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A new version of Carleson measure associated with Hermite operator

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Abstract

Let $L = -\Delta + |x|^2$ be a Hermite operator, where Δ is the Laplacian on $\mathbb{R}^d$. In this paper we define a new version of Carleson measure associated with Hermite operator, which is adapted to the operator L. Then, we will use it to characterize the dual spaces and predual spaces of the Hardy spaces $H_L^p(\mathbb{R}^d)$ associated with L.

MSC: 42B30; 42B35

Keywords: Hermite operator; Carleson measure; Dual space; Hardy space; BMO space

1 Introduction

In recent years, the study of function spaces associated with Hermite operators has inspired great interest. Dziubański [7] introduced the Hardy space $H_L^p(\mathbb{R}^d)$, $0 < p \le 1$, by using the heat maximal function and established its atomic characterization. Dziubański et al. [8] and Yang et al. [20] introduced and studied some BMO spaces and Morrey–Campanato spaces associated with operators. Deng et al. [5] introduced the space $VMO_L(\mathbb{R}^d)$ and proved that $(VMO_L(\mathbb{R}^d))^* = H_L^1(\mathbb{R}^d)$. Moreover, recently, Jiang et al. in [14] defined the predual spaces of Banach completions of Orlicz–Hardy spaces associated with operators. Bui et al. [3] considered the Besov and Triebel–Lizorkin spaces associated with Hermite operators.

One of the main purposes of studying the function spaces is to give the equivalent characterizations of them, for example, square functions characterizations for Hardy spaces [10], Carleson measure characterizations for BMO spaces [8] or Morry–Campanato spaces [6]. The aim of this paper is to give characterizations of the dual spaces and predual spaces of the Hardy spaces $H_L^p(\mathbb{R}^d)$ by a new version of Carleson measure. Now, let us review some known facts about the function spaces for L.

Let L be the basic Schrödinger operator in $\mathbb{R}^d$, $d \ge 1$, the harmonic oscillator $L = -\Delta + |x|^2$. Let $\{T_t^L\}_{t>0}$ be a semigroup of linear operators generated by $-L$ and $K_t^L(x,y)$ be their kernels. The Feynman–Kac formula implies that

$$0 \le K_t^L(x,y) \le \widetilde{T}_t(x,y) = (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{1}$$

Dziubański [7] defined Hardy space $H_L^p(\mathbb{R}^d)$, $0 < p \le 1$ as

$$H_L^p(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : Mf \in L^p(\mathbb{R}^d)\},$$

where

$$Mf(x) = \sup_{t>0} \left| T_t^L f(x) \right|.$$

The norm of Hardy space $H_L^p(\mathbb{R}^d)$ is defined by $\|f\|_{H_L^p} = \|Mf\|_{L^p}$.

Remark 1 For simplicity, we just consider the case of $\frac{d}{d+1} < p \le 1$ in this paper. But all of our results hold for $0 < p \le 1$.

Let $\rho(x) = \frac{1}{1+|x|}$ be the auxiliary function defined in [17]. This auxiliary function plays an important role in the estimates of the operators and in the description of the spaces associated with L. Then, for $\frac{d}{d+1} < p \le 1$ and $1 \le q \le \infty$, a function a is an $H_L^{p,q}$-atom for the Hardy space $H_L^p(\mathbb{R}^d)$ associated with a ball $B(x_0, r)$ if

(1) $\operatorname{supp} a \subset B(x_0, r)$,

(2) $\|a\|_{L^q} \le |B(x_0, r)|^{\frac{1}{q} - \frac{1}{p}}$,

(3) if $r < \rho(x_0)$, then $\int a(x)\,dx = 0$.

The atomic quasi-norm in $H_L^p(\mathbb{R}^d)$ is defined by

$$\|f\|_{L\text{-atom},q} = \inf\left\{ \left(\sum |c_j|^p \right)^{1/p} \right\},$$

where the infimum is taken over all decompositions $f = \sum c_j a_j$ and a_j are $H_L^{p,q}$-atoms.

The atomic decomposition for $H_L^p(\mathbb{R}^d)$ is as follows (see [7]).

Proposition 1 *Let $\frac{d}{d+1} < p \le 1$, we have that the norms $\|f\|_{H_L^p}$ and $\|f\|_{L\text{-atom},q}$ are equivalent, that is, there exists a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H_L^p} \le \|f\|_{L\text{-atom},q} \le C\|f\|_{H_L^p},$$

where $1 \le q \le \infty$.

We define Campanato space associated with L as (cf. [1] or [20]).

Definition 1 Let $0 \le \alpha < 1$, a locally integrable function g on $\mathbb{R}^d$ belongs to Λ_α^L if and only if $\|g\|_{\Lambda_\alpha^L} < \infty$, where

$$\|g\|_{\Lambda_\alpha^L} = \sup_{B \subset \mathbb{R}^d} \left\{ |B|^{-\frac{\alpha}{d}} \left(\int_B \left| g - g(B, x_0) \right|^2 \frac{dx}{|B|} \right)^{1/2} \right\}$$

and

$$g(B,x_0)=\begin{cases}\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}g(y)\,dy, & \text{if } r<\rho(x_0),\\ 0, & \text{if } r\geq\rho(x_0).\end{cases}$$

The duality of $H_L^p(\mathbb{R}^d)$ and $\Lambda_{d(1/p-1)}^L$ can be found in [12] or [20].

In order to give the Carleson measure characterization of $\Lambda_{d(1/p-1)}^L$, we need some notations of the tent spaces (cf. [4]).

Let $0<p<\infty$ and $1\leq q\leq\infty$. Then the tent space T_q^p is defined as the space of functions f on $\mathbb{R}_+^{d+1}$ so that

$$\left(\int_{\Gamma(x)}\left|f(y,t)\right|^q\frac{dy\,dt}{t^{d+1}}\right)^{1/q}\in L^p\left(\mathbb{R}^d\right),\quad \text{when } 1\leq q<\infty,$$

and

$$\sup_{(y,t)\in\Gamma(x)}\left|f(y,t)\right|\in L^p\left(\mathbb{R}^d\right),\quad \text{when } q=\infty,$$

where $\Gamma(x)$ is the standard cone whose vertex is $x\in\mathbb{R}^d$, i.e.,

$$\Gamma(x)=\left\{(y,t):|y-x|<t\right\}.$$

Assume that $B(x_0,r)$ is a ball in $\mathbb{R}^d$, its tent $\widehat{B}$ is defined by $\widehat{B}=\{(x,t):|x-x_0|\leq r-t\}$. A function $a(x,t)$ that is supported in a tent $\widehat{B}$, B is a ball in $\mathbb{R}^d$, is said to be an atom in the tent space T_2^p if it satisfies

$$\left(\int_{\widehat{B}}\left|a(x,t)\right|^2\frac{dx\,dt}{t}\right)^{1/2}\leq|B|^{1/2-1/p}.$$

The atomic decomposition of T_2^p is stated as follows.

