Mondal and Akel *Journal of Inequalities and Applications* (2018) 2018:175
https://doi.org/10.1186/s13660-018-1772-1

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Differential equation and inequalities of the generalized $\mathtt{k}$-Bessel functions

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Abstract

In this paper, we introduce and study a generalization of the $\mathtt{k}$-Bessel function of order ν given by

$$\mathtt{W}^{\mathtt{k}}_{\nu,c}(x) := \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_{\mathtt{k}}(r\mathtt{k}+\nu+\mathtt{k})r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathtt{k}}}.$$

We also indicate some representation formulae for the function introduced. Further, we show that the function $\mathtt{W}^{\mathtt{k}}_{\nu,c}$ is a solution of a second-order differential equation. We investigate monotonicity and log-convexity properties of the generalized $\mathtt{k}$-Bessel function $\mathtt{W}^{\mathtt{k}}_{\nu,c}$, particularly, in the case $c = -1$. We establish several inequalities, including a Turán-type inequality. We propose an open problem regarding the pattern of the zeroes of $\mathtt{W}^{\mathtt{k}}_{\nu,c}$.

MSC: 33B15; 33C10; 26A48; 26D07

Keywords: $\mathtt{k}$-Gamma; $\mathtt{k}$-Bessel; Turán-type inequality; Monotonicity

1 Introductions

Motivated with the repeated appearance of the expression

$$x(x+\mathtt{k})(x+2\mathtt{k})\cdots\bigl(x+(n-1)\mathtt{k}\bigr)$$

in the combinatorics of creation and annihilation operators [13, 14] and the perturbative computation of Feynman integrals (see [12]), a generalization of the well-known Pochhammer symbols is given in [15] as

$$(x)_{n,\mathtt{k}} := x(x+\mathtt{k})(x+2\mathtt{k})\cdots\bigl(x+(n-1)\mathtt{k}\bigr),$$

for all $\mathtt{k} > 0$, calling it the Pochhammer $\mathtt{k}$-symbol. Closely associated functions that have relation with the Pochhammer symbols are the gamma and beta functions. Hence it is useful to recall some facts about the $\mathtt{k}$-gamma and $\mathtt{k}$-beta functions. The $\mathtt{k}$-gamma function, denoted as $\Gamma_{\mathtt{k}}$, is studied in [15] and defined by

$$\Gamma_{\mathtt{k}}(x) := \int_0^{\infty} t^{x-1} e^{-\frac{t^{\mathtt{k}}}{\mathtt{k}}}\,dt \tag{1.1}$$

for $\mathrm{Re}(x) > 0$. Several properties of the $\mathtt{k}$-gamma functions and applications in generalizing other related functions like $\mathtt{k}$-beta and $\mathtt{k}$-digamma functions can be found in [15, 27, 28] and references therein.

The k-digamma functions defined by $\Psi_{\mathtt{k}} := \Gamma'_{\mathtt{k}}/\Gamma_{\mathtt{k}}$ are studied in [28]. These functions have the series representation

$$\Psi_{\mathtt{k}}(t) := \frac{\log(\mathtt{k}) - \gamma_1}{\mathtt{k}} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{n\mathtt{k}(n\mathtt{k}+t)}, \tag{1.2}$$

where γ_1 is the Euler–Mascheroni constant.

A calculation yields

$$\Psi'_{\mathtt{k}}(t) = \sum_{n=0}^{\infty} \frac{1}{(n\mathtt{k}+t)^2}, \quad \mathtt{k} > 0 \text{ and } t > 0. \tag{1.3}$$

Clearly, $\Psi_{\mathtt{k}}$ is increasing on $(0,\infty)$.

The Bessel function of order p given by

$$\mathtt{J}_p(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+p+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+p} \tag{1.4}$$

is a particular solution of the Bessel differential equation

$$x^2 y''(x) + xy'(x) + \bigl(x^2 - p^2\bigr) y(x) = 0. \tag{1.5}$$

Here Γ denotes the gamma function. A solution of the modified Bessel equation

$$x^2 y''(x) + xy'(x) - \bigl(x^2 + \nu^2\bigr) y(x) = 0, \tag{1.6}$$

is the modified Bessel function

$$\mathtt{I}_\nu(x) := \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\nu+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+\nu}. \tag{1.7}$$

The Bessel function has several generalizations (see, e.g., [9, 10]) and is notably investigated in [1, 17]. In [1], a generalized Bessel function is defined in the complex plane, and sufficient conditions for it to be univalent, starlike, close-to-convex, or convex are obtained. This generalization is given by the power series

$$\mathcal{W}_{p,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k (\frac{z}{2})^{2k+p+1}}{\Gamma(k+1)\Gamma(k+p+\frac{b+2}{2})}, \quad p,b,c \in \mathbb{C}. \tag{1.8}$$

In this paper, we consider the function defined by the series

$$\mathtt{W}^{\mathtt{k}}_{\nu,c}(x) := \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_{\mathtt{k}}(r\mathtt{k}+\nu+\mathtt{k})r!} \left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathtt{k}}}, \tag{1.9}$$

where $\mathtt{k} > 0$, $\nu > -1$, and $c \in \mathbb{R}$. As $\mathtt{k} \to 1$, the $\mathtt{k}$-Bessel function $\mathtt{W}^{1}_{\nu,1}$ is reduced to the classical Bessel function J_ν, whereas $\mathtt{W}^{1}_{\nu,-1}$ coincides with the modified Bessel function I_ν. Thus, we call the function $\mathtt{W}^{\mathtt{k}}_{\nu,c}$ the generalized $\mathtt{k}$-Bessel function. Basic properties of the $\mathtt{k}$-Bessel and related functions can be found in recent works [8, 19–21].

Turán [30] proved that the Legendre polynomials $P_n(x)$ satisfy the determinantal inequality

$$\begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n+1}(x) & P_{n+2}(x) \end{vmatrix} \le 0, \quad -1 \le x \le 1, \tag{1.10}$$

where $n = 0, 1, 2, \dots$, and the equality occurs only for $x = \pm 1$. The inequalities similar to (1.10) can be found in the literature [2, 3, 5, 11, 16, 25] for several other functions, for example, ultraspherical polynomials, Laguerre and Hermite polynomials, Bessel functions of the first kind, modified Bessel functions, and the polygamma function. Karlin and Szegö [24] named determinants in (1.10) as Turánians. More details about Turánians can be found in [5, 11, 18, 22, 23, 29].

The aim of this paper is to investigate the influence of the $\Gamma_{\mathtt{k}}$ functions on the properties of the $\mathtt{k}$-Bessel function defined in (1.9). It is shown that the properties of the classical Bessel functions can be extended to the $\mathtt{k}$-Bessel functions. Moreover, we investigate the effects of $\Gamma_{\mathtt{k}}$ instead of Γ on the monotonicity and log-convexity properties and related inequalities of the k-Bessel functions. The outcomes of our investigation are presented as follows.

In Section 2, we derive representation formulae and some recurrence relations for $\mathtt{W}^{\mathtt{k}}_{\nu,c}$. More importantly, the function $\mathtt{W}^{\mathtt{k}}_{\nu,c}$ is shown to be a solution of a certain differential equation of second order, which contains (1.5) and (1.6) for the particular case $\mathtt{k} = 1$ and for particular values of c. At the end of Section 2, we give two types of integral representations for $\mathtt{W}^{\mathtt{k}}_{\nu,c}$.

