Zheng et al. *Journal of Inequalities and Applications* (2018) 2018:170
https://doi.org/10.1186/s13660-018-1762-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Nonlinear impulsive differential and integral inequalities with nonlocal jump conditions

Zhaowen Zheng[1*], Yingjie Zhang[1] and Jing Shao[2,3]

*Correspondence:
zhwzheng@126.com
[1] School of Mathematical Sciences, Qufu Normal University, Qufu, P.R. China
Full list of author information is available at the end of the article

Abstract

Some new nonlinear impulsive differential and integral inequalities with nonlocal integral jump conditions are presented in this paper. Using the method of mathematical induction, we obtain a new upper bound estimation of certain differential and integral inequalities; these inequalities have both nonlocal integral jump and weakly singular kernels. Finally, we give two examples of these inequalities in estimating solutions of certain equations with Riemann–Liouville fractional integral conditions.

Keywords: Nonlinear impulsive; Differential and integral inequalities; Nonlocal intergal jump conditions; Riemann–Liouville fractional derivative

1 Introduction

As is well known, impulsive differential and impulsive integral inequalities play a fundamental part in the study of theory of impulsive equations (see [1–4]). Recently, a lot of experts studied the global existence, uniqueness, bounded-ness, stability, oscillation and other properties of different impulsive inequalities (see [5–18]). For example, in [1], Lakshmikanthan investigated an impulsive differential inequality given as Theorem 1.1.

Let $0 \le t_0 < t_1 < t_2 < \cdots$ be a sequence, $\lim_{k\to\infty} t_k = \infty$, $\mathbb{R}_+ = [0,+\infty)$. For $I \subset \mathbb{R}$, we define the following set of functions:

$PC(\mathbb{R}_+, I) = \{u : \mathbb{R}_+ \to I;\ u(t)$ is continuous for $t \ne t_k$, $u(0^+)$, $u(t_k^+)$, $u(t_k^-)$ exist, and $u(t)$ is left-continuous at t_k, $k = 1,2,\ldots\}$;

$PC^1(\mathbb{R}_+, I) = \{u \in PC(\mathbb{R}_+, I);\ u'(t)$ is continuous for $t \ne t_k$, $u'(0^+)$, $u'(t_k^+)$, $u'(t_k^-)$ exist, and $u'(t)$ is left-continuous at t_k, $k = 1,2,\ldots\}$.

Theorem 1.1 *Assume that*:

(H_0) *the sequence* $\{t_k\}$ *satisfies* $0 \le t_0 < t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = \infty$;

(H_1) $m \in PC^1[\mathbb{R}_+, \mathbb{R}]$ *and* $m(t)$ *is left-continuous at* t_k, $k = 1,2,\ldots$;

(H_2) *for* $k = 1,2,\ldots$, $t \ge t_0$,

$$m'(t) \le p(t)m(t) + q(t), \quad t \ne t_k, \tag{1.1}$$

$$m\bigl(t_k^+\bigr) \le d_k m(t_k) + b_k, \tag{1.2}$$

where $p, q \in C[\mathbb{R}_+, \mathbb{R}]$, $d_k \geq 0$ and b_k $(k = 1, 2, \ldots)$ are constants. Then

$$m(t) \leq m(t_0) \prod_{t_0 < t_k < t} d_k e^{\int_{t_0}^{t} p(\xi)\,d\xi} + \sum_{t_0 < t_k < t} \left(\prod_{t_k < t_j < t} d_j e^{\int_{t_k}^{t} p(\xi)\,d\xi} \right) b_k$$
$$+ \int_{t_0}^{t} \prod_{s < t_k < t} d_k e^{\int_{s}^{t} p(\xi)\,d\xi} q(s)\,ds, \quad t \geq t_0. \tag{1.3}$$

In [12], Thiramanus and Tariboon developed the impulsive inequalities with the following integral jump conditions:

$$m(t_k^+) \leq d_k m(t_k) + c_k \int_{t_k - \tau_k}^{t_k - \sigma_k} m(s)\,ds + b_k, \quad k = 1, 2, \ldots, \tag{1.4}$$

where $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$. In [11], Liengtragulngam et al. generalized further results by replacing the integral jump conditions (1.2) by the following nonlocal jump conditions:

$$m(t_k^+) \leq \frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} m(s)\,ds + d_k m(t_k) + b_k. \tag{1.5}$$

We note that a weak singular kernel is involved in the nonlocal jump conditions. They gave the estimation of $m(t)$ as follows.