Proposition 2 *When $0<p\leq 1$, then every $f\in T_2^p$ can be written as $f=\sum\lambda_k a_k$, where a_k are atoms and $\sum|\lambda_k|^p\leq C\|f\|_{T_2^p}^p$.*

Let

$$T_2^{p,\infty}=\left\{f(x,t):\text{measurable on }\mathbb{R}_+^{d+1}\text{ and }\|f\|_{T_2^{p,\infty}}<\infty\right\},$$

where

$$\|f\|_{T_2^{p,\infty}}=\sup_{B\subset\mathbb{R}^d}\frac{1}{|B|^{1/p-1/2}}\left(\int_{\widehat{B}}\left|f(x,t)\right|^2\frac{dx\,dt}{t}\right)^{1/2}.$$

Assume $0<p\leq 1$, we say a function $f\in T_2^{p,\infty}$ belongs to the space $T_{2,0}^{p,\infty}$ if f satisfies $\eta_1(f)=\eta_1 2(f)=\eta_3(f)=0$, where

$$\eta_1(f)=\lim_{r\to 0}\sup_{B\subset\mathbb{R}^d r_B<r}\frac{1}{|B|^{1/p-1/2}}\left(\int_{\widehat{B}}\left|f(x,t)\right|^2\frac{dx\,dt}{t}\right)^{1/2};$$

$$\eta_2(f)=\lim_{r\to\infty}\sup_{B\subset\mathbb{R}^d r_B\geq r}\frac{1}{|B|^{1/p-1/2}}\left(\int_{\widehat{B}}\left|f(x,t)\right|^2\frac{dx\,dt}{t}\right)^{1/2};$$

$$\eta_3(f)=\lim_{r\to\infty}\sup_{B\subset B(0,r)^c}\frac{1}{|B|^{1/p-1/2}}\left(\int_{\widehat{B}}\left|f(x,t)\right|^2\frac{dx\,dt}{t}\right)^{1/2}.$$

Let $\widetilde{T_2^p}=\{F(x,t)=\sum_i\lambda_i a_i(x,t):a_i(x,t)\text{are } T_2^p \text{ atoms and } \sum_i|\lambda_i|^p<\infty\}$ and $\|F\|_{\widetilde{T_2^p}}=\inf\{(\sum_i|\lambda_i|^p)^{1/p}:F(x,t)=\sum_i\lambda_i a_i(x,t)\}$. Then $\widetilde{T_2^p}$ is a Banach space. In fact, it is the completeness of T_2^p. Especially, $\widetilde{T_2^1}=T_2^1$.

In [19], the author proved the following result.

Proposition 3 *Let* $0<p\leq 1$. *Then*

$$\left(T_{2,0}^{p,\infty}\right)^*=\widetilde{T_2^p},\qquad\left(T_2^p\right)^*=T_2^{p,\infty}.$$

Let $\{P_t^L\}_{t>0}$ be the semigroup of linear operators generated by $-\sqrt{L}$ and $D_t^Lf(x)=t|\nabla P_t^Lf|(x)$, where $\nabla=(\partial_t,\partial_{x_1},\dots,\partial_{x_d})$. The Carleson measure characterization of the Campanato space $\Lambda^L_{d(1/p-1)}$ as (cf. [6]).

Proposition 4 *Let* $\frac{d}{d+1}<p\leq 1$. *Then, for any* $f\in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, *we have*:

(a) *If* $f\in\Lambda^L_{d(1/p-1)}$, *then* $D_t^Lf\in T_2^{p,\infty}$; *moreover, we have*

$$\left\|D_t^Lf\right\|_{T_2^{p,\infty}}\leq C\|f\|_{\Lambda^L_{d(1/p-1)}}.$$

(b) *Conversely, if* $f\in L^1((1+|x|)^{-(d+1)}\,dx)$ *and* $D_t^Lf\in T_2^{p,\infty}$, *then* $f\in\Lambda^L_{d(1/p-1)}$ *and*

$$\|f\|_{\Lambda^L_{d(1/p-1)}}\leq C\left\|D_t^Lf\right\|_{T_2^{p,\infty}}.$$

The predual space of the classical Hardy space has been studied in [19] and [16].

Definition 2 Let $\alpha>0$, we will say a function f of Λ^L_α is in λ^L_α if it satisfies $\gamma_1(f)=\gamma_2(f)=\gamma_3(f)=0$, where

$$\gamma_1(f)=\lim_{r\to 0}\sup_{B\subset\mathbb{R}^d r_B<r}\frac{1}{|B|^{\alpha/d+1/2}}\left(\int_{\widehat{B}}\left|f-f(B,V)\right|^2\frac{dx\,dt}{t}\right)^{1/2};$$

$$\gamma_2(f)=\lim_{r\to\infty}\sup_{B\subset\mathbb{R}^d r_B\geq r}\frac{1}{|B|^{\alpha/d+1/2}}\left(\int_{\widehat{B}}\left|f-f(B,V)\right|^2\frac{dx\,dt}{t}\right)^{1/2};$$

$$\gamma_3(f)=\lim_{r\to\infty}\sup_{B\subset B(0,r)^c}\frac{1}{|B|^{\alpha/d+1/2}}\left(\int_{\widehat{B}}\left|f-f(B,V)\right|^2\frac{dx\,dt}{t}\right)^{1/2}.$$

The dual space of $\lambda^L_{d(1/p-1)}$ is $B^p_L(\mathbb{R}^d)$, which is the completeness of $H^p_L(\mathbb{R}^d)$ (cf. [14]). We can give a Carleson measure characterization of $\lambda^L_{d(1/p-1)}$ as follows (see [14]).

Proposition 5 *Let* $\frac{d}{d+1}<p\leq 1$. *Then, for any* $f\in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, *we have*:

(a) *If* $f\in\lambda^L_{d(1/p-1)}$, *then* $D_t^Lf\in T_2^{p,\infty}$; *moreover, we have*

$$\left\|D_t^Lf\right\|_{T_2^{p,\infty}}\leq C\|f\|_{\lambda^L_{d(1/p-1)}}.$$

(b) *Conversely, if* $f \in L^1((1+|x|)^{-(d+1)}\,dx)$ *and* $D_t^L f \in T_2^{p,\infty}$, *then* $f \in \lambda^L_{d(1/p-1)}$ *and*

$$\|f\|_{\lambda^L_{d(1/p-1)}} \le C\big\|D_t^L f\big\|_{T_2^{p,\infty}}.$$

Let $A_j = \partial_{x_j} + x_j$ and $A_{-j} = \partial_{x_j} - x_j$ for $j = 1, 2, \ldots, d$. Then

$$L = \sum_{j=1}^{d} A_j A_{-j} + A_{-j} A_j.$$

Therefore, in the harmonic analysis associated with L, the operators A_j play the role of the classical partial derivatives ∂_{x_j} in the Euclidean harmonic analysis (see [2, 11, 18]). Now, it is natural to consider the derivatives A_i other than ∂_{x_j}. In [13], the author defined the Lusin area integral operator by A_j and characterized the Hardy space $H^1_L(\mathbb{R}^d)$. As a continuous study of the function spaces associated with L, in this paper we will define the Carleson measure by A_j and characterize the dual spaces and predual spaces of $H^p_L(\mathbb{R}^d)$. Moreover, let $Q_t^L f(x) = t|\tilde{\nabla} P_t^L f|(x)$, where $\tilde{\nabla} = (\partial_t, A_{-1}, \ldots, A_{-d}, A_1, \ldots, A_d)$. Then the main results of this paper can be stated as follows.

Theorem 1 *Let* $\frac{d}{d+1} < p \le 1$. *Then, for every* $f \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, *we have*:

(a) *If* $f \in \Lambda^L_{d(1/p-1)}$, *then* $Q_t^L f \in T_2^{p,\infty}$; *moreover, we have*

$$\big\|Q_t^L f\big\|_{T_2^{p,\infty}} \le C\|f\|_{\Lambda^L_{d(1/p-1)}}.$$

(b) *Conversely, if* $f \in L^1((1+|x|)^{-(d+1)}\,dx)$ *and* $Q_t^L f \in T_2^{p,\infty}$, *then* $f \in \Lambda^L_{d(1/p-1)}$ *and*

$$\|f\|_{\Lambda^L_{d(1/p-1)}} \le C\big\|Q_t^L f\big\|_{T_2^{p,\infty}}.$$

Remark 2 In [8], the authors characterize the case $p = 0$, i.e., BMO_L, by the heat semigroup with the classical derivatives. In [15], the authors characterize the space BMO_L by the Poisson semigroup with the classical derivatives. In this paper, we will use the new derivatives A_j of the Poisson semigroup to characterize the space $\Lambda^L_{d(1/p-1)}$ for $\frac{d}{d+1} < p \le 1$.