Section 3 is devoted to the investigation of monotonicity and log-convexity properties of the functions $\mathtt{W}^{\mathtt{k}}_{\nu,c}$ and to relation between two $\mathtt{k}$-Bessel functions of different order. As a consequence, we deduce Turán-type inequalities.

In Section 4, we give concluding remarks and list two tables for the zeroes of $\mathtt{W}^{\mathtt{k}}_{\nu,c}$, leading to an open problem for future studies.

2 Representations for the $\mathtt{k}$-Bessel function

2.1 The $\mathtt{k}$-Bessel differential equation

In this section, we find differential equations corresponding to the functions $\mathtt{W}^{\mathtt{k}}_{\nu,c}$.

Proposition 2.1 *Let* $\mathtt{k} > 0$ *and* $\nu > -k$. *Then the function* $\mathtt{W}^{\mathtt{k}}_{\nu,c}$ *is a solution of the homogeneous differential equation*

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \frac{1}{\mathtt{k}^2}\bigl(cx^2 \mathtt{k} - \nu^2\bigr) y = 0. \tag{2.1}$$

Proof Differentiating both sides of (1.9) with respect to x, it follows that

$$\frac{d}{dx}\mathtt{W}^{\mathtt{k}}_{\nu,c}(x) = \sum_{r=0}^{\infty} \frac{(-c)^r (2r + \frac{\nu}{\mathtt{k}})}{\Gamma_{\mathtt{k}}(r\mathtt{k} + \nu + \mathtt{k}) r!} \left(\frac{x^{2r+\frac{\nu}{\mathtt{k}}-1}}{2^{2r+\frac{\nu}{\mathtt{k}}}} \right).$$

This implies

$$x\frac{d}{dx}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x)=\sum_{r=0}^{\infty}\frac{(-c)^r(2r+\frac{\nu}{\mathrm{k}})}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu+\mathrm{k})r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathrm{k}}}. \tag{2.2}$$

Now differentiating (2.2) with respect to x and then using the property $\Gamma_{\mathrm{k}}(z+\mathrm{k})=z\Gamma_{\mathrm{k}}(z)$ of the k-gamma function yield

$$\begin{aligned}
&x^2\frac{d^2}{dx^2}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x)+x\frac{d}{dx}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x)\\
&\quad=\sum_{r=0}^{\infty}\frac{(-c)^r(2r+\frac{\nu}{\mathrm{k}})^2}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu+\mathrm{k})r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathrm{k}}}\\
&\quad=\sum_{r=1}^{\infty}\frac{(-c)^r4r(r+\frac{\nu}{\mathrm{k}})}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu+\mathrm{k})r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathrm{k}}}\\
&\qquad+\frac{\nu^2}{\mathrm{k}^2}\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu+\mathrm{k})r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathrm{k}}}\\
&\quad=\frac{4}{\mathrm{k}}\sum_{r=1}^{\infty}\frac{(-c)^r}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu)(r-1)!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathrm{k}}}+\frac{\nu^2}{\mathrm{k}^2}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x)\\
&\quad=-\frac{cx^2}{\mathrm{k}}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x)+\frac{\nu^2}{\mathrm{k}^2}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x).
\end{aligned}$$

A further simplification leads to the differential equation (2.1). □

2.2 Recurrence relations

From (2.2) we have

$$\begin{aligned}
x\frac{d}{dx}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x)&=\frac{1}{\mathrm{k}}\sum_{r=0}^{\infty}\frac{(-c)^r(2r\mathrm{k}+\nu)}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu+\mathrm{k})r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathrm{k}}}\\
&=\frac{\nu}{\mathrm{k}}\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu+\mathrm{k})r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathrm{k}}}\\
&\quad+2\sum_{r=1}^{\infty}\frac{(-c)^r}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu+\mathrm{k})(r-1)!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathrm{k}}}\\
&=\frac{\nu}{\mathrm{k}}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x)+2\sum_{r=0}^{\infty}\frac{(-c)^{r+1}}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu+2\mathrm{k})r!}\left(\frac{x}{2}\right)^{2r+2+\frac{\nu}{\mathrm{k}}}\\
&=\frac{\nu}{\mathrm{k}}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x)-xc\mathrm{W}^{\mathrm{k}}_{\nu+\mathrm{k},c}(x).
\end{aligned}$$

Thus we have the difference equation

$$x\frac{d}{dx}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x)=\frac{\nu}{\mathrm{k}}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x)-xc\mathrm{W}^{\mathrm{k}}_{\nu+\mathrm{k},c}(x). \tag{2.3}$$

Again, rewrite (2.2) as

$$\begin{aligned}
x\frac{d}{dx}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x) &= \frac{1}{\mathrm{k}}\sum_{r=0}^{\infty}\frac{(-c)^r(2r\mathrm{k}+2\nu)-\nu}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu+\mathrm{k})r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathrm{k}}} \\
&= -\frac{\nu}{\mathrm{k}}\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu+\mathrm{k})r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathrm{k}}} + 2\sum_{r=0}^{\infty}\frac{(-c)^r(r\mathrm{k}+\nu)}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu+\mathrm{k})r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathrm{k}}} \\
&= -\frac{\nu}{\mathrm{k}}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x) + \frac{x}{\mathrm{k}}\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu-\mathrm{k}+\mathrm{k})r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu-\mathrm{k}}{\mathrm{k}}} \\
&= -\frac{\nu}{\mathrm{k}}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x) + \frac{x}{\mathrm{k}}\mathrm{W}^{\mathrm{k}}_{\nu-\mathrm{k},c}(x).
\end{aligned}$$

This gives us the second difference equation

$$x\frac{d}{dx}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x) = \frac{x}{\mathrm{k}}\mathrm{W}^{\mathrm{k}}_{\nu-\mathrm{k},c}(x) - \frac{\nu}{\mathrm{k}}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x). \tag{2.4}$$

Thus (2.3) and (2.4) lead to the following recurrence relations.

Proposition 2.2 *Let* $\mathrm{k} > 0$ *and* $\nu > -\mathrm{k}$. *Then*

$$2\nu\mathrm{W}^{\mathrm{k}}_{\nu,c}(x) = x\mathrm{W}^{\mathrm{k}}_{\nu-\mathrm{k},c}(x) + xc\mathrm{k}\mathrm{W}^{\mathrm{k}}_{\nu+\mathrm{k},c}(x), \tag{2.5}$$

$$\mathrm{W}^{\mathrm{k}}_{\nu-\mathrm{k},c}(x) = \frac{2}{x}\sum_{r=0}^{\infty}(-1)^r(\nu+2r\mathrm{k})\mathrm{W}^{\mathrm{k}}_{\nu+2r\mathrm{k},c}(x), \tag{2.6}$$

$$\frac{d}{dx}\left(x^{\frac{\nu}{\mathrm{k}}}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x)\right) = \frac{x^{\frac{\nu}{\mathrm{k}}}}{\mathrm{k}}\mathrm{W}^{\mathrm{k}}_{\nu-\mathrm{k},c}(x), \tag{2.7}$$

$$\frac{d}{dx}\left(x^{-\frac{\nu}{\mathrm{k}}}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x)\right) = -cx^{-\frac{\nu}{\mathrm{k}}}\mathrm{W}^{\mathrm{k}}_{\nu+\mathrm{k},c}(x), \tag{2.8}$$

$$\frac{d^m}{dx^m}\left(\mathrm{W}^{\mathrm{k}}_{\nu,c}(x)\right) = \frac{1}{2^m\mathrm{k}^m}\sum_{n=0}^{m}(-1)^n\binom{m}{n}c^n\mathrm{k}^n\mathrm{W}^{\mathrm{k}}_{\nu-m\mathrm{k}+2n\mathrm{k},c}(x) \quad \textit{for all } m\in\mathbb{N}. \tag{2.9}$$

Proof Relation (2.5) follows by subtracting (2.4) from (2.3).