Theorem 1.2 *Let* (H_0) *and* (H_1) *be true. Suppose that $p, q \in C[\mathbb{R}_+, \mathbb{R}]$ and for $k = 1, 2, \ldots$, $t \geq t_0$,*

$$\begin{cases} m'(t) \leq p(t)m(t) + q(t), & t \neq t_k, \\ m(t_k^+) \leq \frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} m(s)\,ds + d_k m(t_k) + b_k, \end{cases}$$

where $c_k, d_k \geq 0$, $\beta_k > 0$ and b_k $(k = 1, 2, \ldots)$ are constants. Then, for all $t \geq t_0$,

$$m(t) \leq \Biggl\{ m(t_0) \prod_{t_0 < t_k < t} \left(\frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} e^{\int_{t_{k-1}}^{s} p(\xi)\,d\xi}\,ds + d_k e^{\int_{t_{k-1}}^{t_k} p(\xi)\,d\xi} \right)$$
$$+ \sum_{t_0 < t_k < t} \Biggl[\prod_{t_0 < t_j < t} \left(\frac{c_j}{\Gamma(\beta_j)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta_j - 1} e^{\int_{t_{j-1}}^{s} p(\xi)\,d\xi}\,ds + d_j e^{\int_{t_{j-1}}^{t_j} p(\xi)\,d\xi} \right)$$
$$\times \Biggl(\frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} (t_k - s)^{\beta_k - 1} q(\nu) e^{\int_{\nu}^{s} p(\xi)\,d\xi}\,d\nu\,ds$$
$$+ d_k \int_{t_{k-1}}^{t_k} q(s) e^{\int_{s}^{t_k} p(\xi)\,d\xi}\,ds + b_k \Biggr) \Biggr] \Biggr\} e^{\int_{t_l}^{t} p(\xi)\,d\xi} + \int_{t_l}^{t} q(s) e^{\int_{s}^{t} p(\xi)\,d\xi}\,ds, \tag{1.6}$$

where $t_l = \max\{t_k : t \geq t_k, k = 1, 2, \ldots\}$.

These results play fundamental roles in the global existence, uniqueness, stability and other properties of various linear impulsive differential and integral equations.

A lot of authors just study the qualitative properties of linear impulsive inequalities. However, most of the phenomena in the world do not change linearly, such as heart beat, blood pressure, and so on. Hence the nonlinear impulsive differential and integral theories are more accurate than linear impulsive theories in various aspects.

In this paper, we extend the theories of linear impulsive system to nonlinear impulsive inequalities with nonlocal jump conditions. We consider the following nonlinear inequality:

$$m'(t) \le p(t)m(t) + q(t)m^{\alpha}(t), \quad t \ne t_k,$$

with different nonlocal jump conditions, we give the upper bound estimation of the inequality, and an estimation of solutions of certain nonlinear equations is also involved.

For convenience, we give the following lemmas.

Lemma 1.1 ([9]) *Assume that $a, b \in \mathbb{R}$, $p \ge 0$. Then*

$$\big(|a| + |b|\big)^p \le C_p\big(|a|^p + |b|^p\big),$$

where $C_p = 1$ for $0 \le p \le 1$, and $C_p = 2^{p-1}$ for $p > 1$.