Theorem 2 *Let* $\frac{d}{d+1} < p \le 1$. *Then, for any* $f \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, *we have*:

(a) *If* $f \in \lambda^L_{d(1/p-1)}$, *then* $Q_t^L f \in T_{2,0}^{p,\infty}$; *moreover, we have*

$$\big\|Q_t^L f\big\|_{T_{2,0}^{p,\infty}} \le C\|f\|_{\lambda^L_{d(1/p-1)}}.$$

(b) *Conversely, if* $f \in L^1((1+|x|)^{-(d+1)}\,dx)$ *and* $Q_t^L f \in T_{2,0}^{p,\infty}$, *then* $f \in \lambda^L_{d(1/p-1)}$ *and*

$$\|f\|_{\lambda^L_{d(1/p-1)}} \le C\big\|Q_t^L f\big\|_{T_{2,0}^{p,\infty}}.$$

The paper is organized as follows. In Sect. 2, we give some estimates of the kernels. In Sect. 3, we give the proof of Theorem 1. The proofs of Theorem 2 will be given in Sect. 4.

Throughout the article, we will use A and C to denote the positive constants, which are independent of the main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \le \frac{B_1}{B_2} \le C$.

2 Estimates of the kernels

In this section, we give some estimates of the kernels, which we will use in the sequel.

The proofs of these estimates can be found in [9].

Lemma 1

(a) *For every $N \in \mathbb{N}$, there is a constant $C_N > 0$ such that*

$$0 \le K_t^L(x,y) \le C_N t^{-\frac{d}{2}} e^{-(5t)^{-1}|x-y|^2} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}. \tag{2}$$

(b) *There exists $C > 0$ such that, for every $N > 0$, there is a constant $C_N > 0$ so that, for all $|h| \le \sqrt{t}$,*

$$\left|K_t^L(x+h,y) - K_t^L(x,y)\right| \le C_N \frac{|h|}{\sqrt{t}} t^{-\frac{d}{2}} e^{-At^{-1}|x-y|^2} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}. \tag{3}$$

By subordination formula, we can give the following estimates about the Poisson kernel.

Lemma 2

(a) *For every N, there is a constant $C_N > 0$, $A > 0$ such that*

$$0 \le P_t^L(x,y) \le C_N \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}. \tag{4}$$

(b) *Let $|h| < \frac{|x-y|}{2}$. Then, for any $N > 0$, there exist constants $C > 0$, $C_N > 0$ such that*

$$\left|P_t^L(x+h,y) - P_t^L(x,y)\right| \le C_N \frac{|h|}{\sqrt{t}} \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}. \tag{5}$$

Duong et al. [6] proved the following estimates about the kernel $D_t^L(x,y)$.

Lemma 3 *There exist constants C such that, for every N, there is a constant $C_N > 0$, so that*

(a) $$\left|D_t^L(x,y)\right| \le C_N t^{-d} e^{-Ct^{-2}|x-y|^2} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N};$$

(b) $$\left|D_t^L(x+h,y) - D_t^L(x,y)\right| \le C_{k,N} \left(\frac{|h|}{t}\right)^{\delta'} t^{-d} e^{-Ct^{-2}|x-y|^2} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N},$$

for all $|h| \le t$;

(c) $$\left|\int_{\mathbb{R}^d} D_t^L(x,y)\,dy\right| \le C_N \frac{t/\rho(x)}{(1 + t/\rho(x))^N}.$$

Let $t = \frac{1}{2}\ln\frac{1+s}{1-s}$, $s \in (0,1)$. Then

$$K_t^L(x,y) = \left(\frac{1-s^2}{4\pi s}\right)^{d/2} \exp\left(-\frac{1}{4}\left(s|x+y|^2 + \frac{1}{s}|x-y|^2\right)\right) \doteq K_s(x,y). \tag{6}$$

The following estimations are very important for the proofs of the main result in this paper.

Lemma 4 *There is $C > 0$ for $N \in \mathbb{N}$ and $|x - x'| \le \frac{|x-y|}{2}$, any $j = -1, \dots, -d, 1, \dots, d$, we can find $C_N > 0$ such that*

$$\text{(a)}\quad \left|\sqrt{t}A_j K_t^L(x,y)\right| \le C_N t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{Ct}\right)\left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N};$$

$$\begin{aligned}\text{(b)}\quad &\left|\sqrt{t}A_j K_t^L(x,y) - \sqrt{t}A_j K_t^L(x',y)\right| \\ &\quad \le C_N \frac{|x-x'|}{t} t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{Ct}\right)\left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N};\end{aligned}$$

$$\text{(c)}\quad \left|\int_{\mathbb{R}^d} \sqrt{t}A_j K_t^L(x,y)\,dy\right| \le C \frac{t/\rho(x)}{(1 + t/\rho(x))^N}.$$

Proof By

$$\begin{aligned}\left|A_j K_t^L(x,y)\right| &= \left|\frac{\partial}{\partial x_j} K_t^L(x,y) + x_j K_t^L(x,y)\right| \\ &\le \left|\frac{\partial}{\partial x_j} K_t^L(x,y)\right| + \left|x_j K_t^L(x,y)\right| \doteq I_1 + I_2,\end{aligned}$$

and $t = \frac{1}{2}\ln\frac{1+s}{1-s} \sim s$, $s \to 0^+$, for $s \in (0, \frac{1}{2}]$, we have

$$\begin{aligned}I_2 &\le C|x_j| s^{-\frac{d}{2}} \exp\left(-\frac{1}{4}s|x+y|^2\right)\exp\left(-\frac{1}{4}\frac{|x-y|^2}{s}\right) \\ &\le C|x| s^{-\frac{d}{2}} \exp\left(-\frac{1}{4}s|x+y|^2\right)\exp\left(-\frac{1}{4}\frac{|x-y|^2}{s}\right).\end{aligned}$$

If $x \cdot y \le 0$, then $|x| \le |x-y|$. So

$$\begin{aligned}I_2 &\le C s^{-\frac{d}{2}} |x-y| \exp\left(-\frac{1}{4}\frac{|x-y|^2}{s}\right) \le C s^{-\frac{d-1}{2}} \exp\left(-\frac{|x-y|^2}{8s}\right) \\ &\le C t^{-\frac{d-1}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right).\end{aligned}$$

If $x \cdot y \ge 0$, then $|x| \le |x+y|$. So

$$\begin{aligned}I_2 &\le C s^{-\frac{d}{2}} |x+y| \exp\left(-\frac{1}{4}s|x+y|^2\right)\exp\left(-\frac{1}{4}\frac{|x-y|^2}{s}\right) \\ &\le C s^{-\frac{d+1}{2}} \exp\left(-\frac{|x-y|^2}{4s}\right) \le C t^{-\frac{d+1}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right).\end{aligned}$$

Therefore,

$$|\sqrt{t} I_2| \le C(1+t) t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right) \le C t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right). \tag{7}$$

Since

$$\lim_{t \to \infty} t^2 \left(1 - \left(\frac{e^{2t}-1}{e^{2t}+1}\right)^2\right) = 0,$$

we get $(\frac{1-s^2}{4\pi s})^{d/2} \le t^{-d}$ for $s \in [\frac{1}{2}, 1)$.