Next to establish (2.6), let us rewrite (2.5) as

$$\mathrm{W}^{\mathrm{k}}_{\nu-\mathrm{k},c}(x) + c\mathrm{k}\mathrm{W}^{\mathrm{k}}_{\nu+\mathrm{k},c}(x) = 2\frac{\nu}{x}\mathrm{W}^{\mathrm{k}}_{\nu,c}(x). \tag{2.10}$$

Now multiply both sides of (2.10) by $-c\mathrm{k}$ and replace ν by $\nu+2\mathrm{k}$. Then we have

$$-c\mathrm{k}\mathrm{W}^{\mathrm{k}}_{\nu+\mathrm{k},c}(x) - c^2\mathrm{k}^2\mathrm{W}^{\mathrm{k}}_{\nu+3\mathrm{k},c}(x) = -2c\mathrm{k}\frac{\nu+2\mathrm{k}}{x}\mathrm{W}^{\mathrm{k}}_{\nu+2\mathrm{k},c}(x). \tag{2.11}$$

Similarly, multiplying both sides of (2.10) by $c^2\mathrm{k}^2$ and replacing ν by $\nu+4\mathrm{k}$ give

$$c^2\mathrm{k}^2\mathrm{W}^{\mathrm{k}}_{\nu+3\mathrm{k},c}(x) + c^3\mathrm{k}^3\mathrm{W}^{\mathrm{k}}_{\nu+5\mathrm{k},c}(x) = 2c^2\mathrm{k}^2\frac{\nu+4\mathrm{k}}{x}\mathrm{W}^{\mathrm{k}}_{\nu+4\mathrm{k},c}(x). \tag{2.12}$$

Continuing and adding them lead to (2.6).

From definition (1.9) it is clear that

$$x^{\frac{\nu}{k}}\mathtt{W}^{k}_{\nu,c}(x)=\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_k(rk+\nu+k)2^{2r+\frac{\nu}{k}}r!}(x)^{2r+\frac{2\nu}{k}}. \tag{2.13}$$

The derivative of (2.13) with respect to x is

$$\begin{aligned}\frac{d}{dx}\bigl(x^{\frac{\nu}{k}}\mathtt{W}^{k}_{\nu,c}(x)\bigr)&=\sum_{r=0}^{\infty}\frac{(-c)^r(2r+\frac{2\nu}{k})}{\Gamma_k(rk+\nu+k)2^{2r+\frac{\nu}{k}}r!}(x)^{2r+\frac{2\nu}{k}-1}\\&=\frac{x^{\frac{\nu}{k}}}{k}\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_k(rk+\nu)r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{k}-1}\\&=\frac{x^{\frac{\nu}{k}}}{k}\mathtt{W}^{k}_{\nu-k,c}(x).\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{d}{dx}\bigl(x^{-\frac{\nu}{k}}\mathtt{W}^{k}_{\nu,c}(x)\bigr)&=\sum_{r=1}^{\infty}\frac{(-c)^r2r}{\Gamma_k(rk+\nu+k)2^{2r+\frac{\nu}{k}}r!}(x)^{2r-1}\\&=x^{-\frac{\nu}{k}}\sum_{r=1}^{\infty}\frac{(-c)^r}{\Gamma_k(rk+\nu+k)(r-1)!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{k}-1}\\&=x^{-\frac{\nu}{k}}\sum_{r=0}^{\infty}\frac{(-c)^{r+1}}{\Gamma_k(rk+\nu+2k)r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{k}+1}\\&=-cx^{-\frac{\nu}{k}}\mathtt{W}^{k}_{\nu+k,c}(x).\end{aligned}$$

Identity (2.9) can be proved by using mathematical induction on m. Recall that

$$\binom{r}{r}=\binom{r}{0}=1$$

and

$$\binom{r}{n}+\binom{r}{n-1}=\binom{r+1}{n}.$$

For $m=1$, the proof of identity (2.9) is equivalent to showing that

$$2k\frac{d}{dx}\mathtt{W}^{k}_{\nu,c}(x)=\mathtt{W}^{k}_{\nu-k,c}(x)-ck\mathtt{W}^{k}_{\nu+k,c}(x). \tag{2.14}$$

This relation can be obtained by simply adding (2.3) and (2.4). Thus, identity (2.9) holds for $m=1$.

Assume that identity (2.9) also holds for any $m=r\geq 2$, that is,

$$\frac{d^r}{dx^r}\bigl(\mathtt{W}^{k}_{\nu,c}(x)\bigr)=\frac{1}{2^mk^r}\sum_{n=0}^{r}(-1)^n\binom{r}{n}c^nk^n\mathtt{W}^{k}_{\nu-rk+2nk,c}(x).$$

This implies, for $m = r + 1$,

$$
\begin{aligned}
&\frac{d^{r+1}}{dx^{r+1}}\left(\mathtt{W}^{\mathtt{k}}_{\nu,c}(x)\right)\\
&= \frac{1}{2^r\mathtt{k}^r}\sum_{n=0}^{r}(-1)^n\binom{r}{n}c^n\mathtt{k}^n\frac{d}{dr}\mathtt{W}^{\mathtt{k}}_{\nu-r\mathtt{k}+2n\mathtt{k},c}(x)\\
&= \frac{1}{2^{r+1}\mathtt{k}^{r+1}}\sum_{n=0}^{r}(-1)^n\binom{r}{n}c^n\mathtt{k}^n\left(\mathtt{W}^{\mathtt{k}}_{\nu-(r+1)\mathtt{k}+2n\mathtt{k},c}(x) - c\mathtt{k}\mathtt{W}^{\mathtt{k}}_{\nu-(r-1)\mathtt{k}+2n\mathtt{k},c}(x)\right)\\
&= \frac{1}{2^{r+1}\mathtt{k}^{r+1}}\sum_{n=0}^{r}(-1)^n\binom{r}{n}c^n\mathtt{k}^n\mathtt{W}^{\mathtt{k}}_{\nu-(r+1)\mathtt{k}+2n\mathtt{k},c}(x)\\
&\quad - \frac{1}{2^{r+1}\mathtt{k}^{r+1}}\sum_{n=0}^{r}(-1)^n\binom{r}{n}c^{n+1}\mathtt{k}^{n+1}\mathtt{W}^{\mathtt{k}}_{\nu-(r-1)\mathtt{k}+2n\mathtt{k},c}(x)\\
&= \frac{1}{2^{r+1}\mathtt{k}^{r+1}}\Bigg[\mathtt{W}^{\mathtt{k}}_{\nu-(r+1)\mathtt{k},c}(x) + \sum_{n=1}^{r}(-1)^r\left(\binom{r}{n}+\binom{r}{n-1}\right)\mathtt{W}^{\mathtt{k}}_{\nu-(r+1)\mathtt{k}+2n\mathtt{k},c}(x)\\
&\quad - (-1)^r c^{r+1}\mathtt{k}^{r+1}\mathtt{W}^{\mathtt{k}}_{\nu+(r+1)\mathtt{k},c}(x)\Bigg]\\
&= \frac{1}{2^{r+1}\mathtt{k}^{r+1}}\Bigg[\binom{r+1}{0}\mathtt{W}^{\mathtt{k}}_{\nu-(r+1)\mathtt{k},c}(x)\\
&\quad + \sum_{n=1}^{r}(-1)^r\binom{r+1}{n}\mathtt{W}^{\mathtt{k}}_{\nu-(r+1)\mathtt{k}+2n\mathtt{k},c}(x)\\
&\quad + (-1)^{r+1}\binom{r+1}{r+1}c^{r+1}\mathtt{k}^{r+1}\mathtt{W}^{\mathtt{k}}_{\nu-(r+1)\mathtt{k}+2(r+1)\mathtt{k},c}(x)\Bigg]\\
&= \frac{1}{2^{r+1}\mathtt{k}^{r+1}}\sum_{n=0}^{r+1}(-1)^r\binom{r+1}{n}\mathtt{W}^{\mathtt{k}}_{\nu-(r+1)\mathtt{k}+2n\mathtt{k},c}(x).
\end{aligned}
$$

