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On the evolutionary p-Laplacian equation with a partial boundary value condition

Huashui Zhan[1*]

*Correspondence:
huashuizhan@163.com
[1]School of Applied Mathematics,
Xiamen University of Technology,
Xiamen, China

Abstract

Consider the equation

$$u_t = \operatorname{div}(d^\alpha |\nabla u|^{p-2}\nabla u) + \frac{\partial b_i(u,x,t)}{\partial x_i}, \quad (x,t) \in \Omega \times (0,T),$$

where Ω is a bounded domain, $d(x)$ is the distance function from the boundary $\partial\Omega$. Since the nonlinearity, the boundary value condition cannot be portrayed by the Fichera function. If $\alpha < p - 1$, a partial boundary value condition is portrayed by a new way, the stability of the weak solutions is proved by this partial boundary value condition. If $\alpha > p - 1$, the stability of the weak solutions may be proved independent of the boundary value condition.

MSC: 35L65; 35L85; 35R35

Keywords: Nonlinearity; Stability; p-Laplacian equation; Partial boundary value condition

1 Introduction and the main results

Benedikt et al. [1] considered the equation

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + q(x)|u|^{\gamma-1}u, \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

and showed that the uniqueness of the solution is not true [1]. Here, $0 < \gamma < 1$, Ω is a bounded domain in R^N with appropriately smooth boundary, $q(x) \geq 0$ and at least there is a $x_0 \in \Omega$ such that $q(x_0) > 0$. Zhan [2] had shown that the stability of the solutions to the equation

$$u_t = \operatorname{div}(d^\alpha |\nabla u|^{p-2}\nabla u) + f(u,x,t), \quad (x,t) \in Q_T, \tag{1.2}$$

is true, where $d(x) = \operatorname{dist}(x, \partial\Omega)$ is distance function, $\alpha > 0$ is a constant. The result of [2] is in complete antithesis to that of [1]. So, when the well-posedness of the solutions is considered, the degeneracy of the diffusion coefficient d^α plays an important role.

Yin and Wang [3, 4] studied the equation

$$u_t = \operatorname{div}(d^\alpha |\nabla u|^{p-2}\nabla u), \quad (x,t) \in Q_T, \tag{1.3}$$

and showed that there is a constant $\gamma > 1$ such that, if $\alpha < p - 1$, then

$$\iint_{Q_T} |\nabla u|^{\gamma} \, dx \, dt < \infty. \tag{1.4}$$

Recently, Zhan [5] had generalized the Yin and Wang result to the equation

$$u_t = \operatorname{div}\left(d^{\alpha} |\nabla u|^{p-2} \nabla u\right) + \sum_{i=1}^{N} \frac{\partial b_i(u)}{\partial x_i}, \quad (x, t) \in Q_T. \tag{1.5}$$

In this paper, we continue to consider a more general equation,

$$u_t = \operatorname{div}\left(d^{\alpha} |\nabla u|^{p-2} \nabla u\right) + \sum_{i=1}^{N} \frac{\partial b_i(u, x, t)}{\partial x_i}, \quad (x, t) \in Q_T, \tag{1.6}$$

and study the well-posedness of the weak solutions. As usual, the initial value

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{1.7}$$

is necessary. But, since the coefficient d^{α} is degenerate on the boundary, when $\alpha < p - 1$, though (1.4) is true, and the boundary value condition

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T), \tag{1.8}$$

can be imposed in the sense of the trace, it may be overdetermined. While $\alpha \geq p - 1$, it is almost impossible to prove (1.4). How to impose a suitable boundary value condition to match up with Eq. (1.6) becomes very troublesome [4]. Stated succinctly, instead of the Dirichlet boundary value condition (1.8), only a partial boundary value condition,

$$u(x, t) = 0, \quad (x, t) \in \Sigma_p \times (0, T), \tag{1.9}$$

is needed, where $\Sigma_p \subseteq \partial\Omega$ is a relatively open subset. The main difficulty comes from the fact that, since Eq. (1.6) is a nonlinear parabolic equation, Σ_p cannot be expressed by the Fichera function (one can refer to Sect. 6 of this paper). In this paper, we will try to depict the geometric characteristic of Σ_1, and establish the stability of the weak solutions based on the partial boundary value condition (1.9).

We denote

$$W_{\alpha}^{1,p} = \left\{ u \in W_{\text{loc}}^{1,p}(\Omega) : \int_{\Omega} d^{\alpha} |\nabla u|^p \, dx < \infty \right\}.$$

Definition 1.1 Let

$$u \in L^{\infty}(Q_T), \qquad u_t \in L^2(Q_T), \qquad d^{\alpha} |\nabla u|^p \in L^{\infty}\left(0, T; L^1(\Omega)\right), \tag{1.10}$$

and

$$\iint_{Q_T} \left[u_t(\varphi_1 \varphi_2) + d^{\alpha} |\nabla u|^{p-2} \nabla u \cdot \nabla(\varphi_1 \varphi_2) + \sum_{i=1}^{N} b_i(u, x, t)(\varphi_1 \varphi_2)_{x_i} \right] dx \, dt = 0. \tag{1.11}$$

Here $\varphi_1 \in C_0^1(Q_T)$, $\varphi_2(x,t) \in W_\alpha^{1,p}$ for any given t, and $|\varphi_2(x,t)| \le c$ for any given x. If the initial value (1.7) is satisfied in the sense of

$$\lim_{t \to 0} \int_\Omega |u(x,t) - u_0(x)| \, dx = 0, \tag{1.12}$$

then we say $u(x,t)$ is a solution of Eq. (1.6) with the initial condition (1.7).

Theorem 1.2 *If $p > 2$ and $\alpha < \frac{p-2}{2}$, for any $i \in \{1, 2, \ldots, N\}$, $b_i(s,x,t)$ is a C^1 function, and there are constants β, c such that*

$$\left| b_i(s,x,t) \right| \le c|s|^{1+\beta}, \qquad \left| \frac{\partial b_i(s,x,t)}{\partial s} \right| \le c|s|^\beta, \qquad \left| \frac{b_i(s,x,t)}{\partial x_i} \right| \le c,$$

$$i = 1, 2, \ldots, N, \tag{1.13}$$

$$u_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega), \tag{1.14}$$

then there is a solution of Eq. (1.6) with the initial value (1.7).

Certainly, we suggest that the conditions in Theorem 1.2 are not the optimal, we only provide a basic result of the existence here. The main aim of this paper is to research the stability of the weak solutions.