Lemma 1.2 *Let $\{a_n\}$, $\{b_n\}$ be two sequences of numbers. Then we have*

$$\left(\sum_{k=1}^{n-1}\prod_{j=k+1}^{n-1} a_j b_k\right)a_n + b_n = \sum_{k=1}^{n}\left[\prod_{j=k+1}^{n} a_j\right]b_k.$$

2 Nonlinear impulsive inequalities with nonlocal jump conditions

In this section, we present and prove some new nonlinear impulsive differential and integral inequalities with nonlocal jump conditions. Let $t_l = \max\{t_k : t \ge t_k, k = 1, 2, \ldots\}$.

Theorem 2.1 *Let* (H_0) *and* (H_1) *hold. Suppose that $p, q \in C[\mathbb{R}_+, \mathbb{R}]$ and for $k = 1, 2, \ldots$, $t \ge t_0$,*

$$m'(t) \le p(t)m(t) + q(t)m^{\alpha}(t), \quad t \ne t_k, \tag{2.1}$$

$$m^{1-\alpha}\big(t_k^+\big) \le \frac{c_k}{\Gamma(\beta_k)}\int_{t_{k-1}}^{t_k}(t_k - s)^{\beta_k - 1} m^{1-\alpha}(s)\,ds + d_k m^{1-\alpha}(t_k) + b_k, \tag{2.2}$$

where $0 < \alpha < 1$, $c_k, d_k \ge 0$, $\beta_k > 0$ and b_k $(k = 1, 2, \ldots)$ are constants. Then, for $t \ge t_0$, we have

$$\begin{aligned} m(t) \le \Bigg\{\Bigg\{ & m^{1-\alpha}(t_0)\prod_{t_0<t_k<t}\left(\frac{c_k}{\Gamma(\beta_k)}\int_{t_{k-1}}^{t_k}(t_k - s)^{\beta_k-1}e^{\int_{t_{k-1}}^{s}(1-\alpha)p(\xi)\,d\xi}\,ds \right. \\ & \left. + d_k e^{\int_{t_{k-1}}^{t_k}(1-\alpha)p(\xi)\,d\xi}\right) \\ & + \sum_{t_0<t_k<t}\left[\prod_{t_k<t_j<t}\left(\frac{c_j}{\Gamma(\beta_j)}\int_{t_{j-1}}^{t_j}(t_j - s)^{\beta_j-1}e^{\int_{t_{j-1}}^{s}(1-\alpha)p(\xi)\,d\xi}\,ds \right.\right. \\ & \left.\left. + d_j e^{\int_{t_{j-1}}^{t_j}(1-\alpha)p(\xi)\,d\xi}\right)\right] \end{aligned}$$

$$\times \left(\frac{c_k}{\Gamma(\beta_k)} (1-\alpha) \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} (t_k - s)^{\beta_k - 1} q(\nu) e^{\int_\nu^s (1-\alpha) p(\xi)\,d\xi}\,d\nu\,ds \right.$$
$$\left. + (1-\alpha) d_k \int_{t_{k-1}}^{t_k} q(s) e^{\int_s^{t_k} (1-\alpha) p(\xi)\,d\xi}\,ds + b_k \right) \Bigg\} e^{\int_{t_l}^{t} (1-\alpha)(1-\alpha) p(\xi)\,d\xi}$$
$$+ (1-\alpha) \int_{t_l}^{t} q(s) e^{\int_s^t (1-\alpha) p(\xi)\,d\xi}\,ds \Bigg\}^{\frac{1}{1-\alpha}}. \tag{2.3}$$

Proof Let

$$\nu(t) = m^{1-\alpha}(t).$$

Then applying (2.1), we can get

$$\begin{aligned} \nu'(t) &= (1-\alpha) m^{-\alpha}(t) m'(t) \\ &\le (1-\alpha) m^{-\alpha}(t) \big[p(t) m(t) + q(t) m^{\alpha}(t) \big] \\ &= (1-\alpha) p(t) \nu(t) + (1-\alpha) q(t). \end{aligned} \tag{2.4}$$

The inequality (2.2) can be written as

$$\nu\big(t_k^+\big) \le \frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} \nu(s)\,ds + d_k \nu(t_k) + b_k. \tag{2.5}$$