Hence, identity (2.9) is concluded by the mathematical induction on m. □

2.3 Integral representations of k-Bessel functions

Now we will derive two integral representations of the functions $\mathtt{W}^{\mathtt{k}}_{\nu,c}$. For this purpose, we need to recall the k-Beta functions from [15]. The k version of the beta functions is defined by

$$
\mathtt{B}_{\mathtt{k}}(x,y) = \frac{\Gamma_{\mathtt{k}}(x)\Gamma_{\mathtt{k}}(y)}{\Gamma_{\mathtt{k}}(x+y)} = \frac{1}{\mathtt{k}}\int_0^1 t^{\frac{x}{\mathtt{k}}-1}(1-t)^{\frac{y}{\mathtt{k}}-1}\,dt. \tag{2.15}
$$

Substituting t by t^2 on the integral in (2.15), it follows that

$$
\mathtt{B}_{\mathtt{k}}(x,y) = \frac{2}{\mathtt{k}}\int_0^1 t^{\frac{2x}{\mathtt{k}}-1}\left(1-t^2\right)^{\frac{y}{\mathtt{k}}-1}\,dt. \tag{2.16}
$$

Let $x = (r+1)\mathrm{k}$ and $y = \nu$. Then from (2.15) and (2.16) we have

$$\frac{1}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu+\mathrm{k})} = \frac{2}{\Gamma_{\mathrm{k}}((r+1)\mathrm{k})\Gamma_{\mathrm{k}}(\nu)} \int_0^1 t^{2r+1}\left(1-t^2\right)^{\frac{\nu}{\mathrm{k}}-1} dt. \tag{2.17}$$

According to [15], we have the identity $\Gamma_{\mathrm{k}}(\mathrm{k}x) = \mathrm{k}^{x-1}\Gamma(x)$. This gives

$$\frac{1}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu+\mathrm{k})} = \frac{2}{\mathrm{k}^r\Gamma(r+1)\Gamma_{\mathrm{k}}(\nu)} \int_0^1 t^{2r+1}\left(1-t^2\right)^{\frac{\nu}{\mathrm{k}}-1} dt. \tag{2.18}$$

Now (1.9) and (2.18) together yield the first integral representation

$$\begin{aligned}
\mathtt{W}^{\mathrm{k}}_{\nu,c}(x) &= \frac{2}{\Gamma_{\mathrm{k}}(\nu)}\left(\frac{x}{2}\right)^{\frac{\nu}{\mathrm{k}}} \int_0^1 t\left(1-t^2\right)^{\frac{\nu}{\mathrm{k}}-1} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma(r+1)r!}\left(\frac{xt}{2\sqrt{\mathrm{k}}}\right)^{2r} dt \\
&= \frac{2}{\Gamma_{\mathrm{k}}(\nu)}\left(\frac{x}{2}\right)^{\frac{\nu}{\mathrm{k}}} \int_0^1 t\left(1-t^2\right)^{\frac{\nu}{\mathrm{k}}-1} \mathcal{W}_{0,1,c}\left(\frac{xt}{\sqrt{\mathrm{k}}}\right) dt,
\end{aligned} \tag{2.19}$$

where $\mathcal{W}_{p,b,c}$ is defined in (1.8).

For the second integral representation, substitute $x = r + \mathrm{k}/2$ and $y = \nu + \mathrm{k}/2$ into (2.16). Then (2.17) can be rewritten as

$$\frac{1}{\Gamma_{\mathrm{k}}(r\mathrm{k}+\nu+\mathrm{k})} = \frac{2}{\Gamma_{\mathrm{k}}((r+\frac{1}{2})\mathrm{k})\Gamma_{\mathrm{k}}(\nu+\frac{\mathrm{k}}{2})} \int_0^1 t^{2r}\left(1-t^2\right)^{\frac{\nu}{\mathrm{k}}-\frac{1}{2}} dt. \tag{2.20}$$

Again, the identity $\Gamma_{\mathrm{k}}(\mathrm{k}x) = \mathrm{k}^{x-1}\Gamma(x)$ yields

$$\Gamma_{\mathrm{k}}\left(\left(r+\frac{1}{2}\right)\mathrm{k}\right) = \mathrm{k}^{r-\frac{1}{2}}\Gamma\left(r+\frac{1}{2}\right). \tag{2.21}$$

Further, the Legendre duplication formula (see [4, 6])

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) \tag{2.22}$$

shows that

$$\Gamma\left(r+\frac{1}{2}\right)r! = r\Gamma\left(r+\frac{1}{2}\right)\Gamma(r) = \frac{\sqrt{\pi}(2r)!}{2^{2r}}.$$

This, together with (2.20) and (2.21), reduces the series (1.9) of $\mathtt{W}^{\mathrm{k}}_{\nu,c}$ to

$$\begin{aligned}
\mathtt{W}^{\mathrm{k}}_{\nu,c}(x) &= \frac{2\sqrt{\mathrm{k}}}{\Gamma_{\mathrm{k}}(\nu+\frac{\mathrm{k}}{2})}\left(\frac{x}{2}\right)^{\frac{\nu}{\mathrm{k}}} \int_0^1 \left(1-t^2\right)^{\frac{\nu}{\mathrm{k}}-\frac{1}{2}} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma(r+1)r!}\left(\frac{xt}{2\sqrt{\mathrm{k}}}\right)^{2r} dt \\
&= \frac{2\sqrt{\mathrm{k}}}{\sqrt{\pi}\,\Gamma_{\mathrm{k}}(\nu+\frac{\mathrm{k}}{2})}\left(\frac{x}{2}\right)^{\frac{\nu}{\mathrm{k}}} \int_0^1 \left(1-t^2\right)^{\frac{\nu}{\mathrm{k}}-\frac{1}{2}} \sum_{r=0}^{\infty} \frac{(-c)^r}{(2r)!}\left(\frac{xt}{\sqrt{\mathrm{k}}}\right)^{2r} dt.
\end{aligned} \tag{2.23}$$

