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New Poisson inequality for the Radon transform of infinitely differentiable functions

Ziyao Sun[1*]

*Correspondence: ziyaos@qq.com
[1]School of Economic Mathematics, Southwestern University of Finance and Economic, Chengdu, China

Abstract

Poisson inequality for the Radon transform is a key tool in signal analysis and processing. An analogue of the Hardy–Littlewood–Poisson inequality for the Radon transform of infinitely differentiable functions is proved. The result is related to a paper of Luan and Vieira (J. Inequal. Appl. 2017:12, 2017) and to a previous paper by Yang and Ren (Proc. Indian Acad. Sci. Math. Sci. 124(2):175-178, 2014).

Keywords: Poisson inequality; Radon transform; Infinitely differentiable functions

1 Introduction

The Radon transform $\mathfrak{P}\mathfrak{I}$, which is defined as the Cauchy principal value of the following singular integral

$$(\mathfrak{P}\mathfrak{I}h)(x) := p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{h(y)}{x-y}\,dy = \lim_{\epsilon\to 0}\int_{|y-x|\geq\epsilon} \frac{h(y)}{x-y}\,dy$$

for any $x \in \mathbb{R}$, has been widely used in physics, engineering, and mathematics. The following Poisson inequality

$$\mathfrak{P}\mathfrak{I}(hg) \leq h\mathfrak{P}\mathfrak{I}g \tag{1.1}$$

was first studied in [1–3, 5]. It was proved that (1.1) holds if $h, g \in L^2(\mathbb{R})$ satisfy that $\operatorname{supp}\hat{f} \subseteq \mathbb{R}_+$ ($\mathbb{R}_+ = [0,\infty)$) and $\operatorname{supp}\hat{g} \subseteq \mathbb{R}_+$ in [21].

In 2014, Yang and Ren also obtained more general sufficient conditions by weakening the above condition in [24]. Recently, Luan and Vieria established the first necessary and sufficient condition in the time domain and a parallel result in the frequency domain for the Poisson inequality in [16].

It is natural that there have been attempts to define the complex signal and prove the Poisson inequality in a multidimensional case.

Definition 1.1 The partial Radon transform $\mathfrak{P}\mathfrak{I}_j$ of a function $h \in L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ is given by

$$(\mathfrak{P}\mathfrak{I}_j h)(x) := p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{h(y)}{x_j-y_j}\,dy_j.$$

The total Radon transform $\mathfrak{P}\mathfrak{I}$ of a function $h \in L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ is defined as follows:

$$(\mathfrak{P}\mathfrak{I}h)(x) := p.v.\frac{1}{\pi^n}\int_{\mathbb{R}^n}\frac{h(y)}{\prod_{j=1}^n(x_j - y_j)}\,dy$$
$$= \lim_{max\epsilon_j\to 0}\int_{|y_j-x_j|\geq\epsilon_j>0, j=1,2,\ldots,n}\frac{h(y)}{\prod_{j=1}^n(x_j - y_j)}\,dy.$$

The existence of the singular integral above and its boundedness property

$$\|\mathfrak{P}\mathfrak{I}h\|_{L^p(\mathbb{R}^n)} \leq C_p^n\|h\|_{L^p(\mathbb{R}^n)}$$

were proved in [10, 19]. The iterative nature of the Radon transform in $L^p(\mathbb{R}^n)$ $(p > 1)$ was shown in [6]. It was shown that

$$\mathfrak{P}\mathfrak{I} = \prod_{j=1}^n \mathfrak{P}\mathfrak{I}_j.$$

The operations $\mathfrak{P}\mathfrak{I}_i$ and $\mathfrak{P}\mathfrak{I}_j$ commute with each other, where $i, j = 1, 2, \ldots, n$.

It is known that the Fourier transform $\hat{h}$ of $h \in L^1(\mathbb{R}^n)$ is defined as follows (see [7]):

$$\hat{h}(x) = \int_{\mathbb{R}^n} h(t)e^{-ix.t}\,dt,$$

where $x \in \mathbb{R}^n$.

Let $\mathcal{D}(\mathbb{R}^n)$ be the space of infinitely differentiable functions in $\mathbb{R}^n$ with a compact support and $\mathcal{D}'(\mathbb{R}^n)$ be the space of distributions, that is, the dual of $\mathcal{D}(\mathbb{R}^n)$ (see [15, 23]). This definition is consistent with the ordinary one when T is a continuous function.