Next, we set

$$E_k = \frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} e^{\int_{t_{k-1}}^{s} (1-\alpha) p(\xi)\,d\xi}\,ds + d_k e^{\int_{t_{k-1}}^{t_k} (1-\alpha) p(\xi)\,d\xi}, \tag{2.6}$$

$$\begin{aligned} G_k &= \frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} (t_k - s)^{\beta_k - 1} (1-\alpha) q(\nu) e^{\int_\nu^s p(\xi)\,d\xi}\,d\nu\,ds \\ &\quad + (1-\alpha) d_k \int_{t_{k-1}}^{t_k} q(s) e^{\int_s^{t_k} (1-\alpha) p(\xi)\,d\xi}\,ds + b_k, \end{aligned} \tag{2.7}$$

then (2.3) reduces to

$$\begin{aligned} m(t) \le{}& \left\{ \left\{ m^{1-\alpha}(t_0) \prod_{t_0 < t_k < t} E_k + \sum_{t_0 < t_k < t} \left[\prod_{t_k < t_j < t} E_j \right] G_k \right\} e^{\int_{t_l}^t (1-\alpha) p(\xi)\,d\xi} \right. \\ & \left. + (1-\alpha) \int_{t_l}^t q(s) e^{\int_s^t (1-\alpha) p(\xi)\,d\xi}\,ds \right\}^{\frac{1}{1-\alpha}}. \end{aligned} \tag{2.8}$$

By the definition of $\nu(t)$, we just need to prove

$$\begin{aligned} \nu(t) \le{}& \left\{ \nu(t_0) \prod_{t_0 < t_k < t} E_k + \sum_{t_0 < t_k < t} \left[\prod_{t_k < t_j < t} E_j \right] G_k \right\} e^{\int_{t_l}^t (1-\alpha) p(\xi)\,d\xi} \\ & + (1-\alpha) \int_{t_l}^t q(s) e^{\int_s^t (1-\alpha) p(\xi)\,d\xi}\,ds. \end{aligned} \tag{2.9}$$

We prove it by induction. For $t \in [t_0, t_1]$, inequality (2.4) can be written as

$$\frac{d}{dt}\left[v(t)e^{-\int_{t_0}^{t}(1-\alpha)p(\xi)\,d\xi}\right] \le (1-\alpha)q(t)e^{-\int_{t_0}^{t}(1-\alpha)p(\xi)\,d\xi}. \tag{2.10}$$

Integrating (2.10) from t_0 to t, for $t \in [t_0, t_1]$, we have

$$v(t) \le v(t_0)e^{\int_{t_0}^{t}(1-\alpha)p(\xi)\,d\xi} + (1-\alpha)\int_{t_0}^{t} q(s)e^{\int_{s}^{t}(1-\alpha)p(\xi)\,d\xi}\,ds. \tag{2.11}$$

Hence (2.9) is valid on $[t_0, t_1]$. Assume that (2.9) holds for $t \in [t_0, t_n]$, for some integer $n > 1$. Then, for $t \in [t_n, t_{n+1}]$, it follows from (2.4) and (2.11) that

$$v(t) \le v\big(t_n^+\big)e^{\int_{t_n}^{t}(1-\alpha)p(\xi)\,d\xi} + (1-\alpha)\int_{t_n}^{t} q(s)e^{\int_{s}^{t}(1-\alpha)p(\xi)\,d\xi}\,ds. \tag{2.12}$$

Applying (2.5) with (2.12), we get

$$\begin{aligned} v(t) \le{}& \left(\frac{c_n}{\Gamma(\beta_n)}\int_{t_{n-1}}^{t_n}(t_n-s)^{\beta_n-1}v(s)\,ds + d_n v(t_n) + b_n\right)e^{\int_{t_n}^{t}(1-\alpha)p(\xi)\,d\xi} \\ &+ (1-\alpha)\int_{t_n}^{t} q(s)e^{\int_{s}^{t}(1-\alpha)p(\xi)\,d\xi}\,ds. \end{aligned} \tag{2.13}$$