Finally, for $c = \pm\alpha^2$, $\alpha \in \mathbb{R}$, representation (2.23) respectively leads to

$$\mathtt{W}^{\mathrm{k}}_{\nu,\alpha^2}(x) = \frac{2\sqrt{\mathrm{k}}}{\sqrt{\pi}\,\Gamma_{\mathrm{k}}(\nu+\frac{\mathrm{k}}{2})}\left(\frac{x}{2}\right)^{\frac{\nu}{\mathrm{k}}} \int_0^1 \left(1-t^2\right)^{\frac{\nu}{\mathrm{k}}-\frac{1}{2}} \cos\left(\frac{\alpha x t}{\sqrt{\mathrm{k}}}\right) dt \tag{2.24}$$

and

$$\mathtt{W}^{\mathtt{k}}_{\nu,-\alpha^2}(x) = \frac{2\sqrt{\mathtt{k}}}{\sqrt{\pi}\Gamma_{\mathtt{k}}(\nu+\frac{\mathtt{k}}{2})}\left(\frac{x}{2}\right)^{\frac{\nu}{\mathtt{k}}}\int_0^1 \left(1-t^2\right)^{\frac{\nu}{\mathtt{k}}-\frac{1}{2}}\cosh\left(\frac{\alpha x t}{\sqrt{\mathtt{k}}}\right)dt. \tag{2.25}$$

Example 2.1 If $\nu = \mathtt{k}/2$, then from (2.24) computations give the relation between sine and generalized $\mathtt{k}$-Bessel functions by

$$\sin\left(\frac{\alpha x}{\sqrt{\mathtt{k}}}\right) = \frac{\alpha}{\mathtt{k}}\sqrt{\frac{\pi x}{2}}\mathtt{W}^{\mathtt{k}}_{\frac{\nu}{\mathtt{k}},\alpha^2}(x).$$

Similarly, the relation

$$\sinh\left(\frac{\alpha x}{\sqrt{\mathtt{k}}}\right) = \frac{\alpha}{\mathtt{k}}\sqrt{\frac{\pi x}{2}}\mathtt{W}^{\mathtt{k}}_{\frac{\nu}{\mathtt{k}},-\alpha^2}(x)$$

can be derived from (2.25).

3 Monotonicity and log-convexity properties

This section is devoted to discuss the monotonicity and log-convexity properties of the modified $\mathtt{k}$-Bessel function $\mathtt{W}^{\mathtt{k}}_{\nu,-1} = \mathtt{I}^{\mathtt{k}}_{\nu}$. As consequences of those results, we derive several functional inequalities for $\mathtt{I}^{\mathtt{k}}_{\nu}$.

The following result of Biernacki and Krzyż [7] will be required.

Lemma 3.1 ([7]) *Consider the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$, where $a_k \in \mathbb{R}$ and $b_k > 0$ for all k. Further, suppose that both series converge on $|x| < r$. If the sequence $\{a_k/b_k\}_{k\geq 0}$ is increasing (or decreasing), then the function $x \mapsto f(x)/g(x)$ is also increasing (or decreasing) on $(0, r)$.*

The lemma still holds when both f and g are even or both are odd functions.

We now state and prove our main results in this section. Consider the functions

$$\mathcal{I}^{\mathtt{k}}_{\nu}(x) := \left(\frac{2}{x}\right)^{\frac{\nu}{\mathtt{k}}}\Gamma_{\mathtt{k}}(\nu+\mathtt{k})\mathtt{I}^{\mathtt{k}}_{\nu}(x) = \sum_{r=0}^{\infty} f_r(\nu)x^{2r}, \tag{3.1}$$

where

$$\begin{aligned} \mathtt{I}^{\mathtt{k}}_{\nu}(x) = \mathtt{W}^{\mathtt{k}}_{\nu,-1}(x) &= \sum_{r=0}^{\infty}\frac{1}{\Gamma_{\mathtt{k}}(r\mathtt{k}+\nu+\mathtt{k})r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathtt{k}}} \quad \text{and} \\ f_r(\nu) &= \frac{\Gamma_{\mathtt{k}}(\nu+\mathtt{k})}{\Gamma_{\mathtt{k}}(r\mathtt{k}+\nu+\mathtt{k})4^r r!}. \end{aligned} \tag{3.2}$$

Then we have the following properties.

Theorem 3.1 *Let $\mathtt{k} > 0$. The following results are true for the modified $\mathtt{k}$-Bessel functions:*

(a) *If $\nu \geq \mu > -\mathtt{k}$, then the function $x \mapsto \mathcal{I}^{\mathtt{k}}_{\mu}(x)/\mathcal{I}^{\mathtt{k}}_{\nu}(x)$ is increasing on $\mathbb{R}$.*

(b) *The function $\nu \mapsto \mathcal{I}^{\mathtt{k}}_{\nu+\mathtt{k}}(x)/\mathcal{I}^{\mathtt{k}}_{\nu}(x)$ is increasing on $(-\mathtt{k}, \infty)$, that is, for $\nu \geq \mu > -\mathtt{k}$,*

$$\mathcal{I}^{\mathtt{k}}_{\nu+\mathtt{k}}(x)\mathcal{I}^{\mathtt{k}}_{\mu}(x) \geq \mathcal{I}^{\mathtt{k}}_{\nu}(x)\mathcal{I}^{\mathtt{k}}_{\mu+\mathtt{k}}(x) \tag{3.3}$$

for any fixed $x > 0$ *and* $\mathrm{k} > 0$.

(c) *The function* $\nu \mapsto \mathcal{I}_\nu^{\mathrm{k}}(x)$ *is decreasing and log-convex on* $(-\mathrm{k}, \infty)$ *for each fixed* $x > 0$.

Proof (a) From (3.1) it follows that

$$\frac{\mathcal{I}_\nu^{\mathrm{k}}(x)}{\mathcal{I}_\mu^{\mathrm{k}}(x)} = \frac{\sum_{r=0}^{\infty} f_r(\nu)x^{2r}}{\sum_{r=0}^{\infty} f_r(\mu)x^{2r}}.$$

Denote $w_r := f_r(\nu)/f_r(\mu)$. Then

$$w_r = \frac{\Gamma_{\mathrm{k}}(\nu + \mathrm{k})\Gamma_{\mathrm{k}}(r\mathrm{k} + \mu + \mathrm{k})}{\Gamma_{\mathrm{k}}(\mu + \mathrm{k})\Gamma_{\mathrm{k}}(r\mathrm{k} + \nu + \mathrm{k})}.$$

Now, using the property $\Gamma_{\mathrm{k}}(y + \mathrm{k}) = y\Gamma_{\mathrm{k}}(y)$, we can show that

$$\frac{w_{r+1}}{w_r} = \frac{\Gamma_{\mathrm{k}}(r\mathrm{k} + \nu + \mathrm{k})\Gamma_{\mathrm{k}}(r\mathrm{k} + \mu + 2\mathrm{k})}{\Gamma_{\mathrm{k}}(r\mathrm{k} + \mu + \mathrm{k})\Gamma_{\mathrm{k}}(r\mathrm{k} + \nu + 2\mathrm{k})} = \frac{r\mathrm{k} + \mu + \mathrm{k}}{r\mathrm{k} + \nu + \mathrm{k}} \le 1$$

for all $\nu \ge \mu > -\mathrm{k}$. Hence, conclusion (a) follows from the Lemma 3.1.

