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n+1} x\right)\right\|_{\mathbb{A}} \\
&\qquad + |\mu| \left\| h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n+1} x\right) - \frac{2+\alpha}{\alpha} h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right)\right\|_{\mathbb{A}} \\
&\quad \le 2\phi\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x, \left(\frac{2+\alpha}{\alpha}\right)^{n} x, \left(\frac{2+\alpha}{\alpha}\right)^{n} x\right)
\end{aligned}$$

for all $x \in \mathbb{A}$. This implies that

$$\begin{aligned}
&\left\| \left(\frac{\alpha}{2+\alpha}\right)^{n} h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} \mu x\right) - \left(\frac{\alpha}{2+\alpha}\right)^{n} \mu h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right)\right\|_{\mathbb{A}} \\
&\quad \le \frac{2\alpha}{2+\alpha}\left(\frac{\alpha}{2+\alpha}\right)^{n} \phi\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x, \left(\frac{2+\alpha}{\alpha}\right)^{n} x, \left(\frac{2+\alpha}{\alpha}\right)^{n} x\right)
\end{aligned}$$

for all $x \in \mathbb{A}$. By (2.2), we have

$$\lim_{n\to\infty} \left\| \left(\frac{\alpha}{2+\alpha}\right)^{n} h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} \mu x\right) - \left(\frac{\alpha}{2+\alpha}\right)^{n} \mu h\left(\left(\frac{2+\alpha}{\alpha}\right)^{n} x\right)\right\|_{\mathbb{A}} = 0$$

for all $x \in \mathbb{A}$. That is,

$$\theta(\mu x) = \mu\theta(x)$$

for all $x \in \mathbb{A}$. By Lemma 1.5, we obtain that θ is a $\mathbb{C}$-linear mapping. Thus, $\delta : \mathbb{A} \to \mathbb{A}$ is generalized θ-derivation satisfying (3.5). □

Corollary 3.2 *Let $p \in [0,1)$, $\varepsilon \in [0,\infty)$ and f be a mapping of $\mathbb{A}$ into itself such that*

$$\|E_\mu f(x,y,z)\|_{\mathbb{A}} \le \varepsilon\big(\|x\|_{\mathbb{A}}^p + \|y\|_{\mathbb{A}}^p + \|z\|_{\mathbb{A}}^p\big), \tag{3.9}$$

$$\|f(xyz) - f(xy)\theta(z) + \theta(x)f(y)\theta(z) - \theta(x)f(yz)\|_{\mathbb{A}} \le \varepsilon\big(\|x\|_{\mathbb{A}}^p + \|y\|_{\mathbb{A}}^p + \|z\|_{\mathbb{A}}^p\big), \tag{3.10}$$

$$\left\|\mu h\left(\frac{2+\alpha}{2\alpha}(x+y)\right) - \frac{2+\alpha}{2\alpha}\big(h(\mu x) + h(\mu y)\big)\right\|_{\mathbb{A}} \le \varepsilon\big(\|x\|_{\mathbb{A}}^p + \|y\|_{\mathbb{A}}^p + \|x\|_{\mathbb{A}}^p\big), \tag{3.11}$$

$$\|f(x^*) - f(x)^*\|_{\mathbb{A}} \le 3\varepsilon\|x\|_{\mathbb{A}}^r \tag{3.12}$$

for all $\mu \in \mathbb{S}$ and for all $x,y,z \in \mathbb{A}$. Then there exist unique $\mathbb{C}$-linear mappings $\delta,\theta : \mathbb{A} \to \mathbb{A}$ such that

$$\|f(x) - \delta(x)\|_{\mathbb{A}} \le \frac{3\varepsilon}{(1-(\frac{2+\alpha}{\alpha})^{p-1})(2+\alpha)}\|x\|_{\mathbb{A}}^p,$$

$$\|h(x) - \theta(x)\|_{\mathbb{A}} \le \frac{\varepsilon\alpha}{(1-(\frac{2+\alpha}{\alpha})^{p-1})(2+\alpha)}\|x\|_{\mathbb{A}}^p,$$

for all $x \in \mathbb{A}$. Moreover, $\delta : \mathbb{A} \to \mathbb{A}$ is a generalized θ-derivation on $\mathbb{A}$.