Theorem 1.3 *Let $\alpha > p - 1 > 0$, b_i satisfy*

$$\left| b_i(u,x,t) - b_i(v,x,t) \right| \le cd^{\frac{\alpha}{p}} |u - v|, \quad i = 1, 2, \ldots, N. \tag{1.15}$$

If u and v are two solutions of Eq. (1.6) with the initial values $u_0(x)$ and $v_0(x)$, respectively, then

$$\int_\Omega |u(x,t) - v(x,t)| \, dx \le c \int_\Omega |u_0(x) - v_0(x)|, \quad \forall t \in [0,T). \tag{1.16}$$

Remark 1.4 If $\alpha < p - 1$, we can prove the stability of the weak solutions for the initial-boundary value problem (1.6), (1.7), and (1.8) in a standard way [6]. We ask whether the spatial variable x in the nonlinear convection term $b_i(u,x,t)$ can bring about the essential change. In particular, when $b_i(s,x,t) \equiv 0$, then only if $\alpha \ge p - 1$, Yin and Wang [3] had shown that

$$\int_\Omega |u(x,t) - v(x,t)|^2 \, dx \le \int_\Omega |u_0(x) - v_0(x)|^2 \, dx.$$

Without the condition (1.15), we can prove a result of the local stability of the weak solutions. This is the following theorem.

Theorem 1.5 *Let $p > 1$, $b_i(s,x,t)$ be a Lipschitz function. If u and v are two solutions of Eq. (1.6), then there exists a constant β large enough such that*

$$\int_\Omega d^\beta |u(x,t) - v(x,t)|^2 \, dx \le \int_\Omega d^\beta |u_0(x) - v_0(x)|^2 \, dx. \tag{1.17}$$

Theorem 1.5 implies that the uniqueness of the weak solutions is true only if $\alpha > 0$. When $b_i(u, x, t) = b_i(x)D_i u$, i.e., the convection term is just linear, Theorem 1.5 had been proved in paper [7]. When $b_i(u, x, t) = b_i(u)$, Theorem 1.5 had been proved in [8] very recently. For the sake of simplicity, we will not give the details of the proof of Theorem 1.5 in this paper.

Once more, by introducing a new kind of the weak solutions, choosing a suitable test function, we can prove the following theorems.

Theorem 1.6 *Let $\alpha > p - 1$, $p > 2$, b_i satisfying*

$$\left| b_i(u, x, t) - b_i(v, x, t) \right| \le cd(x)|u - v|, \quad i = 1, 2, \ldots, N. \tag{1.18}$$

If u and v are two solutions of Eq. (1.6) with the initial values $u_0(x)$ and $v_0(x)$, respectively, then

$$\int_\Omega \left| u(x, t) - v(x, t) \right| dx \le \int_\Omega \left| u_0(x) - v_0(x) \right| dx, \quad \forall t \in [0, T). \tag{1.19}$$

Theorem 1.6 seems just a minor version of Theorem 1.3. However, on the right hand side of (1.19), there is no constant c as in (1.16).

Last but no the least, we will prove the stability of the solutions based on a partial boundary value condition.

Theorem 1.7 *Let $b(s, x, t)$ be a Lipschitz function, u and v be two weak solutions of Eq. (1.6) with the same partial homogeneous boundary value*

$$u|_{\Sigma_p \times (0, T)} = 0 = v|_{\Sigma_p \times (0, T)}. \tag{1.20}$$

If

$$p > 3, \qquad p - 1 > \alpha \ge \frac{p - 1}{p - 2}, \tag{1.21}$$

and there is nonnegative function $a_i(x)$ such that

$$\left| b_i(u, v, t) - b_i(v, x, t) \right| \le a_i(x)|u - v|, \tag{1.22}$$

then

$$\int_\Omega \left| u(x, t) - v(x, t) \right| dx \le \int_\Omega \left| u_0(x) - v_0(x) \right| dx, \quad \forall t \in [0, T). \tag{1.23}$$

Here,

$$\Sigma_p = \left\{ x \in \partial\Omega : \sum_{i=1}^N a_i(x) \ne 0 \right\}. \tag{1.24}$$

The paper is arranged as follows. In Sect. 1, we have given the basic definition and introduced the main results. In Sect. 2, we prove the existence of the solution to Eq. (1.6)

with initial value (1.7). In Sect. 3, we prove Theorem 1.3. In Sect. 4, we give another kind of the weak solutions. By this new definition, we can prove Theorem 1.6. In Sect. 5, we will prove Theorem 1.7. In Sect. 7, we will give an explanation of the reasonableness of the partial boundary value condition.

2 The proof of existence

Consider the regularized equation

$$u_t = \text{div}\big((d^\alpha + \varepsilon)|\nabla u|^{p-2}\nabla u\big) + \sum_{i=1}^{N} \frac{\partial b_i(u,x,t)}{\partial x_i}, \quad (x,t) \in Q_T, \tag{2.1}$$

with the initial boundary conditions

$$u(x,0) = u_{0\varepsilon}(x), \quad x \in \Omega, \tag{2.2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{2.3}$$

Here, $u_{0\varepsilon} \in C_0^\infty(\Omega)$ and $u_{0\varepsilon}$ converges to u_0 in $W_0^{1,p}(\Omega)$.

Proof of Theorem 1.2 Similar to [9], we can easily prove that there exists a weak solution $u_\varepsilon \in L^\infty(0,T; W_0^{1,p}(\Omega))$ of the initial-boundary value problem (2.1)–(2.3),

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \le c. \tag{2.4}$$

Multiplying (2.1) by u_ε and integrating it over Q_T, by the fact

$$\iint_{Q_T} u_\varepsilon \frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}\,dx\,dt$$

$$= -\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial x_i} b_i(u_\varepsilon,x,t)\,dx\,dt$$

$$= -\iint_{Q_T} \frac{\partial}{\partial x_i}\int_0^{u_\varepsilon} b_i(s,x,t)\,ds\,dx\,dt + \iint_{Q_T}\int_0^{u_\varepsilon} b_{ix_i}(s,x,t)\,ds\,dx\,dt$$

$$= \iint_{Q_T}\int_0^{u_\varepsilon} b_{ix_i}(s,x,t)\,ds\,dx\,dt$$

$$= 0,$$

we are able to deduce that

$$\iint_{Q_T} d^\alpha |\nabla u_\varepsilon|^p\,dx\,dt \le \iint_{Q_T} \big(d^\alpha + \varepsilon\big)|\nabla u_\varepsilon|^p\,dx\,dt \le c. \tag{2.5}$$

Then

$$\int_0^T \int_{\Omega_\lambda} |\nabla u_\varepsilon|^p\,dx\,dt \le c(\lambda) \tag{2.6}$$

for any $\Omega_\lambda = \{x \in \Omega, d(x,\partial\Omega) > \lambda\} \subseteq \Omega$, λ being a small constant.