When $s \in [\frac{1}{2}, 1)$, we get $t = \frac{1}{2} \ln \frac{1+s}{1-s} > s$. Therefore

$$\begin{aligned}
I_2 &\leq C|x_j| \exp\left(-\frac{1}{4}\left(s|x+y|^2 + \frac{|x-y|^2}{s}\right)\right) \\
&\leq Ct^{-d}|x| \exp\left(-\frac{1}{4}\left(s|x+y|^2 + \frac{|x-y|^2}{s}\right)\right) \\
&\leq Ct^{-d}\big(|x+y| + |x-y|\big) \exp\left(-\frac{1}{4}\left(s|x+y|^2 + \frac{|x-y|^2}{s}\right)\right) \\
&\leq Ct^{-d} \exp\left(-\frac{|x-y|^2}{8s}\right) \\
&\leq Ct^{-d} \exp\left(-\frac{|x-y|^2}{8t}\right).
\end{aligned}$$

Then

$$|\sqrt{t} I_2| \leq Ct^{-d+\frac{1}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right) \leq Ct^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right). \tag{8}$$

By (6), we get

$$\frac{\partial}{\partial x_j} K_s(x,y) = -\frac{1}{2}\left(s(x_j + y_j) + \frac{1}{s}(x_j - y_j)\right) K_s(x,y)$$

and

$$I_1 \leq C\left(s|x_j + y_j| + \frac{1}{s}|x_j - y_j|\right) K_s(x,y) \leq C\left(s|x+y| + \frac{1}{s}|x-y|\right) K_s(x,y).$$

Therefore, when $s \in (0, \frac{1}{2}]$, we have

$$I_1 \leq Cs^{-\frac{d}{2}}(1+s) \exp\left(-\frac{|x-y|^2}{8s}\right) \leq Ct^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right).$$

When $s \in [\frac{1}{2}, 1)$, we have

$$I_1 \leq Ct^{-d} \exp\left(-\frac{|x-y|^2}{8s}\right) \leq Ct^{-d} \exp\left(-\frac{|x-y|^2}{8t}\right).$$

Then

$$\left|\sqrt{t} \frac{\partial}{\partial x_j} K_t^L(x,y)\right| \leq Ct^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right). \tag{9}$$

By (7)–(9), we get

$$\left|\sqrt{t} A_j K_t^L(x,y)\right| \leq Ct^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right). \tag{10}$$

Similar to the proof of (10), for any $N > 0$, we can prove

$$\left(\sqrt{t}|x|\right)^N \left|\sqrt{t} A_j K_t^L(x,y)\right| \leq C_N t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right)$$

and

$$t^N\left|\sqrt{t}A_jK_t^L(x,y)\right| \le C_N t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right).$$

Since $\rho(x) = \frac{1}{1+|x|}$, we get $\frac{\sqrt{t}}{\rho(x)} = \sqrt{t}(1+|x|)$. Then, for $N > 0$,

$$\left(\frac{\sqrt{t}}{\rho(x)}\right)^N \left|\sqrt{t}A_jK_t^L(x,y)\right| \le C_N t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right). \tag{11}$$

Since x and y are symmetric, we also have

$$\left(\frac{\sqrt{t}}{\rho(y)}\right)^N \left|tA_jK_t^L(x,y)\right| \le C_N t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right). \tag{12}$$

Then (a) follows from (10)–(12).

(b) Note that

$$\begin{aligned}
&\left|\sqrt{t}A_jK_t^L(x',y) - \sqrt{t}A_jK_t^L(x,y)\right| \\
&\quad \le \left|\sqrt{t}\frac{\partial}{\partial x_j}K_t^L(x',y) - \sqrt{t}\frac{\partial}{\partial x_j}K_t^L(x,y)\right| + \left|\sqrt{t}x_j'K_t^L(x',y) - \sqrt{t}x_jK_t^L(x,y)\right| \\
&\quad \doteq J_1 + J_2.
\end{aligned}$$

For J_2, let

$$\varphi(z) = \varphi_{y,s}(z) = z_j \exp\left(-\frac{1}{4}\alpha(s,z,y)\right),$$

where $\alpha(s,z,y) = s|z+y|^2 + \frac{1}{s}|z-y|^2$.

Then

$$\frac{\partial\varphi}{\partial z_k}(z) = \left(\delta_{jk} - \frac{s}{2}z_j(z_k+y_k) - \frac{1}{2s}z_j(z_k-y_k)\right)\exp\left(-\frac{1}{4}\alpha(s,z,y)\right).$$

Therefore

$$\begin{aligned}
\left|\frac{\partial\varphi}{\partial z_k}(z)\right| &\le C\left(1 + s|z||z+y| + \frac{1}{s}|z||z-y|\right)\exp\left(-\frac{1}{4}\alpha(s,z,y)\right) \\
&\le C\left(1 + s^{1/2}|z| + \frac{1}{s^{1/2}}|z|\right)\exp\left(-\frac{1}{8}\alpha(s,z,y)\right) \\
&\le C\left(1 + s^{1/2}\left(|z-y| + |z+y|\right) + \frac{1}{s^{1/2}}\left(|z-y| + |z+y|\right)\right)\exp\left(-\frac{1}{8}\alpha(s,z,y)\right) \\
&\le C\left(1 + s + \frac{1}{s}\right)\exp\left(-\frac{1}{16s}|z-y|^2\right) \\
&\le Cs^{-1}\exp\left(-\frac{1}{16s}|z-y|^2\right).
\end{aligned} \tag{13}$$

Let $\theta = \lambda x + (1-\lambda)x'$, $0 < \lambda < 1$. Then

$$\begin{aligned}
J_2 &\le Ct^{-d/2}\left|x'_j K_s(x',y) - x_j K_s(x,y)\right| \\
&\le Ct^{-d/2}\left|x - x'\right| \sup_\theta \left|\nabla\varphi(\theta)\right| \\
&\le Ct^{-d/2}\frac{|x-x'|}{s}| \sup_\theta \exp\left(-\frac{|\theta - y|^2}{16s}\right) \\
&\le Ct^{-d/2}\frac{|x-x'|}{t}| \sup_\theta \exp\left(-\frac{|\theta - y|^2}{16t}\right).
\end{aligned}$$

When $|x - x'| \le \frac{|x-y|}{2}$, we can get $|\theta - y| \sim |x - y|$. Therefore, there exists $A > 0$ such that

$$J_2 \le Ct^{-d/2}\frac{|x-x'|}{t} \exp\left(-\frac{|x-y|^2}{At}\right). \tag{14}$$

For J_1,

$$\begin{aligned}
J_1 &= \left|\sqrt{t}\frac{\partial}{\partial x_j}K_t^L(x',y) - \sqrt{t}\frac{\partial}{\partial x_j}K_t^L(x,y)\right| \\
&= \sqrt{t}\left|\frac{\partial}{\partial x_j}K_s(x',y) - \frac{\partial}{\partial x_j}K_s(x,y)\right| \\
&= \sqrt{t}\left|\left(s(x_j + y_j) + \frac{1}{s}(x_j - y_j)\right)\exp\left(-\frac{1}{4}\alpha(s,x,y)\right)\right. \\
&\quad \left. - \left(s(x'_j + y_j) + \frac{1}{s}(x'_j - y_j)\right)\exp\left(-\frac{1}{4}\alpha(s,x',y)\right)\right|.
\end{aligned}$$

Let

$$\psi(z) = \psi_{y,s}(z) = \left(s(z_j + y_j) + \frac{1}{s}(z_j - y_j)\right)\exp\left(-\frac{1}{4}\alpha(s,z,y)\right).$$

Then

$$\begin{aligned}
\frac{\partial\psi}{\partial z_k}(z) = \Bigg[&\left(s + \frac{1}{s}\right)\delta_{jk} - \frac{1}{2}\left(s(z_j + y_j) + \frac{1}{s}(z_j - y_j)\right) \\
&s(z_k + y_k) + \frac{1}{s}(z_k - y_k)\Bigg]\exp\left(-\frac{1}{4}\alpha(s,z,y)\right).
\end{aligned}$$

Therefore, similar to the proofs of (13) and (14), we can prove

$$\left|\frac{\partial\psi}{\partial z_k}(z)\right| \le Cs^{-1}\exp\left(-\frac{1}{4}\alpha(s,z,y)\right)$$

and

$$\begin{aligned}
J_1 &\le C\sup_\theta\left|\nabla\psi(\theta)\right|\left|x - x'\right| \\
&\le Ct^{-d/2}\frac{|x-x'|}{t}| \exp\left(-\frac{|x-y|^2}{At}\right).
\end{aligned} \tag{15}$$