Put

$$D_+ = \left\{x : x \in \mathbb{R}^n, \operatorname{sgn}(-x) = \prod_{j=1}^n \operatorname{sgn}(-x_j) = 1\right\},$$
$$D_- = \left\{x : x \in \mathbb{R}^n, \operatorname{sgn}(-x) = \prod_{j=1}^n \operatorname{sgn}(-x_j) = -1\right\},$$

and

$$D_0 = \left\{x : x \in \mathbb{R}^n, \operatorname{sgn}(-x) = \prod_{j=1}^n \operatorname{sgn}(-x_j) = 0\right\}.$$

We denote by $\mathcal{D}_{D_+}(\mathbb{R}^n)$, $\mathcal{D}_{D_-}(\mathbb{R}^n)$ and $\mathcal{D}_{D_0}(\mathbb{R}^n)$ the set of functions in $\mathcal{D}(\mathbb{R}^n)$ that are supported on D_+, D_-, and D_0, respectively.

The Schwartz class $\mathcal{S}(\mathbb{R}^n)$ consists of all infinitely differentiable functions φ on $\mathbb{R}^n$ satisfying

$$\sup_{x\in\mathbb{R}^n}\left|x^\alpha D^\beta\varphi(x)\right| < \infty$$

for all $\alpha, \beta \in \mathbb{Z}_+^n$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, α_j $(j = 1, 2, \ldots, n)$ and β_j $(j = 1, 2, \ldots, n)$ are nonnegative integers.

The Fourier transform $\hat{\varphi}$ is a linear homeomorphism from $S(\mathbb{R}^n)$ onto itself. Meanwhile, the following identity holds:

$$(\mathfrak{P}\mathfrak{I}\varphi)^\wedge(x) = (-i)\operatorname{sgn}(x)\hat{\varphi},$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

The Fourier transform $F : \mathbb{S}'(\mathbb{R}^n) \to \mathbb{S}'(\mathbb{R}^n)$ defined as

$$\langle \hat{\nu}, \varphi \rangle = \langle \nu, \hat{\varphi} \rangle$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a linear isomorphism from $\mathbb{S}'(\mathbb{R}^n)$ onto itself. For the detailed properties of $\mathbb{S}(\mathbb{R}^n)$ and $\mathbb{S}'(\mathbb{R}^n)$, we refer the readers to [18, 20].

For $\nu \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it is obvious that

$$\langle \tilde{\tilde{\nu}}, \varphi \rangle = \langle \tilde{\nu}, \hat{\varphi} \rangle = \langle \nu, \check{\tilde{\varphi}} \rangle = \langle \hat{\nu}, \varphi \rangle = \langle \nu, \hat{\varphi} \rangle$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, where

$$\tilde{\varphi}(x) = \varphi(-x)$$

and $\tilde{\nu}$ is defined as follows:

$$\langle \check{\nu}, \varphi \rangle = \langle \nu, \tilde{\varphi} \rangle.$$

So we obtain that

$$\tilde{\tilde{\nu}} = \hat{\nu}$$

in the distributional sense.

Following the definition in [16], a function φ belongs to the space $\mathcal{D}_{L^p}(\mathbb{R}^n)$ $(1 \le p < \infty)$ if and only if

(I) $\varphi \in C^\infty(\mathbb{R}^n)$;

(II) $D^k \varphi \in L^p(\mathbb{R}^n)$ $(k = 1, 2, \ldots)$, where $C^\infty(\mathbb{R}^n)$ consists of infinitely differentiable functions,

$$D^k \varphi(x) = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \varphi(x),$$

where $|k| = k_1 + k_2 + \cdots + k_n$ and $k = (k_1, k_2, \ldots, k_n)$.

In the sequel, we denote by $\mathcal{D}'_{L^p}(\mathbb{R}^n)$ the dual of the corresponding spaces

$$\mathcal{D}_{L^{p'}}(\mathbb{R}^n),$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

As a consequence, we have

$$\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{D}_{L^p}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$$

and

$$L^p(\mathbb{R}^n) \subseteq \mathcal{D}'_{L^p}(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n).$$

Definition 1.2 Let $h \in \mathcal{D}'_{L^p}(\mathbb{R}^n)$, where $1 < p < \infty$. Then the Radon transform of h is defined by (see [8])

$$\langle \mathfrak{P}\mathfrak{I}h, \varphi\rangle = \bigl\langle f, (-1)^n \mathfrak{P}\mathfrak{I}\varphi\bigr\rangle$$

for any $\varphi \in \mathcal{D}_{L^{p'}}(\mathbb{R}^n)$.