Multiplying (2.5) by $u_{\varepsilon t}$, integrating it over Q_T, then it yields

$$
\iint_{Q_T} (u_{\varepsilon t})^2 \, dx \, dt = \iint_{Q_T} \operatorname{div}\big((d^\alpha + \varepsilon)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\big) \cdot u_{\varepsilon t} \, dx \, dt
$$
$$
+ \sum_{i=1}^{N} \iint_{Q_T} u_{\varepsilon t} \frac{\partial b_i(u_\varepsilon, x, t)}{\partial x_i} \, dx \, dt. \tag{2.7}
$$

Notice that

$$
|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla u_{\varepsilon t} = \frac{1}{2}\frac{d}{dt}\int_0^{|\nabla u_\varepsilon|^2} s^{\frac{p-2}{2}} \, ds.
$$

Thus,

$$
\iint_{Q_T} \operatorname{div}\big((d^\alpha + \varepsilon)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\big) \cdot u_{\varepsilon t} \, dx \, dt
$$
$$
= -\iint_{Q_T} (d^\alpha + \varepsilon)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla u_{\varepsilon t} \, dx \, dt
$$
$$
= -\frac{1}{2}\iint_{Q_T} (d^\alpha + \varepsilon)\frac{d}{dt}\int_0^{|\nabla u_\varepsilon|^2} s^{\frac{p-2}{2}} \, ds \, dx \, dt. \tag{2.8}
$$

By condition (1.13),

$$
\iint_{Q_T} u_{\varepsilon t} \frac{\partial}{\partial x_i} b_i(u_\varepsilon, x, t) \, dx \, dt
$$
$$
\leq \iint_{Q_T} \left|\frac{b_i(u_\varepsilon, x, t)}{\partial u}\right| |u_{\varepsilon x_i}||u_{\varepsilon t}| \, dx \, dt + \iint_{Q_T} \left|\frac{b_i(u_\varepsilon, x, t)}{\partial x_i}\right| |u_{\varepsilon x_i}||u_{\varepsilon t}| \, dx \, dt
$$
$$
\leq \frac{1}{4}\iint_{Q_T} (u_{\varepsilon t})^2 \, dx \, dt + c\iint_{Q_T} |u_\varepsilon|^{2\beta}|\nabla u_\varepsilon|^2 \, dx \, dt
$$
$$
+ \frac{1}{4}\iint_{Q_T} (u_{\varepsilon t})^2 \, dx \, dt + c\iint_{Q_T} |\nabla u_\varepsilon|^2 \, dx \, dt. \tag{2.9}
$$

By Hölder's inequality and $\alpha \leq \frac{p-2}{2}$,

$$
\iint_{Q_T} |u_\varepsilon|^{2\beta}|\nabla u_\varepsilon|^2 \, dx \, dt
$$
$$
\leq c\iint_{Q_T} |\nabla u_\varepsilon|^2 \, dx \, dt
$$
$$
= c\iint_{Q_T} d^{-\frac{2\alpha}{p}} \cdot d^{\frac{2\alpha}{p}}|\nabla u_\varepsilon|^2 \, dx \, dt
$$
$$
\leq c\left(\iint_{Q_T} d^{-\frac{2\alpha}{p-2}} \, dx \, dt\right)^{\frac{p-2}{p}} \cdot \left(\iint_{Q_T} d^\alpha|\nabla u_\varepsilon|^p \, dx \, dt\right)^{\frac{2}{p}}
$$
$$
\leq c. \tag{2.10}
$$

Combining (2.7)–(2.10), we have

$$\iint_{Q_T} (u_{\varepsilon t})^2 \, dx \, dt + \iint_{Q_T} (d^\alpha + \varepsilon) \frac{d}{dt} \int_0^{|\nabla u_\varepsilon|^2} s^{\frac{p-2}{2}} \, ds \, dx \, dt \le c,$$

by the inequality, we have

$$\iint_{Q_T} (u_{\varepsilon t})^2 \, dx \, dt \le c + c \int_\Omega (d^\alpha + \varepsilon) |\nabla u_{0\varepsilon}|^p \, dx \le c. \tag{2.11}$$

Hence, by (2.4), (2.6), (2.11), there exist a function u and a n-dimensional vector $\overrightarrow{\zeta} = (\zeta_1, \dots, \zeta_n)$ satisfying

$$u \in L^\infty(Q_T), \qquad \frac{\partial u}{\partial t} \in L^2(Q_T), \qquad |\overrightarrow{\zeta}| \in L^1\left(0, T; L^{\frac{p}{p-1}}(\Omega)\right), \tag{2.12}$$

and $u_\varepsilon \to u$ a.e. $\in Q_T$,

$$u_\varepsilon \rightharpoonup u, \quad \text{weakly star in } L^\infty(Q_T),$$

$$u_\varepsilon \to u, \quad \text{in } L^2\left(0, T; L^r_{\text{loc}}(\Omega)\right),$$

$$\frac{\partial u_\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^2(Q_T),$$

$$d^\alpha |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \rightharpoonup \overrightarrow{\zeta} \quad \text{in } L^1\left(0, T; L^{\frac{p}{p-1}}(\Omega)\right).$$

Here, if $p \ge 2$, $r = 2$, while $1 < p < 2$, $1 < r < \frac{Np}{N-p}$.

In order to prove that u is the solution of Eq. (1.6), for any function $\varphi \in C_0^1(Q_T)$, we have

$$\iint_{Q_T} \left[u_{\varepsilon t} \varphi + (d^\alpha + \varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \varphi + \sum_{i=1}^N b_i(u_\varepsilon, x, t) \varphi_{x_i} \right] dx \, dt = 0,$$

we let $\varepsilon \to 0$.