(b) Let $\nu \ge \mu > -\mathrm{k}$. It follows from part (a) that

$$\frac{d}{dx}\left(\frac{\mathcal{I}_\nu^{\mathrm{k}}(x)}{\mathcal{I}_\mu^{\mathrm{k}}(x)}\right) \ge 0$$

on $(0, \infty)$. Thus

$$\left(\mathcal{I}_\nu^{\mathrm{k}}(x)\right)'\left(\mathcal{I}_\mu^{\mathrm{k}}(x)\right) - \left(\mathcal{I}_\nu^{\mathrm{k}}(x)\right)\left(\mathcal{I}_\mu^{\mathrm{k}}(x)\right)' \ge 0. \tag{3.4}$$

It now follows from (2.8) that

$$\frac{x}{2}\left(\mathcal{I}_{\nu+k}^{\mathrm{k}}(x)\mathcal{I}_\mu^{\mathrm{k}}(x) - \mathcal{I}_{\mu+k}^{\mathrm{k}}(x)\mathcal{I}_\nu^{\mathrm{k}}(x)\right) \ge 0,$$

whence $\mathcal{I}_{\nu+k}^{\mathrm{k}}/\mathcal{I}_\nu^{\mathrm{k}}$ is increasing for $\nu > -\mathrm{k}$ and for some fixed $x > 0$, which concludes (b).

(c) It is clear that, for all $\nu > -\mathrm{k}$,

$$f_r(\nu) = \frac{\Gamma_{\mathrm{k}}(\nu + \mathrm{k})}{\Gamma_{\mathrm{k}}(r\mathrm{k} + \nu + \mathrm{k})4^r r!} > 0.$$

A logarithmic differentiation of $f_r(\nu)$ with respect to ν yields

$$\frac{f_r'(\nu)}{f_r(\nu)} = \Psi_{\mathrm{k}}(\nu + \mathrm{k}) - \Psi_{\mathrm{k}}(r\mathrm{k} + \nu + \mathrm{k}) \le 0$$

since Ψ_{k} are increasing functions on $(-\mathrm{k}, \infty)$. This implies that $f_r(\nu)$ is decreasing.

Thus, for $\mu \ge \nu > -\mathrm{k}$, it follows that

$$\sum_{r=0}^{\infty} f_r(\nu)x^{2r} \ge \sum_{r=0}^{\infty} f_r(\mu)x^{2r},$$

which is equivalent to say that the function $\nu \mapsto \mathcal{I}_\nu^{\mathrm{k}}$ is decreasing on $(-\mathrm{k}, \infty)$ for some fixed $x > 0$.

The twice logarithmic differentiation of $f_r(\nu)$ yields

$$\begin{aligned}\frac{\partial^2}{\partial \nu^2}\big(\log\big(f_r(\nu)\big)\big) &= \Psi_k'(\nu + \mathrm{k}) - \Psi_k'(r\mathrm{k} + \nu + \mathrm{k}) \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{(n\mathrm{k} + \nu + \mathrm{k})^2} - \frac{1}{(n\mathrm{k} + r\mathrm{k} + \nu + \mathrm{k})^2}\right) \\ &= \sum_{n=0}^{\infty}\frac{r\mathrm{k}(2n\mathrm{k} + r\mathrm{k} + 2\nu + 2\mathrm{k})}{(n\mathrm{k} + \nu + \mathrm{k})^2(n\mathrm{k} + r\mathrm{k} + \nu + \mathrm{k})^2} \geq 0\end{aligned}$$

for all $\mathrm{k} > 0$ and $\nu > -\mathrm{k}$. Since, a sum of log-convex functions is log-convex, it follows that $\nu \to \mathcal{I}_\nu^{\mathrm{k}}$ is log-convex on $(-\mathrm{k}, \infty)$ for each fixed $x > 0$. □

Remark 3.1 One of the most significance consequences of the Theorem 3.1 is the Turán-type inequality for the function $\mathcal{I}_\nu^{\mathrm{k}}$. From the definition of log-convexity it follows from Theorem 3.1(c) that

$$\mathcal{I}^{\mathrm{k}}_{\alpha\nu_1 + (1-\alpha)\nu_2}(x) \leq \big(\mathcal{I}^{\mathrm{k}}_{\nu_1}\big)^{\alpha}(x)\big(\mathcal{I}^{\mathrm{k}}_{\nu_2}\big)^{1-\alpha}(x),$$

for $\alpha \in [0,1]$, $\nu_1, \nu_2 > -\mathrm{k}$, and $x > 0$. For any $a \in \mathbb{R}$ and $\nu \geq -k$, by choosing $\alpha = 1/2$, $\nu_1 = \nu - a$, and $\nu_2 = \nu + a$, this inequality yields the reverse Turán-type inequality

$$\big(\mathcal{I}^{\mathrm{k}}_{\nu}(x)\big)^2 - \mathcal{I}^{\mathrm{k}}_{\nu-\mathrm{a}}(x)\mathcal{I}^{\mathrm{k}}_{\nu+\mathrm{a}}(x) \leq 0 \tag{3.5}$$

for any $\nu \geq |a| - \mathrm{k}$.

Our final result is based on the Chebyshev integral inequality [26, p. 40], which states the following: suppose f and g are two integrable functions and monotonic in the same sense (either both decreasing or both increasing). Let $q : (a,b) \to \mathbb{R}$ be a positive integrable function. Then

$$\left(\int_a^b q(t)f(t)\,dt\right)\left(\int_a^b q(t)g(t)\,dt\right) \leq \left(\int_a^b q(t)\,dt\right)\left(\int_a^b q(t)f(t)g(t)\,dt\right). \tag{3.6}$$

Inequality (3.6) is reversed if f and g are monotonic in the opposite sense.