In [16], Luan and Vieira proved that the total Radon transform is a linear homeomorphism from $\mathcal{D}_{L^p}(\mathbb{R}^n)$ onto itself, as well as if $h \in \mathcal{D}'_{L^p}(\mathbb{R}^n)$ $(1 < p < \infty)$, then $\mathfrak{P}\mathfrak{I}h \in \mathcal{D}'_{L^p}(\mathbb{R}^n)$ and the Radon transform H defined above is a linear isomorphism from $\mathcal{D}'_{L^p}(\mathbb{R}^n)$ onto itself.

Note that if $\nu \in L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, then we have

$$\begin{aligned}
\bigl\langle (H\nu)^\wedge, \varphi\bigr\rangle &= \langle H\nu, \hat{\varphi}\rangle \\
&= (-1)^n \langle \nu, H\hat{\varphi}\rangle \\
&= (-1)^n \bigl\langle \check{\nu}, (H\hat{\varphi})^\wedge\bigr\rangle \\
&= (-1)^n \bigl\langle \check{\nu}, (-i)^n \operatorname{sgn}(\cdot)\hat{\hat{\varphi}}\bigr\rangle \\
&= \bigl\langle \check{\nu}, (i)^n \operatorname{sgn}(\cdot)\tilde{\varphi}\bigr\rangle \\
&= \bigl\langle \tilde{\check{\nu}}, (i)^n \operatorname{sgn}(\cdot)\varphi\bigr\rangle \\
&= \bigl\langle (-i)^n \operatorname{sgn}(\cdot)\hat{\nu}, \varphi\bigr\rangle
\end{aligned}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

So the following inequality holds:

$$(H\nu)^\wedge(x) = (-i)^n \operatorname{sgn}(\cdot)\hat{\nu}(x)$$

in the distributional sense.

Let Ω be a nonempty subset of $\mathbb{R}$, define (see [16])

$$t\Omega = \{tx : x \in \Omega\},$$

where t is a nonzero real number. Hence we have

$$\operatorname{supp}\left(u\left(\frac{x}{t}\right)\right) = t \operatorname{supp}(u).$$

For a subset $A \subseteq \mathbb{R}$, define

$$A\Omega = \bigcup_{t\in A} t\Omega.$$

2 Main lemmas

In this section, we shall introduce some lemmas.

Lemma 2.1 *Let $h \in L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$ and $g \in \mathcal{S}(\mathbb{R}^n)$. Then the Radon transform of function hg satisfies the Poisson inequality $\mathfrak{P}\mathfrak{I}(hg) \le h\mathfrak{P}\mathfrak{I}g$ if and only if*

$$p.v. \int_{\mathbb{R}^n} \frac{h(x) - h(y)}{\prod_{j=1}^{n}(x_j - y_j)} g(y)\,dy = 0,$$

where $x \in \mathbb{R}^n$.

Proof We have

$$\mathfrak{P}\mathfrak{I}(hg)(x) = \frac{1}{(\pi)^n} p.v. \int_{\mathbb{R}^n} \frac{h(y)g(y)}{\prod_{j=1}^{n}(x_j - y_j)}\,dy$$

and

$$h(x)\mathfrak{P}\mathfrak{I}g(x) = \frac{1}{(\pi)^n} p.v. \int_{\mathbb{R}^n} \frac{h(x)g(y)}{\prod_{j=1}^{n}(x_j - y_j)}\,dy$$

for $x \in \mathbb{R}^n$ from the total Radon transform.

It is clear that the Poisson inequality is satisfied if and only if

$$p.v. \int_{\mathbb{R}^n} \frac{h(x)g(y)}{\prod_{j=1}^{n}(x_j - y_j)}\,dy = p.v. \int_{\mathbb{R}^n} \frac{h(y)g(y)}{\prod_{j=1}^{n}(x_j - y_j)}\,dy.$$

So

$$p.v. \int_{\mathbb{R}^n} \frac{h(x) - h(y)}{\prod_{j=1}^{n}(x_j - y_j)} g(y)\,dy = 0,$$

where $x \in \mathbb{R}^n$. □

We use $W^{k,p}(\mathbb{R})$ to denote the Sobolev space

$$W^{k,p}(\mathbb{R}) = \bigl\{ f \in L^p(\mathbb{R}) : D^m f \in L^p(\mathbb{R}), |m| \le k \bigr\},$$

where the derivative $D^m f$ is understood in the distributional sense.