Since as $\varepsilon \to 0$, by $d(x) > 0$, $x \in \Omega$, then $c > \sup_{\text{supp}\,\varphi} \frac{|\nabla \varphi|}{d^\alpha} > 0$ due to $\varphi \in C_0^1(Q_T)$, we have

$$\varepsilon \left| \iint_{Q_T} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \varphi \, dx \, dt \right|$$

$$\le \varepsilon \sup_{\text{supp}\,\varphi} \frac{|\nabla \varphi|}{d^\alpha} \iint_{Q_T} \left(|\nabla u_\varepsilon|^p + c \right) dx \, dt \to 0.$$

By this note, we have

$$\iint_{Q_T} \vec{\zeta} \cdot \nabla \varphi \, dx \, dt$$

$$= \lim_{\varepsilon \to 0} \iint_{Q_T} d^\alpha |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \varphi \, dx \, dt$$

$$= \lim_{\varepsilon \to 0} \iint_{Q_T} (d^\alpha + \varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \varphi \, dx \, dt - \lim_{\varepsilon \to 0} \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \varphi \, dx \, dt$$

$$= \lim_{\varepsilon \to 0} \iint_{Q_T} (d^\alpha + \varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \varphi \, dx \, dt.$$

Now, similar to the general evolutionary p-Laplician equation [6], we are able to prove that (the details are omitted here)

$$\iint_{Q_T} \left[u\varphi_t + \vec{\zeta} \cdot \nabla\varphi + b_i(u,x,t)\varphi_{x_i} \right] dx\,dt = 0 \tag{2.13}$$

and

$$\iint_{Q_T} d^\alpha |\nabla u|^{p-2} \nabla u \cdot \nabla\varphi\,dx\,dt = \iint_{Q_T} \vec{\zeta} \cdot \nabla\varphi\,dx\,dt, \tag{2.14}$$

for any function $\varphi \in C_0^1(Q_T)$. Then

$$\iint_{Q_T} \left[u_t\varphi + d^\alpha |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \sum_{i=1}^N b_i(u,x,t)\varphi_{x_i} \right] dx\,dt = 0. \tag{2.15}$$

If for any given $t \in [0,T)$, we denote $\Omega_\varphi = \operatorname{supp}\varphi$, then

$$\int_0^T \int_{\Omega_\varphi} \left[u_t\varphi + d^\alpha |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \sum_{i=1}^N b_i(u,x,t)\varphi_{x_i} \right] dx\,dt = 0. \tag{2.16}$$

Now, for any $\varphi_1 \in C_0^1(Q_T)$, $\varphi_2(x,t) \in W_\alpha^{1,p}$ for any given t, and $|\varphi_2(x,t)| \le c$ for any given x, it is clear that $\varphi_2 \in W^{1,p}(\Omega_{\varphi_1})$. By the fact that $C_0^\infty(\Omega_{\varphi_1})$ is dense in $W^{1,p}(\Omega_{\varphi_1})$, by a process of limits, we have

$$\int_0^T \int_{\Omega_{\varphi_1}} \Bigg[u_t(\varphi_1\varphi_2) + d^\alpha |\nabla u|^{p-2}\nabla u \cdot \nabla(\varphi_1\varphi_2)$$
$$+ \sum_{i=1}^N b_i(u,x,t)(\varphi_1\varphi_2)_{x_i} \Bigg] dx\,dt = 0, \tag{2.17}$$

which implies that

$$\int_0^T \int_\Omega \left[u_t(\varphi_1\varphi_2) + d^\alpha |\nabla u|^{p-2}\nabla u \cdot \nabla(\varphi_1\varphi_2) + b_i(u,x,t)(\varphi_1\varphi_2)_{x_i} \right] dx\,dt = 0. \tag{2.18}$$

Then u satisfies Eq. (1.6) in the sense of Definition 1.1. $\qquad\square$

3 Proof of Theorem 1.3

Proof Let u and v be two weak solutions of Eq. (1.6) with the initial values $u_0(x)$ and $v_0(x)$, respectively. For large enough $n > 0$, let

$$g_n(s) = \int_0^s h_n(\tau)\,d\tau, h_n(s) = 2n\big(1 - |ns|\big)_+. \tag{3.1}$$

Obviously $h_n(s) \in C(\mathbb{R})$, and

$$h_n(s) \ge 0, \qquad \big|sh_n(s)\big| \le 1, \qquad \big|g_n(s)\big| \le 1;$$
$$\lim_{n\to\infty} g_n(s) = \operatorname{sign} s, \qquad \lim_{n\to\infty} sh_n(s) = 0. \tag{3.2}$$

We define

$$d_n(x) = \begin{cases} nd(x), & d(x) < \frac{1}{n}, \\ 1, & d(x) \geq \frac{1}{n}. \end{cases}$$

Since for any given t, $\varphi_1 = g_n(u - v) \in W_\alpha^{1,p}$, by a process of limit, we can choose $d_n g_n(u - v)$ as the test function, then

$$\int_\Omega d_n(x) g_n(u - v) \frac{\partial(u - v)}{\partial t} \, dx$$

$$+ \int_\Omega d^\alpha \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla(u - v) h_n(u - v) d_n(x) \, dx$$

$$+ \int_\Omega d^\alpha \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right)(u - v) g_n(u - v) \nabla d_n \, dx$$

$$+ \int_\Omega \left(b_i(u, x, t) - b_i(v, x, t) \right) \cdot (u - v)_{x_i} h_n(u - v) d_n(x) \, dx$$

$$+ \int_\Omega \left(b_i(u, x, t) - b_i(v, x, t) \right) \cdot g_n(u - v) d_{n x_i}(x) \, dx = 0. \tag{3.3}$$

Thus

$$\lim_{n \to \infty} \int_\Omega d_n(x) g_n(u - v) \frac{\partial(u - v)}{\partial t} \, dx = \frac{d}{dt} \|u - v\|_{L^1(\Omega)}, \tag{3.4}$$

$$\int_\Omega d^\alpha \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla(u - v) h_n(u - v) d_n(x) \, dx \geq 0. \tag{3.5}$$

Denoting $D_n = \{x \in \Omega : d(x) > \frac{1}{n}\}$, $q = \frac{p}{p-1}$, clearly

$$\left\| n d^{\frac{\alpha}{p}} \right\|_{L^p(\Omega \setminus D_n)} = n \left\| d^{\frac{\alpha}{p}} \right\|_{L^p(\Omega \setminus D_n)}$$

$$= n \left(\int_{\Omega \setminus D_n} d^\alpha \, dx \right)^{\frac{1}{p}} \leq c n^{1 - \frac{1+\alpha}{p}},$$

which goes to zero since that $\alpha > p - 1$.