Proof The proof follows from Theorem 3.1 by taking

$$\phi(x,y,z) = \varepsilon\big(\|x\|_{\mathbb{A}}^p + \|y\|_{\mathbb{A}}^p + \|z\|_{\mathbb{A}}^p\big)$$

for all $x,y,z \in \mathbb{A}$. Then $k = (\frac{2+\alpha}{\alpha})^{p-1}$ and we get the desired results. □

Theorem 3.3 *Let $\phi : \mathbb{A}^3 \to [0,\infty)$ such that there exists a $k < 1$ satisfying*

$$\phi(x,y,z) \le \left(\frac{\alpha}{2+\alpha}\right)^3 k\phi\left(\frac{2+\alpha}{\alpha}x, \frac{2+\alpha}{\alpha}y, \frac{2+\alpha}{\alpha}z\right)$$

for all $x,y,z \in \mathbb{A}$. Let f,h be mappings of $\mathbb{A}$ into itself satisfying (3.1), (3.2), (3.3) and (3.4). Then there exist unique $\mathbb{C}$-linear mappings $\delta,\theta : \mathbb{A} \to \mathbb{A}$ such that

$$\|f(x) - \delta(x)\|_{\mathbb{A}} \le \frac{\alpha^2 k}{(1-k)(2+\alpha)^3}\phi(x,x,x),$$

$$\|h(x) - \theta(x)\|_{\mathbb{A}} \le \frac{k}{1-k}\left(\frac{\alpha}{2+\alpha}\right)^3\phi(x,x,x)$$

for all $x \in \mathbb{A}$. Moreover, $\delta : \mathbb{A} \to \mathbb{A}$ is a generalized θ-derivation on $\mathbb{A}$.

Proof The proof is similar to the proofs of Theorem 2.3 and Theorem 3.1. □

Corollary 3.4 *Let $p \in (3,\infty]$, $\varepsilon \in [0,\infty)$ and f be a mapping of $\mathbb{A}$ into itself satisfying (3.9), (3.10), (3.11) and (3.12). Then there exist unique $\mathbb{C}$-linear mappings $\delta,\theta : \mathbb{A} \to \mathbb{A}$*

such that

$$\left\| f(x) - \delta(x) \right\|_{\mathbb{A}} \leq \frac{3\alpha^2\varepsilon}{((\frac{2+\alpha}{\alpha})^{p-3} - 1)(2+\alpha)^3} \|x\|_{\mathbb{A}}^p,$$

$$\left\| h(x) - \theta(x) \right\|_{\mathbb{A}} \leq \frac{\varepsilon}{(\frac{2+\alpha}{\alpha})^{p-3} - 1} \cdot \left(\frac{\alpha}{2+\alpha}\right)^3 \|x\|_{\mathbb{A}}^p$$

for all $x \in \mathbb{A}$. *Moreover,* $\delta : \mathbb{A} \to \mathbb{A}$ *is a generalized* θ*-derivation* $\mathbb{A}$.

Proof The proof follows from Theorem 3.3 by taking

$$\phi(x, y, z) = \varepsilon\left(\|x\|_{\mathbb{A}}^p + \|y\|_{\mathbb{A}}^p + \|z\|_{\mathbb{A}}^p\right)$$

for all $x, y, z \in \mathbb{A}$. Then $k = (\frac{\alpha}{2+\alpha})^{p-3}$ and we get the desired results. □

We recall definition of generalized derivations on C^*-algebra.

Definition 3.2 ([13]) A generalized derivation $\delta : \mathbb{A} \to \mathbb{A}$ is involutive $\mathbb{C}$-linear and satisfies

$$\delta(xyz) = \delta(xy)z - x\delta(y)z + x\delta(yz)$$

for all $x, y, z \in \mathbb{A}$.

Remark 3.5 According to Definition 3.1, If $\theta = I$, I is identity mapping on $\mathbb{A}$, then a generalized θ-derivation is a generalized derivation. If the mapping h is identity mapping and $\alpha = 2$, Then Theorem 3.1 and Theorem 3.3 we recover Theorem 3.2 and Theorem 3.4 in [10], respectively. Moreover, if we set the mapping h is identity mapping, $\alpha = 2$ and $\phi(x, y, z) = \varepsilon \cdot \|x\|_{\mathbb{A}}^{\frac{p}{3}} \cdot \|y\|_{\mathbb{A}}^{\frac{p}{3}} \cdot \|z\|_{\mathbb{A}}^{\frac{p}{3}}$ in Theorem 3.1 where $p \in [0, 1)$ and $\varepsilon \in [0, \infty)$, then Theorem 3.1 one recovers Corollary 3.3 in [10] with $k = (\frac{2+\alpha}{\alpha})^{p-1}$.

4 Conclusions

In the first section of main results, we prove Hyers–Ulam–Rassias stability of C^*-algebra homomorphisms for the generalized Cauchy–Jensen equation C^*-algebras by using fixed point alternative theorem. In the second section of main results, we introduce and investigate the Hyers–Ulam–Rassias stability of generalized θ-derivation for such function C^*-algebras by the same method. By our main results we recover partial results of Park and An in [10] by Remark 2.5 and Remark 3.5.

Acknowledgements
The author has greatly benefited from the referee's report and would like to thank the referees for their valuable comments and kind suggestions, which have considerably contributed to the improvement of this work.

Funding
Not applicable.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
All authors equally contributed to this work. All authors read and approved the final manuscript.

Publisher's Note

Received: 29 January 2018 Accepted: 21 August 2018 Published online: 12 September 2018

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