Lemma 2.2 *Suppose that $1 < p \le 2$. Then, for fixed $x \in \mathbb{R}$, the function*

$$\nu_x(y) = \frac{\mu(x) - \mu(y)}{x - y}$$

for any $y \in \mathbb{R}$ and $\mu \in W^{1,p}(\mathbb{R})$ is in $L^p(\mathbb{R})$ and

$$\hat{\nu}(w) = ie^{-ixw}\int_0^1 \frac{w}{t^2}e^{\frac{ixw}{t}}\hat{\mu}\left(\frac{w}{t}\right)dt.$$

Proof Since $\mu \in W^{1,p}(\mathbb{R})$, we have

$$\nu_x(y) = \int_0^1 \mu'\bigl(ty + (1-t)x\bigr)\,dt.$$

Now we prove that $\nu \in L^p(\mathbb{R})$. We observe that

$$\begin{aligned}
\|\nu\|_p &= \left(\int_{\mathbb{R}}\left\|\int_0^1 \mu'\bigl(ty + (1-t)x\bigr)\,dt\right\|^p\right)^{\frac{1}{p}} \\
&\le \int_0^1\left(\int_{\mathbb{R}}\bigl\|\mu'\bigl(ty + (1-t)x\bigr)\bigr\|^p\,dy\right)^{\frac{1}{p}}dt \\
&= \bigl\|\mu'\bigr\|_p\int_0^1 \frac{1}{\sqrt[p]{t}}\,dt \\
&= p'\bigl\|\mu'\bigr\|_p \\
&< \infty
\end{aligned}$$

for fixed $x \in \mathbb{R}$ by using the generalized Minkowski inequality, which involves that $\nu \in L^p(\mathbb{R})$.

Since (see [9])

$$\nu = \mathfrak{P}\mathfrak{I}(u) = \int_{1/\sqrt{k\sigma}}^{u}\sigma(s)\,ds,$$

it follows that

$$\nabla\nu = \sigma(u)\nabla u = \bigl(ku^2 - 1\bigr)^{1/2}\nabla u,$$

which yields that

$$\nabla u = \bigl(ku^2 - 1\bigr)^{-1/2}\nabla\nu.$$

Thus we have (see [11, 22])

$$\bigl(1 - ku^2\bigr)\nabla u\nabla\varphi = -\bigl(ku^2 - 1\bigr)^{1/2}\nabla\nu\nabla\varphi \tag{2.1}$$

for each $\varphi \in C_0^1(\mathbb{R}^n)$.

On the other hand, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\bigl(ku^2 - 1\bigr)^{1/2}\nabla\nu\nabla\varphi \\
&\quad= \int_{\mathbb{R}^n}\nabla\nu\nabla\bigl\{\bigl(ku^2 - 1\bigr)^{1/2}\varphi\bigr\} - \int_{\mathbb{R}^n}\frac{ku}{ku^2 - 1}|\nabla\nu|^2\varphi
\end{aligned}$$

$$= -\int_{\mathbb{R}^{\mathbb{N}}} a(x)\frac{g(u)}{\sigma(u)}\left(ku^2-1\right)^{1/2}\varphi - \int_{\mathbb{R}^{\mathbb{N}}} ku|\nabla u|^2\varphi$$
$$= -\int_{\mathbb{R}^{\mathbb{N}}} a(x)g(u)\varphi - \int_{\mathbb{R}^{\mathbb{N}}} ku|\nabla u|^2\varphi.$$

So

$$\begin{aligned}\hat{\nu}(w) &= \int_0^1 \left[\mu'\left(ty+(1-t)x\right)\right]^{\wedge}(w)\,dt\\ &= e^{-ixv}\int_0^1 \frac{1}{t}e^{\frac{ixw}{t}}\hat{\mu}'\left(\frac{w}{t}\right)dt\\ &= ie^{-ixv}\int_0^1 \frac{v}{t^2}e^{\frac{ixw}{t}}\hat{\mu}\left(\frac{w}{t}\right)dt\end{aligned}$$

from the definition of $W^{1,p}(\mathbb{R})$, which is the desired result. □

3 Poisson inequality for $W^{1,p}(\mathbb{R})$ functions

In this section, we develop a characterization of $W^{1,p}(\mathbb{R})$ functions which satisfy the Poisson inequality $\mathfrak{P}\mathfrak{I}(hg) \le h\mathfrak{P}\mathfrak{I}g$.