By this fact, $|\nabla d_n| = n$, $x \in \Omega \setminus D_n$, we have

$$\left| \int_\Omega d^\alpha \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla d_n g_n(u - v) \, dx \right|$$

$$= \left| \int_{\Omega \setminus D_n} d^\alpha \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla d_n g_n(u - v) \, dx \right|$$

$$\leq \left\| d^{\alpha \frac{p-1}{p}} \left(|\nabla u|^{p-1} + |\nabla v|^{p-1} \right) \right\|_{L^q(\Omega \setminus D_n)} \left\| n d^{\alpha \frac{1}{p}} \right\|_{L^p(\Omega \setminus D_n)}$$

$$\leq c \left[\left(\int_{\Omega \setminus D_n} d^\alpha |\nabla u|^p \, dx \right)^{\frac{1}{q}} + \left(\int_{\Omega \setminus D_n} d^\alpha |\nabla u|^p \, dx \right)^{\frac{1}{q}} \right], \tag{3.6}$$

which goes to 0 as $n \to 0$.

Once more, since

$$\int_\Omega \left| d^{\frac{\alpha}{p}} (u-v)_{x_i} \right| dx \le c \left(\int_\Omega d^\alpha \left(|\nabla u|^p + |\nabla v|^p \right) dx \right)^{\frac{1}{p}} \le c,$$

by the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \left| \int_\Omega \left(b_i(u,x,t) - b_i(v,x,t) \right) d_n(x) h_n(u-v)(u-v)_{x_i} \, dx \right|$$

$$\le \lim_{n\to\infty} \int_\Omega \left| b_i(u,x,t) - b_i(v,x,t) \right| \left| h_n(u-v)(u-v)_{x_i} \right| dx$$

$$\le c \lim_{n\to\infty} \int_\Omega \left| (u-v)h_n(u-v) \right| \left| d^{\frac{\alpha}{p}}(u-v)_{x_i} \right| dx = 0. \tag{3.7}$$

Once again,

$$\lim_{n\to\infty} \left| \int_\Omega \left(b_i(u,x,t) - b_i(v,x,t) \right) \cdot g_n(u-v) d_{nx_i}(x) \, dx \right| \le c \int_\Omega |u-v| \, dx. \tag{3.8}$$

Now, let $n \to \infty$ in (3.3). Then

$$\frac{d}{dt} \|u-v\|_{L^1(\Omega)} \le c\|u-v\|_{L^1(\Omega)}.$$

It implies that

$$\int_\Omega \left| u(x,t) - v(x,t) \right| dx \le c \int_\Omega |u_0 - v_0| \, dx, \quad \forall t \in [0,T).$$

Theorem 1.3 is proved. □

4 Another kind of weak solution

In this section, we introduce another kind of weak solution and prove another stability theorem.

Definition 4.1 If a function $u(x,t)$ satisfies (1.10), and

$$\iint_{Q_T} \left[u_t g(\varphi) + d^\alpha |\nabla u|^{p-2} \nabla u \cdot \nabla g(\varphi) + \sum_{i=1}^N b_i(u,x,t) g_{x_i}(\varphi) \right] dx \, dt = 0, \tag{4.1}$$

for $\varphi \in C_0^1(Q_T)$, $g(s)$ is a C^1 function with $g(0) = 0$, the initial value (1.7) is satisfied in the sense of (1.12), then we say $u(x,t)$ is a weak solution of Eq. (1.6) with the initial value (1.7).

Only if we choose $\varphi_1 = g(\varphi)$, $\varphi_2 = 1$ in Definition 1.1, one can obtain the existence of the weak solutions in the sense of Definition 4.1.

Theorem 4.2 *If b_i is a Lipchitz function,*

$$\left| b_i(u,x,t) - b_i(v,x,t) \right| \le cg(x)|u-v|, \tag{4.2}$$

$$\int_{\Omega} g(x) d^{-1}(x) \, dx \leq c, \tag{4.3}$$

and one of the following conditions is true:

(i) $\alpha \geq p$;

(ii) $p > \alpha > p - 1, p > 2$;

then the stability

$$\int_{\Omega} \left| u(x,t) - v(x,t) \right| dx \leq \int_{\Omega} \left| u_0(x) - v_0(x) \right| dx, \quad \forall t \in [0, T), \tag{4.4}$$

is true for the solutions u and v with the initial values $u_0(x)$ and $v_0(x)$, respectively.

Proof By a process of limit, we may choose $\varphi = \chi_{[\tau,s]} g_n(d^\beta(u-v))$ as a test function, where β is a constant to be chosen later. Then

$$\iint_{Q_{\tau s}} g_n\big((u-v)d^\beta\big) \frac{\partial(u-v)}{\partial t} \, dx \, dt$$

$$= -\iint_{Q_{\tau s}} d^\alpha \big(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v\big) \nabla \big[g_n\big((u-v)d^\beta\big)\big] \, dx \, dt$$

$$- \iint_{Q_{\tau s}} \big[b_i(u,x,t) - b_i(v,x,t)\big] \big[g_n\big((u-v)d^\beta\big)\big]_{x_i} \, dx \, dt. \tag{4.5}$$

Now, let us calculate every term in (4.5). For the first term on the right hand side of (4.5),

$$\iint_{Q_{\tau s}} d^\alpha \big(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v\big) \nabla \big[g_n\big((u-v)d^\beta\big)\big] \, dx \, dt$$

$$= \iint_{Q_{\tau s}} d^{\alpha+\beta} h_n\big((u-v)d^\beta\big) \big(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v\big) \nabla(u-v) \, dx \, dt$$

$$+ \beta \iint_{Q_{\tau s}} d^{\alpha+\beta-1} h_n\big((u-v)d^\beta\big)(u-v)\big(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v\big) \nabla d \, dx \, dt. \tag{4.6}$$

Clearly,

$$\iint_{Q_{\tau s}} d^{\alpha+\beta} h_n\big((u-v)d^\beta\big) \big(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v\big) \nabla(u-v) \, dx \, dt \geq 0. \tag{4.7}$$

By the fact that $|\nabla d| = 1$ is true almost everywhere, $\alpha > p - 1$, we have

$$\iint_{Q_T} d^{\alpha-p} \, dx \, dt \leq c,$$

accordingly, using the Lebesgue dominated convergent theorem and the limit $\lim_{n\to\infty} s \times h_n(s) = 0$, we have

$$\left| \iint_{Q_{\tau s}} d^{\alpha+\beta-1} \big(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v\big)(u-v) h_n\big((u-v)d^\beta\big) \nabla d \, dx \, dt \right|$$

$$\leq c \left(\int_\tau^s \int_\Omega d^\alpha \big(|\nabla u|^p + |\nabla v|^p\big) \, dx \, dt \right)^{\frac{p-1}{p}}$$

$$\times \left(\int_\tau^s \int_\Omega d^\alpha d^{p(\beta-1)} |\nabla d|^p \big| h_n\big((u-v)d^\beta\big)(u-v)\big|^p \, dx\, dt \right)^{\frac{1}{p}}$$

$$\le c \left(\int_\tau^s \int_\Omega d^{\alpha-p} \big| h_n\big((u-v)d^\beta\big)d^\beta(u-v)\big|^p \, dx\, dt \right)^{\frac{1}{p}}, \tag{4.8}$$

which goes to zero as $n \to \infty$.