Inequalities (13) and (15) show

$$\left|\sqrt{t}A_jK_t^L(x,y)-\sqrt{t}A_jK_t^L(x',y)\right|\le C_N\frac{|x-x'|}{t}t^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{At}\right).$$

Then, similar to the proof of (a), we have

$$\left|\sqrt{t}A_jK_t^L(x,y)-\sqrt{t}A_jK_t^L(x',y)\right|\le C_N\frac{|x-x'|}{t}t^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{At}\right)\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

(c) Noting that

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\sqrt{t}A_jK_t^L(x,y)\,dy\right|\\&\quad\le\left|\int_{\mathbb{R}^d}\sqrt{t}\partial_{x_j}K_t^L(x,y)\,dy\right|+\left|\int_{\mathbb{R}^d}\sqrt{t}x_jK_t^L(x,y)\,dy\right|\\&\quad\doteq I+II.\end{aligned}$$

The proof of part I can be found in Lemma 3.9 of [6]. For part II, since $|x_j|\le 1+|x|=\frac{1}{\rho(x)}$ and Lemma 1, we get

$$II\le\frac{\sqrt{t}}{\rho(x)}\int_{\mathbb{R}^d}\sqrt{t}\left|K_t^L(x,y)\right|dy\le\frac{\frac{\sqrt{t}}{\rho(x)}}{(1+\frac{\sqrt{t}}{\rho(x)})^N}.$$

Therefore, part (c) holds and this completes the proof of Proposition 4. □

Lemma 4 and the subordination formula give the following.

Lemma 5 *There is $C>0$ for $N\in\mathbb{N}$ and $|x-x'|\le\frac{|x-y|}{2}$, we can find $C_N>0$ such that*

(a) $$\left|Q_t^L(x,y)\right|\le C_N\frac{t}{(t^2+A|x-y|^2)^{(d+1)/2}}\left(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\right)^{-N};$$

(b) $$\left|Q_t^L(x,y)-Q_t^L(x',y)\right|\le C_N\left(\frac{|x-x'|}{t}\right)\frac{t}{(t^2+A|x-y|^2)^{(d+1)/2}}\left(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\right)^{-N};$$

(c) $$\left|\int_{\mathbb{R}^d}Q_t^L(x,y)\,dy\right|\le C_N\frac{t/\rho(x)}{(1+t/\rho(x))^N}.$$

3 Carleson measure characterization of Λ_α^L

Let s_L denote the Littlewood–Paley g-function associated with L, i.e.,

$$s_Lf(x)=\left(\int_0^\infty\left|Q_t^Lf(x)\right|^2\frac{dt}{t}\right)^{1/2},$$

and A_L denote the Lusin area integral associated with L, i.e.,

$$A_Lf(x)=\left(\int_0^\infty\int_{\Gamma(x)}\left|Q_t^Lf(x)\right|^2\frac{dy\,dt}{t}\right)^{1/2}.$$

Then we can prove the following.

Lemma 6 *The operators s_L and A_L are isometries on $L^2(\mathbb{R}^d)$ up to constant factors. Exactly,*

$$\|s_L f\|_{L^2} = \frac{1}{2}\|f\|_{L^2}, \qquad \|A_L f\|_{L^2} = C_d\|f\|_{L^2}.$$

The proof of Lemma 6 is standard, we omit it.

Let $F(x,t) = Q_t^L f(x)$ and $G(x,t) = Q_t^L g(x)$. Then we have the following lemma.

Lemma 7 *If $g \in L^1((1+|x|)^{-(d+1)}\,dx)$ and f is an $H_L^{p,\infty}$-atom, then*

$$\frac{1}{4}\int_{\mathbb{R}^d} f(x)\overline{g(x)}\,dx = \int_{\mathbb{R}_+^{d+1}} F(x,t)G(x,t)\frac{dx\,dt}{t}.$$

Lemma 8 *There exists $C > 0$ such that, for any $H_L^{p,\infty}$-atom $a(x)$, we have $\|A_L a\|_{L^p} \le C$.*

The proofs of Lemmas 7 and 8 can be found in [8].

Now we can give the proof of Theorem 1.

Proof of Theorem 1 Let $f \in \Lambda^L_{d(1/p-1)}$, then $f \in L^1((1+|x|)^{-(d+1)}\,dx)$. By Lemma 5(a), we know

$$Q_t^L f(x) = \int_{\mathbb{R}^d} Q_t^L(x,y)f(y)\,dy$$

is absolutely convergent. To prove the assertion (a), we need to prove that, for any ball $B = B(x_0, r)$,

$$\frac{1}{|B|^{2/p-1}}\int_{\widehat{B}}\bigl|Q_t^L f(x)\bigr|^2\frac{dx\,dt}{t} \le C\|f\|^2_{\Lambda^L_{d(1/p-1)}}. \tag{16}$$

Set $B_k = B(x_0, 2^k r)$ and

$$f = \bigl(f - f(B_1)\bigr)\chi_{B_1} + \bigl(f - f(B_1)\bigr)\chi_{B_1^c} + f(B_1) = \widetilde{f}_1 + \widetilde{f}_2 + f(B_1).$$

By Lemma 6, we have

$$\begin{aligned}
\frac{1}{|B|^{2/p-1}}\int_{\widehat{B}}\bigl|Q_t^L\widetilde{f}_1(x)\bigr|^2\frac{dx\,dt}{t} &\le \frac{1}{|B|^{2/p-1}}\int_B\bigl|s_L\widetilde{f}_1(x)\bigr|^2\,dx \\
&= \frac{1}{4|B|^{2/p-1}}\|\widetilde{f}_1\|^2_{L^2} = \frac{1}{4|B|^{2/p-1}}\int_{B_1}\bigl|f(g) - f(B_1)\bigr|^2\,dx \\
&\le C\|f\|^2_{\Lambda^L_{d(1/p-1)}}.
\end{aligned} \tag{17}$$

Note that

$$\begin{aligned}
\bigl|f(B_2) - f(B_1)\bigr| &\le 2^d\frac{1}{|B_2|}\int_{B_2}\bigl|f(x) - f(B_2)\bigr|\,dx \\
&\le 2^d\frac{1}{|B_2|^{1/2}}\biggl(\int_{B_2}\bigl|f(x) - f(B_2)\bigr|^2\,dx\biggr)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= 2^d |B_2|^{1/p-1} \frac{1}{|B_2|^{1/p-1/2}} \left(\int_{B_2} \left| f(x) - f(B_2) \right|^2 dx \right)^{1/2} \\
&\leq 2^d |B_2|^{1/p-1} \|f\|_{\Lambda^L_{d(1/p-1)}}.
\end{aligned}
$$

Therefore

$$
\left| f(B_{k+1}) - f(B_1) \right| \leq Ck|B_{k+1}|^{1/p-1} \|f\|_{\Lambda^L_{d(1/p-1)}}. \tag{18}
$$

For $x \in B(x_0, r)$, by Lemma 5(a) and (18),

$$
\begin{aligned}
\left| Q^L_t \widetilde{f}_2(x) \right| &\leq C \int_{\mathbb{R}^d} \frac{t}{(t^2 + C|x-y|^2)^{(d+1)/2}} \left| \widetilde{f}_2(y) \right| dy \\
&\leq C \int_{(B_1)^c} \frac{t}{|x_0 - y|^{(d+1)}} \left| f(y) - f(B_1) \right| dy \\
&\leq C \sum_{k=1}^{\infty} \frac{t}{(2^k r)^{d+1}} \left(\int_{B_{k+1} \setminus B_k} \left| f(y) - f(B_{k+1}) \right| dy + \left(2^k r\right)^d \left| f(B_{k+1}) - f(B_1) \right| \right) \\
&\leq C \frac{t}{r^{1-d(1/p-1)}} \sum_{k=1}^{\infty} 2^{k(d(1/p-1)-1)} (1+k) \|f\|_{\Lambda^L_{d(1/p-1)}} \\
&\leq C \frac{t}{r^{1-d(1/p-1)}} \|f\|_{\Lambda^L_{d(1/p-1)}}.
\end{aligned}
$$

In the last step of the above, we use the facts $\frac{d}{d+1} < p \leq 1$ to get $d(1/p-1) - 1 < 0$.