Theorem 3.1 *Let $h \in W^{1,p}(\mathbb{R})$ $(1 < p \le 2)$ and $g \in L^p(\mathbb{R}) \cap L^{p'}(\mathbb{R})$. Then the Radon transform of the function hg satisfies the Poisson inequality $\mathfrak{P}\mathfrak{I}(hg) \le h\mathfrak{P}\mathfrak{I}g$ if and only if*

$$\int_0^1\int_{\mathbb{R}} \frac{w}{t^2}e^{\frac{-iwx(t-1)}{t}}\hat{h}\left(\frac{w}{t}\right)\hat{g}(-w)\,dw\,dt = 0 \tag{3.1}$$

holds.

Proof By Lemma 2.1, we know that $\mathfrak{P}\mathfrak{I}hg \le h\mathfrak{P}\mathfrak{I}g$ holds if and only if

$$p.v.\int_{\mathbb{R}^n} \frac{h(x)-h(y)}{x-y}g(y)\,dy = 0. \tag{3.2}$$

Since $h \in W^{1,p}(\mathbb{R})$, Lemma 2.2 ensures that

$$\frac{h(x)-h(\cdot)}{x-.} \in L^p(\mathbb{R}).$$

Thus (3.2) holds if and only if

$$\int_{\mathbb{R}^n}\left(\frac{h(x)-h(y)}{x-y}\right)^{\wedge}(w)\left(g(y)\right)^{\vee}(w)\,dw = 0,$$

which yields that $\check{g}(w) = \hat{g}(-w)$. It is known that the above equation is equivalent to

$$\int_{\mathbb{R}^n} ie^{-iwx}\int_0^1 \frac{w}{t^2}e^{\frac{iwx}{t}}\hat{h}\left(\frac{w}{t}\right)dt\hat{g}(-w)\,dw = 0$$

from Lemma 2.2.

Let

$$h(t,w) = \frac{w}{t^2} e^{\frac{(iwx)(t-1)}{t}} \hat{h}\left(\frac{w}{t}\right) \hat{g}(-w).$$

Replacing t by $\frac{1}{y}$, we obtain that (see [14])

$$\begin{aligned}
\int_{\mathbb{R}} \int_0^1 \left|h(t,w)\right| dt\,dw &= \int_{\mathbb{R}} \int_1^{\infty} \left|w\hat{h}(wy)\hat{g}(-w)\right| dy\,dw \\
&\le \left(\int_{\mathbb{R}} \int_1^{\infty} \left|y^{-\frac{1+\delta}{p'}} \hat{g}(-w)\right|^p dy\,dw \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_{\mathbb{R}} \int_1^{\infty} \left|wy^{\frac{1+\delta}{p'}} \hat{h}(yw)\right|^{p'} dy\,dw \right)^{\frac{1}{p'}} \\
&= \left(\frac{p'-1}{-p'+\delta+2} \right)^{\frac{1}{p}} \|\hat{g}\|_p \\
&\quad \times \left(\int_{\mathbb{R}} \int_1^{\infty} \left|wy^{\frac{1+\delta}{p'}} \hat{h}(yw)\right|^{p'} dy\,dw \right)^{\frac{1}{p'}} \\
&\le \left(\frac{p'-1}{-p'+\delta+2} \right)^{\frac{1}{p}} \|\hat{g}\|_p \\
&\quad \times \left(\int_{\mathbb{R}} \int_1^{\infty} \left|\lambda\hat{h}(\lambda)\right|^{p'} y^{\delta-p'} dy\,d\lambda \right)^{\frac{1}{p'}} \\
&\le \left(\frac{p'-1}{-p'+\delta+2} \right)^{\frac{1}{p}} \|\hat{g}\|_p \\
&\quad \times \left\| (f')^{\wedge} \right\|_{p'} \left(\frac{1}{p'-\delta-1} \right)^{\frac{1}{p'}} \\
&\le \left(\frac{p'-1}{-p'+\delta+2} \right)^{\frac{1}{p}} \left(\frac{1}{p'-\delta-1} \right)^{\frac{1}{p'}} \|\hat{g}\|_p \\
&\quad \times \left\| (f') \right\|_p \\
&< \infty,
\end{aligned}$$

where

$$\frac{p'}{p} - 1 < \delta < p' - 1.$$

Set (see [13])

$$\begin{aligned}
&\Delta w_\delta = \overline{a}\big(|x|\big) \frac{g(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))}{h(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))} \quad \text{in } \mathbb{R}^n, \\
&w_\delta(0) = \delta, \\
&\lim_{|x|\to\infty} w_\delta(x) = \infty,
\end{aligned}$$

and

$$\Delta w_\zeta = \underline{a}\big(|x|\big)\frac{g(\mathfrak{P}\mathfrak{I}^{-1}(w_\zeta))}{h(\mathfrak{P}\mathfrak{I}^{-1}(w_\zeta))} \quad \text{in } \mathbb{R}^n,$$
$$w_\zeta(0) = \zeta,$$
$$\lim_{|x|\to\infty} w_\zeta(x) = \infty,$$

respectively.