As for the second term on the right hand side of (5.5),

$$\iint_{Q_{\tau s}} \big[b_i(u,x,t) - b_i(v,x,t) \big] \big[g_n\big((u-v)d^\beta\big) \big]_{x_i} \, dx\, dt$$

$$= \iint_{Q_{\tau s}} \big[b_i(u,x,t) - b_i(v,x,t) \big](u-v) h_n\big((u-v)d^\beta\big) d_{x_i}^\beta \, dx\, dt$$

$$+ \iint_{Q_{\tau s}} \big[b_i(u,x,t) - b_i(v,x,t) \big](u-v)_{x_i} d^\beta h_n\big((u-v)d^\beta\big) \, dx\, dt. \tag{4.9}$$

Since for any given (x,t), $b_i(s,x,t)$ is a Lipschitz function, $u, v \in L^\infty(Q_T)$, we have

$$\iint_{Q_{\tau s}} \big[b_i(u,x,t) - b_i(v,x,t) \big] h_n\big((u-v)d^\beta\big)(u-v) d_{x_i}^\beta \, dx\, dt$$

$$= \beta \int_\tau^s \int_\Omega \big[b_i(u,x,t) - b_i(v,x,t) \big] d^{-1} h_n\big((u-v)d^\beta\big)(u-v) d^\beta d_{x_i} \, dx\, dt, \tag{4.10}$$

which goes to zero when $n \to 0$. This is due to $[b_i(u,x,t) - b_i(v,x,t)]d^{-1}(x) \in L^1(Q_T)$ by (4.2)–(4.3), using the Lebesgue dominated convergent theorem in (4.10) and using $\lim_{n\to\infty} s h_n(s) = 0$ again.

Meanwhile, also using the dominated convergent theorem, we have

$$\left| \iint_{Q_{\tau s}} \big[b_i(u,x,t) - b_i(v,x,t) \big](u-v)_{x_i} d^\beta h_n\big((u-v)d^\beta\big) \, dx\, dt \right|$$

$$\le \left(\int_\tau^s \int_\Omega d^{(-\frac{\alpha}{p})q} \big[h_n\big((u-v)d^\beta\big) d^\beta \big| b_i(u,x,t) - b_i(v,x,t) \big| \big]^q \, dx\, dt \right)^{\frac{1}{q}}$$

$$\times \left(\int_\tau^s \int_\Omega d^\alpha \big(|\nabla u|^p + |\nabla v|^p \big) \, dx\, dt \right)^{\frac{1}{p}}$$

$$\le c \left(\int_\tau^s \int_\Omega d^{(1-\frac{\alpha}{p})q} \big[h_n\big((u-v)d^\beta\big) d^\beta |u-v| \big]^q \, dx\, dt \right)^{\frac{1}{q}}, \tag{4.11}$$

which goes to zero provided that one of the conditions (i) and (ii) is true. Here $q = \frac{p}{p-1}$ as usual.

At last,

$$\lim_{n\to\infty} \iint_{Q_{\tau s}} g_n\big((u-v)d^\beta\big) \frac{\partial(u-v)}{\partial t} \, dx\, dt$$

$$= \iint_{Q_{\tau s}} \mathrm{sign}\big((u-v)d^\beta\big) \frac{\partial(u-v)}{\partial t} \, dx\, dt$$

$$= \iint_{Q_{\tau s}} \mathrm{sign}\big((u-v)\big) \frac{\partial(u-v)}{\partial t} \, dx\, dt$$

$$= \int_\Omega \big| u(x,s) - v(x,s) \big| \, dx - \int_\Omega \big| u(x,\tau) - v(x,\tau) \big| \, dx. \tag{4.12}$$

By (4.6)–(4.12), we have

$$\int_\Omega |u(x,s) - v(x,s)|\, dx \le \int_\Omega |u(x,\tau) - v(x,\tau)|\, dx. \qquad (4.13)$$

Then

$$\int_\Omega |u(x,s) - v(x,s)|\, dx \le \int_\Omega |u_0(x) - v_0(x)|\, dx.$$

The proof is complete. □

Proof of Theorem 1.6 Since $\alpha > p - 1$, $p > 2$ and the condition (1.18) in Theorem 1.6, one can see that (4.2)–(4.3) are all right. Thus, Theorem 1.6 is true. □

5 Proof of Theorem 1.7

Proof For a small positive constant $\lambda > 0$, define

$$\phi(x) = \begin{cases} 1, & \text{if } x \in \Omega_\lambda, \\ \frac{d(x)}{\lambda}, & \text{if } x \in \Omega \setminus \Omega_\lambda, \end{cases} \qquad (5.1)$$

where

$$\Omega_\lambda = \big\{ x \in \Omega : d(x) = \mathrm{dist}(x, \partial\Omega) > \lambda \big\}.$$