Thus we have

$$
\frac{1}{|B|^{2/p-1}} \int_{\widehat{B}} \left| Q^L_t \widetilde{f}_2(x) \right|^2 \frac{dx\,dt}{t} \leq C \|f\|^2_{\Lambda^L_{d(1/p-1)}}. \tag{19}
$$

It remains to estimate the constant term. Assume first that $r < \rho(x_0)$. Taking k_0 such that $2^{k_0} r < \rho(x_0) \leq 2^{k_0+1} r$, we have

$$
\begin{aligned}
\left| f(B_1) \right| &\leq \left| f(B_{k_0+1}) - f(B_1) \right| + \left| f(B_{k_0+1}) \right| \\
&\leq Ck_0 |B_{k_0+1}|^{1/p-1} \|f\|_{\Lambda^L_{d(1/p-1)}} + |B_{k_0+1}|^{1/p-1} \|f\|_{\Lambda^L_{d(1/p-1)}} \\
&\leq C \left(1 + \log_2 \frac{\rho(x_0)}{r} \right) |B_{k_0+1}|^{1/p-1} \|f\|_{\Lambda^L_{d(1/p-1)}}.
\end{aligned}
$$

Note that $\rho(x) \sim \rho(x_0) > r$ for any $x \in B(x_0, r)$, by using Lemma 5(c), we get

$$
\begin{aligned}
&\frac{1}{|B|^{2/p-1}} \int_{\widehat{B}} \left| Q^L_t \left(f(B_1) \mathbf{1} \right)(x) \right|^2 \frac{dx\,dt}{t} \\
&= \frac{|f(B_1)|^2}{|B|^{2/p-1}} \int_{\widehat{B}} \left| \int_{\mathbb{R}^d} Q^L_t(x,y)\,dy \right|^2 \frac{dx\,dt}{t} \\
&\leq \frac{C|f(B_1)|^2}{|B|^{2/p-1}} \int_{\widehat{B}} \left(\frac{t}{\rho(x_0)} \right)^2 \frac{dx\,dt}{t} \\
&\leq C \frac{|B_{k_0+1}|^{2/p-2}}{|B|^{2/p-2}} \left(1 + \log_2 \frac{\rho(x_0)}{r} \right)^2 \left(\frac{r}{\rho(x_0)} \right)^2 \|f\|^2_{\Lambda^L_{d(1/p-1)}}
\end{aligned}
$$

$$
\begin{aligned}
&= C\left(1+\log_2\frac{\rho(x_0)}{r}\right)^2\left(\frac{r}{\rho(x_0)}\right)^{2-2d(1/p-1)}\|f\|^2_{\Lambda^L_{d(1/p-1)}} \\
&\leq C\|f\|^2_{\Lambda^L_{d(1/p-1)}}.
\end{aligned} \tag{20}
$$

In the last step of the above, we use the fact $d(1/p-1)<1$. For $r\geq\rho(x_0)$, we have $|f(B_1)|\leq C|B_1|^{1/p-1}\|f\|_{\Lambda^L_{d(1/p-1)}}$.

Note that $\rho(x)\leq Cr$ for any $x\in B(x_0,r)$, again by Lemma 5(c), we get

$$
\begin{aligned}
&\frac{1}{|B|^{2/p-1}}\int_{\widehat{B}}\left|Q_t^L\big(f(B_1)\mathbf{1}\big)(x)\right|^2\frac{dx\,dt}{t} \\
&\leq \frac{|f(B_1)|^2}{|B|^{2/p-1}}\int_0^\infty\int_B\left|\int_{\mathbb{R}^d}Q_t^L(x,y)\,dy\right|^2\frac{dx\,dt}{t} \\
&\leq \frac{C|f(B_1)|^2}{|B|^{2/p-1}}\left(\int_B\int_0^{\rho(x)}\left(\frac{t}{\rho(x)}\right)^2\frac{dt}{t}\,dx+\int_B\int_{\rho(x)}^\infty\left(\frac{t}{\rho(x)}\right)^{-2}\frac{dt}{t}\,dx\right) \\
&\leq C\|f\|^2_{\Lambda^L_{d(1/p-1)}}.
\end{aligned} \tag{21}
$$

Then (17) follows from (18)–(21). This proves part (a).

Let $f\in L^1((1+|x|)^{-(d+1)}\,dx)$ and $Q_t^Lf(x)\in T_2^{p,\infty}$. We want to prove that $f\in\Lambda^L_{d(1/p-1)}$. By $\Lambda^L_{d(1/p-1)}$ is the dual space of $H_L^p(\mathbb{R}^d)$, it is sufficient to prove that

$$
H_L^p\ni g\mapsto\mathcal{L}_f(g):=\int_{\mathbb{R}^d}f(x)g(x)\,dx
$$

defined on finite linear combinations of $H_L^{p,\infty}$-atoms satisfies the estimate

$$
\left|\mathcal{L}_f(g)\right|\leq C\left\|Q_t^Lf\right\|_{T_2^{p,\infty}}\|g\|_{H_L^p}.
$$

By Lemma 7, Lemma 8, and Proposition 3, we get

$$
\begin{aligned}
\left|\mathcal{L}_f(g)\right| &= \left|\int_{\mathbb{R}^d}f(x)g(x)\,dx\right| \\
&= 4\left|\int_{\mathbb{R}^{d+1}_+}Q_t^Lf(x)Q_t^Lg(x)\frac{dx\,dt}{t}\right| \\
&\leq C\left\|Q_t^Lf\right\|_{T_2^{p,\infty}}\left\|Q_t^Lg\right\|_{T_2^p} \\
&\leq C\left\|Q_t^Lf\right\|_{T_2^{p,\infty}}\|g\|_{H_L^p}.
\end{aligned}
$$

This gives the proof of part (b) and then Theorem 1 is proved. □

4 The predual space of Hardy space $H_L^p(\mathbb{R}^d)$

In this section, we give a Carleson measure characterization of the space $\lambda^L_{d(1/p-1)}(\mathbb{R}^d)$.

Proof of Theorem 2 Let $f\in\lambda^L_{d(1/p-1)}$, then $f\in\Lambda^L_{d(1/p-1)}$. By Theorem 1, we know $f\in L^1((1+|x|)^{-(d+1)}\,dx)$. To prove $Q_t^Lf\in T_{2,0}^{p,\infty}$, we first prove that there exists a constant $C>0$ such that, for any ball $B = B(x_0, r)$, we have

$$\frac{1}{|B|^{2/p-1}} \int_{\widehat{B}} \left|Q_t^L f(x)\right|^2 \frac{dx\,dt}{t} \le \sum_{k=1}^{\infty} 2^{-k(1-d(1/p-1))} \beta_k(f, B), \tag{22}$$

where

$$\beta_k(f, B) = \sup_{B' \subset B_{k+1}} \frac{1}{|B'|^{2/p-1}} \int_{B'} \left|f(y) - f(B')\right|^2 dy.$$

We first assume (22) holds, then we show that $Q_t^k f \in T_{2,0}^{p,\infty}$. In fact, as $f \in \lambda_{d(1/p-1)}^L$, we have $f \in \Lambda_{d(1/p-1)}^L$ and there exists a constant $C > 0$ such that

$$\beta_k(f, B) \le C\|f\|_{\Lambda_{d(1/p-1)}^L}.$$

Then, for any $k \in \mathbb{N}$, we have

$$\lim_{a\to 0} \sup_{B\subset\mathbb{R}^d r_B \le a} \beta_k(f, B) = \lim_{a\to\infty} \sup_{B\subset\mathbb{R}^d r_B \ge a} \beta_k(f, B) = \lim_{a\to\infty} \sup_{B\subset\mathbb{R}^d B\subset B(0,a)^c} \beta_k(f, B) = 0. \tag{23}$$