It follows that

$$\begin{aligned}
w_\delta(r) &\le 2\int_0^r \left(\int_0^t \overline{a}(s)\frac{g(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))}{h(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))}\,ds\right)dt \\
&\le 2g\big(\mathfrak{P}\mathfrak{I}^{-1}\big(w_\delta(r)\big)\big)\int_0^r \left(\int_0^t \frac{\overline{a}(s)}{h(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))}\,ds\right)dt \\
&\le 2g\left(\sqrt{2\sqrt{\frac{\varrho}{(\varrho-1)k}}w_\delta(r)+\frac{\varrho}{k}}\right)\int_0^r \left(\int_0^t \frac{\overline{a}(s)}{h(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))}\,ds\right)dt \\
&\le 2g\left(2\sqrt[4]{\frac{\varrho}{(\varrho-1)k}}\sqrt{w_\delta}\right)\int_0^r \left(\int_0^t \frac{\overline{a}(s)}{h(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))}\,ds\right)dt \\
&\le \frac{2}{\sqrt{\varrho-1}}g\left(2\sqrt[4]{\frac{\varrho}{(\varrho-1)k}}\sqrt{w_\delta}\right)\left[r\left(\int_0^r \overline{a}(t)\,dt\right)-\int_0^r t\overline{a}(t)\,dt\right] \\
&\le \frac{2}{\sqrt{\varrho-1}}g\left(2\sqrt[4]{\frac{\varrho}{(\varrho-1)k}}\sqrt{w_\delta}\right)r\int_0^r \overline{a}(t)\,dt
\end{aligned}$$

for all $r > 0$ sufficiently large, which yields that (see [17])

$$2\sqrt[4]{\frac{\varrho}{(\varrho-1)k}}\sqrt{w_\delta} \le \mathcal{G}^{-1}\left(r\int_0^r \overline{a}(t)\,dt\right)$$

for all $r \gg 0$.

Put

$$0 < S(\zeta) = \sup\big\{r > 0 : w_\delta(r) < w_\zeta(r)\big\} \le \infty.$$

So

$$\begin{aligned}
\zeta_0 &\le \delta + \int_0^{S(\zeta_0)} t^{1-N}\left[\int_0^t s^{N-1}\left(\overline{a}(s)\frac{g(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))}{h(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))} - \underline{a}(s)\frac{g(\mathfrak{P}\mathfrak{I}^{-1}(w_\zeta))}{h(\mathfrak{P}\mathfrak{I}^{-1}(w_\zeta))}\right)ds\right]dt \\
&\le \delta + \int_0^{S(\zeta_0)} t^{1-N}\left[\int_0^t s^{N-1}\left(\overline{a}(s)\frac{g(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))}{h(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))}\right.\right. \\
&\quad \left.\left. - \underline{a}(s)\frac{g(\mathfrak{P}\mathfrak{I}^{-1}(w_\zeta))}{\mathfrak{P}\mathfrak{I}^{-1}(w_\zeta)^\delta}\frac{\mathfrak{P}\mathfrak{I}^{-1}(w_\zeta)^\delta}{h(\mathfrak{P}\mathfrak{I}^{-1}(w_\zeta))}\right)ds\right]dt \\
&\le \delta + \int_0^{S(\zeta_0)} t^{1-N}\left[\int_0^t s^{N-1}\left(\overline{a}(s)\frac{g(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))}{h(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))} - \underline{a}(s)\frac{g(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))}{h(\mathfrak{P}\mathfrak{I}^{-1}(w_\delta))}\right)ds\right]dt. \qquad (3.3)
\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
0 &\le t^{1-N}\left[\int_0^t s^{N-1}\left(\overline{a}(s)\frac{g(\mathfrak{PI}^{-1}(w_\delta))}{h(\mathfrak{PI}^{-1}(w_\delta))}-\underline{a}(s)\frac{g(\mathfrak{PI}^{-1}(w_\delta))}{h(\mathfrak{PI}^{-1}(w_\delta))}\right)ds\right]\chi_{[0,S(\zeta)]}(t) \\
&= t^{1-N}\left[\int_0^t s^{N-1}a_{\mathrm{osc}}(s)\frac{g(\mathfrak{PI}^{-1}(w_\delta))}{h(\mathfrak{PI}^{-1}(w_\delta))}\,ds\right] \\
&\le \frac{1}{\sqrt{\varrho-1}}\left(t^{1-N}\int_0^t s^{N-1}a_{\mathrm{osc}}(s)\,ds\right)g\left(\mathcal{G}^{-1}\left(t\int_0^t \overline{a}(s)\,ds\right)\right) := \mathcal{H}(t)
\end{aligned}
$$