Then

$$\nabla\phi = \frac{1}{\lambda}\nabla d, \quad x \in \Omega \setminus \Omega_\lambda.$$

u and v are two weak solutions of Eq. (1.6) with the same partial homogeneous boundary value (1.20) and with the different initial values $u_0(x)$ and $v_0(x)$, respectively. According to Definition 4.1, we choose $g_n(\phi(u - v))$ as the test function. Thus

$$\int_\Omega g_n\big(\phi(u - v)\big)\frac{\partial(u - v)}{\partial t}\, dx$$

$$+ \int_\Omega d^\alpha \big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big) \cdot \phi\nabla(u - v)h_n\big(\phi(u - v)\big)\, dx$$

$$+ \int_\Omega d^\alpha \big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big) \cdot \nabla\phi(u - v)h_n\big(\phi(u - v)\big)\, dx$$

$$+ \sum_{i=1}^N \int_\Omega \big(b_i(u,x,t) - b_i(v,x,t)\big)(u - v)_{x_i}h_n\big(\phi(u - v)\big)\phi\, dx$$

$$+ \sum_{i=1}^N \int_\Omega \big(b_i(u,x,t) - b_i(v,x,t)\big)\phi_{x_i}(u - v)h_n\big(\phi(u - v)\big)\, dx$$

$$= 0. \qquad (5.2)$$

For the terms on the left hand side of (5.2),

$$\lim_{n \to \infty} \lim_{\lambda \to 0} \int_\Omega g_n\big(\phi(u-v)\big)\frac{\partial(u-v)}{\partial t}\, dx = \frac{d}{dt}\int_\Omega |u-v|\, dx, \tag{5.3}$$

$$\int_\Omega d^\alpha \big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\cdot \phi \nabla(u-v) h_n\big(\phi(u-v)\big)\, dx \ge 0. \tag{5.4}$$

By the fact that

$$\big|(u-v)h_n\big(\phi(u-v)\big)\big| = \big|\phi(u-v)h_n\big(\phi(u-v)\big)\big|\frac{1}{\phi} \le \frac{c}{\phi}, \qquad \frac{|\nabla\phi|}{\phi} \le \frac{c}{\lambda}, \tag{5.5}$$

using the Young inequality, we have

$$\left|\int_\Omega d^\alpha \big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\cdot \nabla\phi(u-v)h_n\big(\phi(u-v)\big)\, dx\right|$$

$$\le \int_{\Omega\setminus\Omega_\lambda} d^\alpha \big(|\nabla u|^{p-1}+|\nabla v|^{p-1}\big)\frac{|\nabla\phi|}{\phi}\big|\phi(u-v)\big|h_n\big(\phi(u-v)\big)\, dx$$

$$\le c\int_{\Omega\setminus\Omega_\lambda} \frac{1}{\lambda}d^\alpha \big(|\nabla u|^{p-1}+|\nabla v|^{p-1}\big)\big|\phi(u-v)\big|h_n\big(\phi(u-v)\big)\, dx$$

$$\le \frac{c}{\lambda}\int_{\Omega\setminus\Omega_\lambda} d^{\alpha-\frac{\alpha}{p-1}}\rho^{\frac{\alpha}{p-1}}\big(|\nabla u|^{p-1}+|\nabla v|^{p-1}\big)\, dx$$

$$\le c\int_{\Omega\setminus\Omega_\lambda}\left[d^\alpha\big(|\nabla u|^p+|\nabla v|^p\big)+\frac{1}{\lambda^p}d^{p(\alpha-\frac{\alpha}{p-1})}\right] dx, \tag{5.6}$$

which goes to 0 as $\lambda \to 0$, by $p-1 > \alpha \ge \frac{p-1}{p-2}$, implying

$$\frac{1}{\lambda^p}d^{p(\alpha-\frac{\alpha}{p-1})} \le \lambda^{[\alpha-1-\frac{\alpha}{p-1}]p} \to 0.$$

Meanwhile,

$$\sum_{i=1}^{N}\left|\int_\Omega \big(b_i(u,x,t)-b_i(v,x,t)\big)h_n\big(\phi(u-v)\big)(u-v)\phi_{x_i}(x)\, dx\right|$$

$$\le c\sum_{i=1}^{N}\int_{\Omega\setminus\Omega_\lambda}\frac{|b_i(u,x,t)-b_i(v,x,t)|}{\lambda}\, dx. \tag{5.7}$$

We use $|b_i(u,x,t)-b_i(v,x,t)| \le a_i(x)|u-v|$. According to the definition of the trace, by the partial boundary value condition (1.6),

$$u(x,t) = v(x,t) = 0, \quad x \in \Sigma_1 = \left\{x \in \partial\Omega : \sum_{i=1}^{N} a_i(x) \ne 0\right\}$$

and

$$\sum_{i=1}^{N} a_i(x) = 0, \quad x \in \Sigma_2 = \left\{x \in \partial\Omega : \sum_{i=1}^{N} a_i(x) = 0\right\},$$

we have

$$\lim_{\lambda \to 0}\left|\int_{\Omega}\big(b_i(u,x,t)-b_i(v,x,t)\big)h_n\big(\phi(u-v)\big)(u-v)\phi_{x_i}(x)\,dx\right|$$

$$\le c\sum_{i=1}^{N}\int_{\partial\Omega}\big|a_i(x)\big||u-v|\,d\Sigma = c\sum_{i=1}^{N}\int_{\Sigma_1\cup\Sigma_2}\big|a_i(x)\big||u-v|\,d\Sigma = 0. \tag{5.8}$$

Moreover, as in [10], we can prove that

$$\lim_{n\to\infty}\lim_{\lambda\to 0}\int_{\Omega}\big(b_i(u,x,t)-b_i(v,x,t)\big)h_n\big(\phi(u-v)\big)(u-v)_{x_i}\phi(x)\,dx = 0. \tag{5.9}$$

In detail,

$$\lim_{\lambda \to 0}\left|\int_{\Omega}\big(b_i(u,x,t)-b_i(v,x,t)\big)h_n\big(\phi(u-v)\big)(u-v)_{x_i}\phi(x)\,dx\right|$$

$$=\left|\int_{\{x\in\Omega:|u-v|<\frac{1}{n}\}}\big[b_i(u,x,t)-b_i(v,x,t)\big]h_n(u-v)(u-v)_{x_i}\,dx\right|$$

$$\le c\int_{\{x\in\Omega:|u-v|<\frac{1}{n}\}}\left|\frac{b_i(u,x,t)-b_i(v,x,t)}{u-v}\right|\big|(u-v)_{x_i}\big|\,dx$$

$$= c\int_{\{x\in\Omega:|u-v|<\frac{1}{n}\}}\left|d^{-\frac{\alpha}{p}}\frac{b_i(u,x,t)-b_i(v,x,t)}{u-v}\right|\big|d^{\frac{\alpha}{p}}(u-v)_{x_i}\big|\,dx$$

$$\le c\left[\int_{\{x\in\Omega:|u-v|<\frac{1}{n}\}}\left|d^{-\frac{\alpha}{p}}\frac{b_i(u,x,t)-b_i(v,x,t)}{u-v}\right|^{\frac{p}{p-1}}dx\right]^{\frac{p-1}{p}}$$

$$\times\left[\int_{\{x\in\Omega:|u-v|<\frac{1}{n}\}}\big|d^{\alpha}\nabla(u-v)\big|^p\,dx\right]^{\frac{1}{p}}. \tag{5.10}$$