The following function is required:

$$\mathcal{J}^{\mathrm{k}}_{\nu}(x) := \left(\frac{2}{x}\right)^{\frac{\nu}{\mathrm{k}}}\Gamma_{\mathrm{k}}(\nu + \mathrm{k})\mathtt{J}^{\mathrm{k}}_{\nu}(x) = \sum_{r=0}^{\infty} g_r(\nu)x^{2r}, \tag{3.7}$$

where

$$\begin{aligned}\mathtt{J}^{\mathrm{k}}_{\nu}(x) &= \mathtt{W}^{\mathrm{k}}_{\nu,1}(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{\Gamma_{\mathrm{k}}(r\mathrm{k} + \nu + \mathrm{k})r!}\left(\frac{x}{2}\right)^{2r+\frac{\nu}{\mathrm{k}}} \quad \text{and} \\ g_r(\nu) &= \frac{(-1)^r\Gamma_{\mathrm{k}}(\nu + \mathrm{k})}{\Gamma_{\mathrm{k}}(r\mathrm{k} + \nu + \mathrm{k})4^r r!}.\end{aligned} \tag{3.8}$$

Theorem 3.2 *Let* $\mathtt{k} > 0$. *Then, for* $\nu \in (-3\mathtt{k}/4, -\mathtt{k}/2] \cup [\mathtt{k}/2, \infty)$,

$$\mathcal{I}_{\nu}^{\mathtt{k}}(x)\mathcal{I}_{\nu+\frac{\mathtt{k}}{2}}^{\mathtt{k}}(x) \leq \frac{\sqrt{\mathtt{k}}}{x}\sin\left(\frac{x}{\mathtt{k}}\right)\mathcal{I}_{2\nu+\frac{\mathtt{k}}{2}}^{\mathtt{k}}(x) \tag{3.9}$$

and

$$\mathcal{J}_{\nu}^{\mathtt{k}}(x)\mathcal{J}_{\nu+\frac{\mathtt{k}}{2}}^{\mathtt{k}}(x) \leq \frac{\sqrt{\mathtt{k}}}{x}\sinh\left(\frac{x}{\mathtt{k}}\right)\mathcal{J}_{2\nu+\frac{\mathtt{k}}{2}}^{\mathtt{k}}(x). \tag{3.10}$$

Inequalities (3.9) *and* (3.10) *are reversed if* $\nu \in (-\mathtt{k}/2, \mathtt{k}/2)$.

Proof Define the functions q, f, and g on $[0,1]$ as

$$q(t) = \cos\left(\frac{xt}{\sqrt{\mathtt{k}}}\right), \qquad f(t) = \left(1-t^2\right)^{\frac{\nu}{\mathtt{k}}-\frac{1}{2}}, \qquad g(t) = \left(1-t^2\right)^{\frac{\nu}{\mathtt{k}}+\frac{1}{2}}.$$

Then, for any $x \geq 0$,

$$\begin{aligned}
&\int_0^1 q(t)\,dt = \int_0^1 \cos\left(\frac{xt}{\sqrt{\mathtt{k}}}\right)dt = \frac{\sqrt{\mathtt{k}}}{x}\sin\left(\frac{x}{\sqrt{\mathtt{k}}}\right),\\
&\int_0^1 q(t)f(t)\,dt = \int_0^1 \cos\left(\frac{xt}{\sqrt{\mathtt{k}}}\right)\left(1-t^2\right)^{\frac{\nu}{\mathtt{k}}-\frac{1}{2}}dt = \mathcal{I}_{\nu}^{\mathtt{k}}(x) \quad \text{if } \nu \geq -\mathtt{k},\\
&\int_0^1 q(t)g(t)\,dt = \int_0^1 \cos\left(\frac{xt}{\sqrt{\mathtt{k}}}\right)\left(1-t^2\right)^{\frac{\nu}{\mathtt{k}}+\frac{1}{2}}dt = \mathcal{I}_{\nu+\mathtt{k}}^{\mathtt{k}}(x) \quad \text{if } \nu \geq -2\mathtt{k},\\
&\int_0^1 q(t)f(t)g(t)\,dt = \int_0^1 \cos\left(\frac{xt}{\sqrt{\mathtt{k}}}\right)\left(1-t^2\right)^{\frac{2\nu}{\mathtt{k}}}dt = \mathcal{I}_{2\nu+\frac{\mathtt{k}}{2}}^{\mathtt{k}}(x) \quad \text{if } \nu \geq -\frac{3\mathtt{k}}{4}.
\end{aligned}$$

Since the functions f and g both are decreasing for $\nu \geq \mathtt{k}/2$ and both are increasing for $\nu \in (-3\mathtt{k}/4, -\mathtt{k}/2]$, inequality (3.6) yields (3.9). On the other hand, if $\nu \in (-\mathtt{k}/2, \mathtt{k}/2)$, then the function f is increasing, but g is decreasing, and hence inequality (3.9) is reversed.

Similarly, inequality (3.10) can be derived from (3.6) by choosing

$$q(t) = \cosh\left(\frac{xt}{\sqrt{\mathtt{k}}}\right), \qquad f(t) = \left(1-t^2\right)^{\frac{\nu}{\mathtt{k}}-\frac{1}{2}}, \qquad g(t) = \left(1-t^2\right)^{\frac{\nu}{\mathtt{k}}+\frac{1}{2}}. \qquad \square$$

4 Conclusion

It is shown that the generalized k-Bessel functions $W_{\nu,c}^{k}$ are solutions of a second-order differential equation, which for $k = 1$ is reduced to the well-known second-order Bessel differential equation. It is also proved that the generalized modified k-Bessel function $\mathcal{I}_{\nu}^{\mathtt{k}}$ is decreasing and log-convex on $(-\mathtt{k}, \infty)$ for each fixed $x > 0$. Several other inequalities, especially the Turán-type inequality and reverse Turán-type inequality for $\mathcal{I}_{\nu}^{\mathtt{k}}$ are established.

Furthermore, we investigate the pattern for zeroes of $\mathcal{W}_{\nu}^{\mathtt{k},1}$ in two ways: (i) with respect to fixed $\mathtt{k}$ and variation of ν and (ii) with respect to fixed ν and variation of $\mathtt{k}$.

From the data in Table 1 and Table 2, we can observe that the zeroes of $\mathtt{W}_{\nu,1}^{\mathtt{k}}$ are increasing in in both cases. However, we have no any analytical proof for this monotonicity of the zeroes of $W_{\nu,1}^{k}$. As there are several works on the zeroes of the classical Bessel functions,

Table 1 Positive zeroes of $\mathtt{W}^{\mathtt{k}}_{\nu,1}$ for fixed ν and different $\mathtt{k}$

$\mathtt{k}$	0.5	1	1.5	2	2.5
$\nu = -0.4$ and $c = 1$					
1st zero	0.662422	1.75098	2.42334	2.95334	3.40423
2nd zero	2.96686	4.87852	6.24148	7.3588	8.32849
3rd zero	5.2018	8.01663	10.0812	11.7913	13.2836
$\nu = 0.5$ and $c = 1$					
1st zero	2.70943	3.14159	3.55493	3.93277	4.28026
2nd zero	4.96077	6.28319	7.38858	8.35255	9.21757
3rd zero	7.19373	9.42478	11.2315	12.7879	14.1752

Table 2 Positive zeroes of $\mathtt{W}^{\mathtt{k}}_{\nu,1}$ for different ν and $\mathtt{k}$