for $t \gg 0$, where $\chi_{[0,S(\zeta)]}$ stands for the characteristic function of $[0,S(\zeta)]$, which yields that (see [12])

$$
\zeta_0 \le \delta + \int_0^\infty \mathcal{H}(s)\,ds \le \delta + \overline{H},
$$

but this is impossible.

Consider the following problem (see [15]):

$$
\begin{aligned}
&\Delta w = a(x)\frac{g(\mathfrak{PI}^{-1}(w))}{h(\mathfrak{PI}^{-1}(w))} \quad \text{in } B_n(0), \\
&w \ge 0 \quad \text{in } B_n(0), \\
&w = w_\delta \quad \text{on } \partial B_n(0).
\end{aligned}
\tag{3.4}
$$

As a consequence, we get

$$
\int_0^1 \int_{\mathbb{R}} \frac{w}{t^2} e^{\frac{-iwx(t-1)}{t}}\hat{h}\left(\frac{w}{t}\right)\hat{g}(-w)\,dw\,dt = 0
$$

by using the Fubini theorem.

This completes the proof. $\square$

Now we give an application of Theorem 3.1.

Theorem 3.2 *Let $h \in W^{1,p}(\mathbb{R})$ $(1 < p \le 2)$ and $g \in \mathfrak{PI}^p(\mathbb{R}) \cap \mathfrak{PI}^{p'}(\mathbb{R})$. If*

$$
\left(I^- \operatorname{supp}\hat{h}\right) \cap \operatorname{supp}\hat{g} = \emptyset,
\tag{3.5}
$$

where $I^- = [-1,0)$, then the Poisson inequality $\mathfrak{PI}(hg) \le h\mathfrak{PI}g$ holds.

Proof By condition (3.5), we obtain that (see [4])

$$
(t \operatorname{supp}\hat{h}) \cap \operatorname{supp}\hat{g} = \emptyset
$$

for any $t \in I^-$, which is equivalent to

$$
\operatorname{supp}\hat{h}\left(\frac{\cdot}{t}\right) \cap \operatorname{supp}\hat{g} = \emptyset
$$

for any $t \in I^-$.

By the embedding theorem and Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{A_{k_{j+1},j+1}} \bigl(h(u)-k_{j+1}\bigr)_+\,dx \\
&\quad\le \left(\int_{A_{k_{j+1},j}} \bigl(\bigl(h(u)-k_{j+1}\bigr)_+\zeta_j^q\bigr)^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}} |A_{k_{j+1},j+1}|^{1/n} \\
&\quad\le \gamma \int_{A_{k_{j+1},j}} \bigl|\nabla\bigl(\bigl(h(u)-k_{j+1}\bigr)_+\zeta_j^q\bigr)\bigr| |A_{k_{j+1},j}|^{1/n} \\
&\quad\le \gamma\left(\int_{A_{k_{j+1},j}} g(u)|\nabla u|\zeta_j^q\,dx\right. \\
&\qquad\left.+\int_{A_{k_{j+1},j}} \bigl(h(u)-k_{j+1}\bigr)_+|\nabla\zeta_j|\zeta_j^{q-1}\,dx\right)|A_{k_{j+1},j}|^{1/n}.
\end{aligned}\tag{3.6}$$