Since $\alpha < p-1$, $|b_i(u,x,t)-b_i(v,x,t)|\le c|u-v|$,

$$\int_{\{x\in\Omega:|u-v|<\frac{1}{n}\}}\left|d^{-\frac{\alpha}{p}}\frac{b_i(u,x,t)-b_i(v,x,t)}{u-v}\right|^{\frac{p}{p-1}}dx \le c\int_{\Omega}d^{-\frac{\alpha}{p-1}}\,dx \le c. \tag{5.11}$$

If $\{x\in\Omega:|u-v|=0\}$ is a set with 0 measure, then

$$\lim_{n\to\infty}\int_{\{x\in\Omega:|u-v|<\frac{1}{n}\}}\big|d^{\frac{\alpha}{p-1}}\big|\,dx = \int_{\{x\in\Omega:|u-v|=0\}}\big|d^{\frac{\alpha}{p-1}}\big|\,dx = 0. \tag{5.12}$$

If the set $\{x\in\Omega:|u-v|=0\}$ has a positive measure, then

$$\lim_{n\to\infty}\int_{\{x\in\Omega:|u-v|<\frac{1}{n}\}}d^{\alpha}\big|\nabla(u-v)\big|^p\,dx = \int_{\{x\in\Omega:|u-v|=0\}}d^{\alpha}\big|\nabla(u-v)\big|^p\,dx = 0. \tag{5.13}$$

Therefore, in both cases, (5.10) goes to 0 as $\eta \to 0$.

Now, after letting $\lambda \to 0$, let $n\to\infty$ in (5.2). Then, by (5.3), (5.4), (5.6), (5.8), and (5.9), we have

$$\frac{d}{dt}\int_{\Omega}|u-v|\,dx \le c\int_{\Omega}|u-v|\,dx,$$

by the Gronwall inequality, we have

$$\int_\Omega \big| u(x,t) - v(x,t) \big| \, dx \le c \int_\Omega \big| u_0(x) - v_0(x) \big| \, dx, \quad \forall t \in [0,T),$$

Theorem 1.7 is proved. □

6 The partial boundary condition

Let us simply review Fichera–Oleĭnik theory. For a linear degenerate elliptic equation,

$$\sum_{r,s=1}^{N+1} a^{rs}(x) \frac{\partial^2 u}{\partial x_r \partial x_s} + \sum_{r=1}^{N+1} b_r(x) \frac{\partial u}{\partial x_r} + c(x)u = f(x), \quad x \in \widetilde{\Omega} \subset \mathbb{R}^{N+1}, \tag{6.1}$$

the symmetric matrix $(a^{rs}(x))$ has nonnegative characteristic value, to study its well-posedness problem, one only needs to give a partial boundary condition. In detail, let $\{n_s\}$ be the unit inner normal vector of $\partial\widetilde{\Omega}$ and denote

$$\Sigma_2 = \big\{ x \in \partial\widetilde{\Omega} : a^{rs} n_r n_s = 0, \big(b_r - a^{rs}_{x_s}\big) n_r < 0 \big\},$$
$$\Sigma_3 = \big\{ x \in \partial\widetilde{\Omega} : a^{rs} n_s n_r > 0 \big\}.$$

Then, to ensure the well-posedness of Eq. (1.7), Fichera–Oleĭnik theory tells us that the suitable boundary condition is

$$u|_{\Sigma_2 \cup \Sigma_3} = g(x). \tag{6.2}$$

In particular, if the matrix (a^{rs}) is positive definite, (6.2) is just the usual Dirichlet boundary condition. Considering the classical parabolic equation

$$u_t = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u = f(x,t), \tag{6.3}$$

with the matrix (a^{ij}) is positive definite, besides the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{6.4}$$

only a parabolic boundary value condition

$$u(x,t) = g(x,t), \quad (x,t) \in \partial\Omega \times [0,T), \tag{6.5}$$

is imposed. However, for Eq. (1.6) considered in this paper, since the equations are strongly nonlinear and degenerate, including the extremely case of $a \equiv 0$, Fichera–Oleĭnik theory is invalid, the corresponding problem becomes more complicated. To show that the partial boundary value condition imposed on the main equation (1.6) is reasonable, we can come back to the linear case. In other words, let us suppose that $p = 2$ and

$$b_i(u,x,t) = a_i(x)u. \tag{6.6}$$

Then Eq. (1.6) has the form

$$u_t = \operatorname{div}\left(d^\alpha \nabla u\right) + \sum_{i=1}^{N} a_i(x) \frac{\partial u}{\partial x_i} + u \operatorname{div} \overrightarrow{a}, \quad (x,t) \in Q_T, \tag{6.7}$$

where $\overrightarrow{a} = \{a_i\}$. According to Fichera–Oleĭnik theory, the optional boundary value condition is

$$u(x,t) = 0, \quad (x,t) \in \Sigma \times [0,T), \tag{6.8}$$

with

$$\Sigma = \left\{ x \in \partial\Omega : a_i(x) n_i(x) < 0 \right\}, \tag{6.9}$$

where $\vec{n} = \{n_i\}$ is the inner normal vector of Ω.

Now, by reviewing the partial boundary value condition (1.24)

$$\Sigma_p = \left\{ x \in \partial\Omega : \sum_{i=1}^{N} a_i(x) \neq 0 \right\},$$

we have found

$$\Sigma \subseteq \Sigma_p. \tag{6.10}$$

Though the condition (1.24) may be not the optimal, it is reasonable.

7 Conclusion

Besides the diffusion coefficient d^α being degenerate on the boundary, Eq. (1.6) has a convection term $\sum_{i=1}^{N} \frac{\partial b_i(u,x,t)}{\partial x_i}$, which depends on the spatial variable x. Such a characteristic can bring about essential changes on the boundary value condition. A reasonable partial boundary value condition is proposed for the first time, the stability of the weak solutions based on this partial boundary value condition is established. One can see that, if the convection term is independent of the spatial variable x, putting up a reasonable partial boundary condition becomes more difficult. We hope we can solve this problem in our follow-up work.

Competing interests
The author declares to have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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