By (22), we have

$$\begin{aligned}
&\frac{1}{|B|^{2/p-1}} \int_{\widehat{B}} \left|Q_t^L f(x)\right|^2 \frac{dx\,dt}{t} \\
&\quad \le C \sum_{k=1}^{k_0} 2^{-k(1-d(1/p-1))} \beta_k(f, B) + C \sum_{k=k_0}^{\infty} 2^{-k(1-d(1/p-1))} \|f\|^2_{\Lambda_{d(1/p-1)}^L} \\
&\quad \le C \sum_{k=1}^{k_0} 2^{-k(1-d(1/p-1))} \beta_k(f, B) + C 2^{-k_0(1-d(1/p-1))} \|f\|^2_{\Lambda_{d(1/p-1)}^L}.
\end{aligned}$$

We can take k_0 large enough such that $2^{-k_0/2}\|f\|^2_{\Lambda_{d(1/p-1)}^L}$ is small. This proves that $\|Q_t^L f\|_{T_2^{p,\infty}} < \infty$ and $\eta_1(f) = \eta_2(f) = \eta_3(f) = 0$ follows from (23). Therefore $Q_t^L f \in T_{2,0}^{p,\infty}$.

Now we give the proof of (22). Set $B_k = B(x_0, 2^k r)$ and

$$f = \left(f - f(B_1)\right)\chi_{B_1} + \left(f - f(B_1)\right)\chi_{(B_1)^c} + f(B_1) = \widetilde{f}_1 + \widetilde{f}_2 + f(B_1).$$

By Lemma 6, we have

$$\begin{aligned}
\frac{1}{|B|^{2/p-1}} \int_{\widehat{B}} \left|Q_t^L \widetilde{f}_1(x)\right|^2 \frac{dx\,dt}{t} &\le \frac{1}{|B|^{2/p-1}} \int_B \left|s_L \widetilde{f}_1(x)\right|^2 dx \\
&= \frac{1}{4|B|^{2/p-1}} \int_{B_1} \left|f(x) - f(B_1)\right|^2 dx \le C\beta_1(f, B).
\end{aligned} \tag{24}$$

By

$$\left|f(B_{k+1}) - f(B_1)\right| \le C \sum_{i=2}^{k+1} |B_i|^{1/p-1} \frac{1}{|B_i|^{1/p-1/2}} \left(\int_{B_i} \left|f(x) - f(B_i)\right|^2 dx\right)^{1/2}$$

and Lemma 5(a), for $x \in B(x_0, r)$,

$$\begin{aligned}
\left|Q_t^L \widetilde{f}_2(x)\right| &\le C \int_{\mathbb{R}^d} \frac{t}{|x_0 - y|^{(d+1)}} \left|\widetilde{f}_2(y)\right| dy \\
&\le C \int_{(B_1)^c} \frac{t}{|x_0 - y|^{(d+1)}} \left|f(y) - f(B_1)\right| dy \\
&\le C \sum_{k=1}^{\infty} \frac{t}{(2^k r)^{d+1}} \left(\int_{B_{k+1} \setminus B_k} \left|f(y) - f(B_{k+1})\right| dy \right. \\
&\quad \left. + \left(2^k r\right)^d \left|f(B_{k+1}) - f(B_1)\right| \right) \\
&\le C \frac{t}{r^{1-d(1/p-1)}} \sum_{k=1}^{\infty} 2^{k(d(1/p-1)-1)} \left(\frac{1}{|B_{k+1}|^{1/p-1/2}} \left(\int_{B_{k+1}} \left|f - f(B_{k+1})\right|^2 dy \right)^{1/2} \right. \\
&\quad \left. + \sum_{i=2}^{k+1} \frac{1}{|B_i|^{1/p-1/2}} \left(\int_{B_i} \left|f - f(B_i)\right|^2 dy \right)^{1/2} \right) \\
&\le C \frac{t}{r^{1-d(1/p-1)}} \sum_{k=1}^{\infty} 2^{k(d(1/p-1)-1)} (1+k) \beta_k(f, B)^{1/2}.
\end{aligned}$$

Therefore

$$\frac{1}{|B|^{2/p-1}} \int_{\widehat{B}} \left|Q_t^L \widetilde{f}_2(x)\right|^2 \frac{dx\,dt}{t} \le C \sum_{k=1}^{\infty} 2^{k(d(1/p-1)-1)} \beta_k(f, B). \tag{25}$$

It remains to estimate the constant term. Assume first that $r < \rho(x_0)$. Taking k_0 such that $2^{k_0} r < \rho(x_0) \le 2^{k_0+1} r$, we have

$$\begin{aligned}
\left|f(B_1)\right| &\le \left|f(B_{k_0+1}) - f(B_1)\right| + \left|f(B_{k_0+1})\right| \\
&\le C \sum_{i=2}^{k_0+1} |B_i|^{1/p-1} \frac{1}{|B_i|^{1/p-1/2}} \left(\int_{B_i} \left|f - f(B_i)\right|^2 dy \right)^{1/2} \\
&\quad + |B_{k_0+1}|^{1/p-1} \frac{1}{|B_{k_0+1}|^{1/p-1/2}} \left(\int_{B_{k_0+1}} |f|^2 dy \right)^{1/2} \\
&\le C |B_{k_0+1}|^{1/p-1} (k_0 + 1) \beta_{k_0}^{1/2}(f, B).
\end{aligned}$$

Note that $\rho(x) \sim \rho(x_0) > r$ for any $x \in B(x_0, r)$, by Lemma 5(c), we get

$$\begin{aligned}
&\frac{1}{|B|^{2/p-1}} \int_{\widehat{B}} \left|Q_t^L \left(f(B_1)\mathbf{1}\right)(x)\right|^2 \frac{dx\,dt}{t} \\
&= \frac{|f(B_1)|^2}{|B|^{2/p-1}} \int_{\widehat{B}} \left| \int_{\mathbb{R}^d} Q_t^L(x, y)\, dy \right|^2 \frac{dx\,dt}{t} \\
&\le \frac{C|f(B_1)|^2}{|B|^{2/p-1}} \int_{\widehat{B}} \left(\frac{t}{\rho(x_0)} \right)^2 \frac{dx\,dt}{t} \\
&\le C \frac{|B_{k_0+1}|^{2/p-2}}{|B|^{2/p-2}} (1 + k_0)^2 \left(\frac{r}{\rho(x_0)} \right)^2 \beta_{k_0}(f, B)
\end{aligned}$$

$$
\begin{aligned}
&\le C2^{-2k_0(1-d(1/p-1))}(1+k_0)^2\beta_{k_0}(f,B) \\
&\le C2^{-k_0(1-d(1/p-1))}\beta_{k_0}(f,B).
\end{aligned} \tag{26}
$$

For $r \ge \rho(x_0)$, we have

$$
\left|f(B_1)\right| \le C|B_1|^{1/p-1}\frac{1}{|B_1|^{1/p-1/2}}\left(\int_{B_1}\left|f-f(B_1)\right|^2\,dy\right)^{1/2}.
$$

Note that $\rho(x) \le Cr$ for any $x \in B(x_0,r)$, again by Lemma 5(c),

$$
\begin{aligned}
&\frac{1}{|B|^{2/p-1}}\int_{\widehat{B}}\left|Q_t^L\big(f(B_1)\mathbf{1}\big)(x)\right|^2\frac{dx\,dt}{t} \\
&\le \frac{|f(B_1)|^2}{|B|^{2/p-1}}\int_0^\infty\int_B\left|\int_{\mathbb{R}^d}Q_t^L(x,y)\,dy\right|^2\frac{dx\,dt}{t} \\
&\le \frac{C|f(B_1)|^2}{|B|^{2/p-1}}\left(\int_B\int_0^{\rho(x)}\left(\frac{t}{\rho(x)}\right)^2\frac{dt}{t}\,dx+\int_B\int_{\rho(x)}^\infty\left(\frac{t}{\rho(x)}\right)^{-2}\frac{dt}{t}\,dx\right) \\
&\le C\frac{|B_1|^{2/p-1}}{|B|^{2/p-1}}\beta_1(f,B) \le 2^{d(1/p-1)-1}\beta_1(f,B).
\end{aligned} \tag{27}
$$

Then (22) follows from (24)–(27).