Let $\ell=\delta(\rho)/\rho$. We estimate the first term on the right-hand side of (3.6) as follows:

$$\begin{aligned}
&\int_{A_{k_{j+1},j}} g(u)|\nabla u|\zeta_j^q\,dx \\
&\quad=\frac{1}{g(\ell)}\int_{A_{k_{j+1},j}} g(u)g(\ell)|\nabla u|\zeta_j^q\,dx \\
&\quad\le \ell\int_{A_{k_{j+1},j}} g(u)\zeta_j^q\,dx+\frac{1}{g(\ell)}\int_{A_{k_{j+1},j}} g(u)G\bigl(|\nabla u|\bigr)\zeta_j^q\,dx \\
&\quad\le 2^j\frac{\ell}{k}\int_{A_{k_{j+1},j}} \bigl(h(u)-k_{j+1}\bigr)_+g(u)\zeta_j^q\,dx+\frac{1}{g(\ell)}\int_{A_{k_{j+1},j}} g(u)G\bigl(|\nabla u|\bigr)\zeta_j^q\,dx.
\end{aligned}\tag{3.7}$$

It follows that

$$\begin{aligned}
&\int_{A_{k_{j+1},j}} g(u)|\nabla u|\zeta_j^q\,dx \\
&\quad\le \gamma(1-\varrho)^{-\gamma}2^{j\gamma}\left(\frac{\ell}{k}+\frac{1}{g(\ell)}\right)\rho^{-1}g\left(\frac{\delta(\rho)}{\rho}\right)\int_{A_{k_j,j}} \bigl(h(u)-k_j\bigr)_+\,dx
\end{aligned}\tag{3.8}$$

from the previous inequality and Lemma 2.2.

Since

$$k\ge G(\ell)=G\left(\frac{\delta(\rho)}{\rho}\right),\tag{3.9}$$

we obtain

$$y_{j+1}=\int_{A_{k_{j+1},j+1}} \bigl(h(u)-k_{j+1}\bigr)\,dx\le \gamma(1-\varrho)^{-\gamma}2^{j\gamma}\rho^{-1}k^{-\frac{1}{n}}y_j^{1+\frac{1}{n}}\tag{3.10}$$

from (3.6) and (3.7), which gives that

$$k\ge \gamma(1-\varrho)^{-\gamma}\rho^{-n}\int_{B_{\frac{1-\varrho}{2}\rho}(\bar{x})} h(u)\,dx.\tag{3.11}$$

(3.8) and (3.9) also imply that

$$h\big(u(\bar{x})\big) \le \gamma(1-\varrho)^{-\gamma} G\left(\frac{\delta(\rho)}{\rho}\right) + \gamma(1-\varrho)^{-\gamma}\rho^{-n}\int_{B_{\frac{1-\varrho}{2}\rho}(\bar{x})} h(u)\,dx. \tag{3.12}$$

Since

$$\begin{aligned}\int_{B_{\frac{1-\varrho}{2}\rho}(\bar{x})} h(u)\,dx &\le \delta(\rho)\int_{B_{(1-\varrho)\rho}(\bar{x})} g(u)\xi^q\,dx \\ &\le \gamma(1-\varrho)^{-1}\frac{\delta(\rho)}{\rho}\int_{B_{(1-\varrho)\rho}(\bar{x})} g\big(|\nabla u|\big)\xi^{q-1}\,dx,\end{aligned}$$

we obtain that

$$\begin{aligned}\int_{B_{\frac{1-\varrho}{2}\rho}(\bar{x})} h(u)\,dx &\le \gamma(1-\varrho)^{-1}\frac{\delta(\rho)}{M(\rho)}\int_{B_{(1-\varrho)\rho}(\bar{x})} G\big(|\nabla u|\big)\xi^q\,dx \\ &\quad + \gamma(1-\varrho)^{-1}\frac{\delta(\rho)}{\rho} g\left(\frac{M(\rho)}{\rho}\right)\rho^n \end{aligned}\tag{3.13}$$

and

$$\int_{B_{(1-\varrho)\rho}(\bar{x})} G\big(|\nabla u|\big)\xi^q\,dx \le \gamma(1-\varrho)^{-\gamma} G\left(\frac{M(\rho)}{\rho}\right)\rho^n. \tag{3.14}$$

Combining (3.13) and (3.14), we have

$$\int_{B_{(1-\varrho)\rho}(\bar{x})} h(u)\,dx \le \gamma(1-\varrho)^{-\gamma}\frac{\delta(\rho)}{\rho} g\left(\frac{M(\rho)}{\rho}\right)\rho^n. \tag{3.15}$$

As a consequence, (3.1) holds. Thus, by invoking Theorem 3.1, the Radon transform of function hg satisfies the Poisson inequality

$$\mathfrak{P}\mathfrak{I}(hg) \le h\mathfrak{P}\mathfrak{I}g. \qquad \square$$

4 Conclusions

This paper was mainly devoted to studying a new Poisson inequality for the Radon transform of infinitely differentiable functions. An application of it was also given.

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