ν	–0.4	–0.3	0	0.5	1	1.5	2	2.5
$\mathtt{k} = 0.5$ and $c = 1$								
1st zero	0.662422	0.97534	1.70047	2.70943	3.63143	4.51146	5.36577	6.20238
2nd zero	2.96686	3.21271	3.90328	4.96077	5.95189	6.90209	7.82393	8.72471
3rd zero	5.2018	5.43751	6.11911	7.19373	8.21647	9.20314	10.1629	11.1017
$\mathtt{k} = 1$ and $c = 1$								
1st zero	1.75098	1.92285	2.40483	3.14159	3.83171	4.49341	5.13562	5.76346
2nd zero	4.87852	5.04213	5.52008	6.28319	7.01559	7.72525	8.41724	9.09501
3rd zero	8.01663	8.17785	8.65373	9.42478	10.1735	10.9041	11.6198	12.3229
$\mathtt{k} = 1.5$ and $c = 1$								
1st zero	2.42334	2.55767	2.9453	3.55493	4.13426	4.69286	5.2362	5.76774
2nd zero	6.24148	6.37291	6.76069	7.38858	7.9979	8.5923	9.1744	9.74613
3rd zero	10.0812	10.2116	10.5986	11.2315	11.8513	12.4599	13.0587	13.6488
$\mathtt{k} = 2$ and $c = 1$								
1st zero	2.95334	3.06754	3.40094	3.93277	4.44288	4.93703	5.41885	5.8908
2nd zero	7.3588	7.47176	7.80657	8.35255	8.88577	9.40825	9.92154	10.4269
3rd zero	11.7913	11.9037	12.2382	12.7879	13.3286	13.8616	14.3875	14.907

the zeroes of $\mathtt{W}^{\mathtt{k}}_{\nu,1}$ would be an interesting topic for future investigations. The monotonicity of the zeroes of $\mathtt{W}^{\mathtt{k}}_{\nu,c}$ with respect to c and fixed $\mathtt{k}$, ν will be another open problem for further studies.

Funding
The work is supported by Deanship of Scientific Research (DSR), King Faisal University, Hofuf, Saudi Arabia, through the project number 150282.

Availability of data and materials
Not applicable. The data in both Tables 1 and 2 are generated using Mathematica 9.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed to each part of this work equally, and they both read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 October 2017 Accepted: 9 July 2018 Published online: 16 July 2018

References

1. Baricz, Á: Generalized Bessel Functions of the First Kind. Lecture Notes in Mathematics, vol. 1994. Springer, Berlin (2010)
2. Baricz, Á: Turán type inequalities for modified Bessel functions. Bull. Aust. Math. Soc. **82**(2), 254–264 (2010)
3. Baricz, Á, Ponnusamy, S., Singh, S.: Turán type inequalities for general Bessel functions. Math. Inequal. Appl. **19**(2), 709–719 (2016)
4. Abramowitz, M., Stegun, I.A. (eds.): A Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables. In Dover Books on Mathematics. Dover, New York (1965)
5. Alzer, H., Gerhold, S., Kauers, M., Lupaş, A.: On Turán's inequality for Legendre polynomials. Expo. Math. **25**(2), 181–186 (2007). https://doi.org/10.1016/j.exmath.2006.11.001
6. Andrews, G.E., Askey, R., Roy, R.: Special Functions. Encyclopedia of Mathematics and Its Applications, vol. 71. Cambridge Univ. Press, Cambridge (1999)
7. Biernacki, M., Krzyż, J.: On the monotonity of certain functionals in the theory of analytic functions. Ann. Univ. Mariae Curie-Skłodowska, Sect. A **9**, 135–147 (1955)
8. Cesarano, C.: Some special classes of Bessel functions and modified Humbert functions. Int. J. Pure Appl. Math. Sci. **111**(4), 589–604 (2016). https://doi.org/10.12732/ijpam.v111i4.5
9. Cesarano, C., Assante, D.: A note on generalized Bessel functions. Int. J. Math. Models Methods Appl. Sci. **7**(6), 625–629 (2013)
10. Cesarano, C., Fornaro, C.: Generalized Bessel functions in terms of generalized Hermite polynomials. Int. J. Pure Appl. Math. Sci. **112**(3), 613–629 (2017). https://doi.org/10.12732/ijpam.v112i3.13
11. Constantinescu, E.: On the inequality of P. Turán for Legendre polynomials. JIPAM. J. Inequal. Pure Appl. Math. **6**(2), Article ID 28 (2005). https://doi.org/10.1016/j.exmath.2006.11.001
12. Deligne, P., Etingof, P., Freed, D., Jeffrey, L., Kazhdan, D., Morgan, J., Morrison, D., Witten, E.: Quantum Fields and Strings: A Course for Mathematicians. Am. Math. Soc. Providence (1999)
13. Díaz, R., Pariguan, E.: Symmetric quantum Weyl algebras. Ann. Math. Blaise Pascal **11**, 187–203 (2004)
14. Díaz, R., Pariguan, E.: Quantum symmetric functions. Commun. Algebra **33**, 1947–1978 (2005)
15. Díaz, R., Pariguan, E.: On hypergeometric functions and Pochhammer *k*-symbol. Divulg. Mat. **15**, 179–192 (2007)
16. Dimitrov, D.K.: Higher order Turán inequalities. Proc. Am. Math. Soc. **126**, 2033–2037 (1998)
17. Galué, L.: A generalized Bessel function. Integral Transforms Spec. Funct. **14**, 395–401 (2003)
18. Gasper, G.: An inequality of Turán type for Jacobi polynomials. Proc. Am. Math. Soc. **32**, 435–439 (1972)
19. Gehlot, K.S.: Differential equation of *K*-Bessel's function and its properties. Nonlinear Anal. Differ. Equ. **2**(2), 61–67 (2014). https://doi.org/10.12988/nade.2014.3821
20. Gehlot, K.S.: Recurrence relations of *K*-Bessel's function. Thai J. Math. **14**, 677–685 (2016)
21. Gehlot, K.S., Purohit, S.D.: Integral representations of the *k*-Bessel's function. Honam Math. J. **38**, 17–23 (2016)
22. Gerhold, S., Kauers, M.: A computer proof of Turán's inequality. JIPAM. J. Inequal. Pure Appl. Math. **7**(2), Article ID 42 (2000)
23. Gerhold, S., Kauers, M.: A procedure for proving special function inequalities involving a discrete parameter. In: Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation (ISSAC'05), Beijing, China, July 24–27, 2005, pp. 156–162. ACM, New York (2005). https://doi.org/10.1145/1073884.1073907
24. Karlin, S., Szegö, G.: On certain determinants whose elements are orthogonal polynomials. J. Anal. Math. **8**(1), 1–157 (1960). https://doi.org/10.1007/BF02786848
25. Laforgia, A., Natalini, P.: Turán-type inequalities for some special functions. JIPAM. J. Inequal. Pure Appl. Math. **7**(1), Article ID 22 (2006)
26. Mitrinović, D.S.: Analytic Inequalities. Springer, New York (1970)
27. Mubeen, S., Naz, M., Rahman, G.: A note on *k*-hypergeometric differential equations. J. Inequal. Spec. Funct. **4**(3), 38–43 (2013)
28. Nantomah, K., Prempeh, E.: Some inequalities for the *k*-digamma function. Math. Æterna **4**, 521–525 (2014)
29. Szwarc, R.: Positivity of Turán determinants for orthogonal polynomials. In: Ross, K.A., Singh, A.I., Anderson, J.M., Sunder, V.S., Litvinov, G.L., Wildberger, N.J. (eds.) Harmonic Analysis and Hypergroups, Delhi, 1995. Trends in Mathematics, pp. 165–182. Birkhäuser, Boston (1998). https://doi.org/10.1007/978-0-8176-4348-5_11
30. Turán, P.: On the zeros of the polynomials of Legendre. Čas. Pěst. Math. Fys. **75**, 113–122 (1950)