For the reverse, by Theorem 1, we get $f \in \Lambda^L_{d(1/p-1)}$ from $Q_t^L f \in T^{p,\infty}_{2,0}$. For any ball $B = B(x_0,r)$,

$$
\begin{aligned}
\left(\int_B\left|f(x)-f(B)\right|^2\,dx\right)^{1/2} &= \sup_{\operatorname{supp} g\subset B,\|g\|_{L^2(B)}\le 1}\left|\int_B\big(f(x)-f(B)\big)g(x)\,dx\right| \\
&= \sup_{\operatorname{supp} g\subset B,\|g\|_{L^2(B)}\le 1}\left|\int_B f(x)\big(g(x)-g(B)\big)\,dx\right|.
\end{aligned}
$$

Let $G(x) = (g(x)-g(B))\chi_B$. Then, by Lemma 7, we obtain

$$
\begin{aligned}
\left|\int_{\mathbb{R}^d}f(x)G(x)\,dx\right| &= 4\left|\int_{\mathbb{R}^{d+1}_+}Q_t^Lf(x)\big(Q_t^Lg(x)-Q_t^Lg(B)(x)\big)\frac{dx\,dt}{t}\right| \\
&\le C\int_{\widehat{B_2}}\left|Q_t^Lf(x)\right|\left|Q_t^LG(x)\right|\frac{dx\,dt}{t} \\
&\quad+\sum_{k=2}^\infty\int_{\widehat{B_{k+1}}\setminus\widehat{B_k}}\left|Q_t^Lf(x)\right|\left|Q_t^LG(x)\right|\frac{dx\,dt}{t} \\
&= E_1+\sum_{k=2}^\infty E_k.
\end{aligned}
$$

By Hölder's inequality and Lemma 6, we have

$$
\begin{aligned}
E_1 &\le \left(\int_{\widehat{B_2}}\left|Q_t^Lf(x)\right|^2\frac{dx\,dt}{t}\right)^{1/2}\left\|\left(\int_0^\infty\left|Q_t^L\big(g-g(B)\big)(x)\right|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^2(B_2)} \\
&\le C\left(\int_{\widehat{B_2}}\left|Q_t^Lf(x)\right|^2\frac{dx\,dt}{t}\right)^{1/2}.
\end{aligned} \tag{28}
$$

Now, we estimate E_k. By Hölder's inequality again, we have that

$$E_k \le F_k \cdot I_k,$$

where

$$F_k = \left(\int_{\widehat{B_{k+1}}\setminus\widehat{B_k}} \left|Q_t^L f(x)\right|^2 \frac{dx\,dt}{t}\right)^{1/2}$$

and

$$I_k = \left(\int_{\widehat{B_{k+1}}\setminus\widehat{B_k}} \left|Q_t^L g(x) - Q_t^L g(B)(x)\right|^2 \frac{dx\,dt}{t}\right)^{1/2}.$$

When $r < \rho(x_0)$, then $\int_B g(x) - g(B)\,dx = 0$. Therefore, by Lemma 5(b),

$$\begin{aligned}
\left|Q_t^L g(x) - Q_t^L g(B)(x)\right| &= \left|\int_B \big(Q_t^L(x,y) - Q_t^L(x,x_0)\big)\big(g(y) - g(B)\big)\,dy\right| \\
&\le C\int_B \frac{t}{(t+|x-y|)^{d+1}} \frac{|x_0-y|}{t} \left|g(y) - g(B)\right|dy \\
&\le C\int_B \frac{t}{(2^k r)^{d+1}} \frac{r}{t} \left|g(y) - g(B)\right|dy \\
&\le C\frac{t}{(2^k r)^{d+1}} \frac{r}{t} \|g\|_{L^1(B)} \le C|B|^{1/2} \frac{t}{(2^k r)^{d+1}} \frac{r}{t}.
\end{aligned}$$

Therefore

$$I_k^2 \le C|B| \int_0^{2^{k+1}r} \int_{B_{k+1}} \frac{t^2}{(2^k r)^{2d+2}} \left(\frac{r}{t}\right)^2 \frac{dx\,dt}{t} \le C|B| \frac{1}{(2^k r)^d} 2^{-2k}.$$

It follows that

$$E_k \le C|B|^{1/2}|B_k|^{-1/2}2^{-k}\left(\int_{\widehat{B_{k+1}}} \left|Q_t^L f(x)\right|^2 \frac{dx\,dt}{t}\right)^{1/2}.$$

When $r \ge \rho(x_0)$, we have $\rho(y) \le Cr$ for $y \in B(x_0, r)$. Then, by Lemma 5(a),

$$\begin{aligned}
\left|Q_t^L g(x) - Q_t^L g(B)(x)\right| &= \left|\int_B Q_t^L(x,y) g(y)\,dy\right| \\
&\le C\int_B \frac{t}{(2^k r)^{d+1}} \frac{\rho(y)}{t} \left|g(y)\right|dy \\
&\le C\frac{t}{(2^k r)^{d+1}} \frac{r}{t} \|g\|_{L^1(B)} \le C|B|^{1/2} \frac{t}{(2^k r)^{d+1}} \frac{r}{t}.
\end{aligned}$$

Then we can get

$$E_k \le C|B|^{1/2}|B_k|^{-1/2}2^{-k}\left(\int_{\widehat{B_{k+1}}} \left|Q_t^L f(x)\right|^2 \frac{dx\,dt}{t}\right)^{1/2}. \tag{29}$$

By (28) and (29), we know

$$\begin{aligned}
&\frac{1}{|B|^{1/p-1/2}}\left(\int_B \left|f(x)-f(B)\right|^2 dx\right)^{1/2} \\
&\quad\leq \frac{C}{|B|^{1/p-1}}\sum_{k=1}^{\infty} 2^{-k}|B_k|^{-1/2}\left(\int_{\widehat{B_k}} \left|Q_t^L f(x)\right|^2 \frac{dx\,dt}{t}\right)^{1/2} \\
&\quad= C\sum_{k=1}^{\infty} 2^{-k}\frac{|B_k|^{1/p-1}}{|B|^{1/p-1}}\frac{1}{|B_k|^{1/p-1/2}}\left(\int_{\widehat{B_k}} \left|Q_t^L f(x)\right|^2 \frac{dx\,dt}{t}\right)^{1/2} \\
&\quad\leq C\sum_{k=1}^{\infty} 2^{-k(1-\alpha)}\sigma_k(f,B),
\end{aligned}$$

where

$$\sigma_k(f,B) = \frac{1}{|B_k|^{1/p-1/2}}\left(\int_{\widehat{B_k}} \left|Q_t^L f(x)\right|^2 \frac{dx\,dt}{t}\right)^{1/2}.$$

Then we can get $\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0$ as the proof of the first part of this theorem. Therefore $f \in \lambda_\alpha^L$ and the proof of Theorem 2 is completed. □

5 Conclusions

This paper defines a new version of Carleson measure associated with Hermite operator, which is adapted to the operator L. Then, we characterize the dual spaces and predual spaces of the Hardy spaces $H_L^p(\mathbb{R}^d)$ associated with L. The main results of this paper are the central problems in harmonic analysis, which can be used in PED or geometry widely.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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