By induction and (2.13), we get

$$\begin{aligned} v(t) \le{}& \left\{\frac{c_n}{\Gamma(\beta_n)}\int_{t_{n-1}}^{t_n}(t_n-s)^{\beta_n-1} \times \left\{\left\{v(t_0)\prod_{t_0<t_k<s}E_k + \sum_{t_0<t_k<s}\left[\prod_{t_k<t_j<s}E_j\right]G_k\right\}\right.\right. \\ &\times e^{\int_{t_{n-1}}^{s}(1-\alpha)p(\xi)\,d\xi} + (1-\alpha)\int_{t_{n-1}}^{s} q(\nu)e^{\int_{\nu}^{s}(1-\alpha)p(\xi)\,d\xi}\,d\nu\bigg\}\,ds \\ &+ d_n\left\{\left\{v(t_0)\prod_{t_0<t_k<t_n}E_k + \sum_{t_0<t_k<t_n}\left[\prod_{t_k<t_j<t_n}E_j\right]G_k\right\}e^{\int_{t_{n-1}}^{t_n}(1-\alpha)p(\xi)\,d\xi}\right. \\ &+ (1-\alpha)\int_{t_{n-1}}^{t_n} q(s)e^{\int_{s}^{t_n}(1-\alpha)p(\xi)\,d\xi}\,ds\bigg\} + b_n\bigg\} \\ &\times e^{\int_{t_n}^{t}(1-\alpha)p(\xi)\,d\xi} + (1-\alpha)\int_{t_n}^{t} q(s)e^{\int_{s}^{t}(1-\alpha)p(\xi)\,d\xi}\,ds \\ ={}& \left\{\left(v(t_0)\prod_{t_0<t_k<t_n}E_k + \sum_{t_0<t_k<t_n}\left[\prod_{t_k<t_j<t_n}E_j\right]G_k\right)\frac{c_n}{\Gamma(\beta_n)}\int_{t_{n-1}}^{t_n}(t_n-s)^{\beta_n-1}\right. \\ &\times e^{\int_{t_{n-1}}^{s}(1-\alpha)p(\xi)\,d\xi}\,ds \\ &+ \frac{c_n}{\Gamma(\beta_n)}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{s}(t_n-s)^{\beta_n-1}(1-\alpha)q(\nu)e^{\int_{\nu}^{s}(1-\alpha)p(\xi)\,d\xi}\,d\nu\,ds \\ &+ \left(v(t_0)\prod_{t_0<t_k<t_n}E_k + \sum_{t_0<t_k<t_n}\left[\prod_{t_k<t_j<t_n}E_j\right]G_k\right)d_n e^{\int_{t_{n-1}}^{t_n}(1-\alpha)p(\xi)\,d\xi} \\ &+ d_n(1-\alpha)\int_{t_{n-1}}^{t_n} q(s)e^{\int_{s}^{t_n}(1-\alpha)p(\xi)\,d\xi}\,ds + b_n\bigg\}e^{\int_{t_n}^{t}(1-\alpha)p(\xi)\,d\xi} \end{aligned}$$

$$
\begin{aligned}
&+ (1-\alpha)\int_{t_n}^{t} q(s)e^{\int_s^t (1-\alpha)p(\xi)\,d\xi}\,ds \\
&= \biggl\{\biggl(\nu(t_0)\prod_{t_0<t_k<t_n} E_k + \sum_{t_0<t_k<t_n}\biggl[\prod_{t_k<t_j<t_n} E_j\biggr]G_k\biggr)E_n + G_n\biggr\}e^{\int_{t_n}^t (1-\alpha)p(\xi)\,d\xi} \\
&+ (1-\alpha)\int_{t_n}^{t} q(s)e^{\int_s^t (1-\alpha)p(\xi)\,d\xi}\,ds \\
&= \biggl\{\nu(t_0)\prod_{t_0<t_k<t} E_k + \sum_{t_0<t_k<t}\biggl[\prod_{t_k<t_j<t} E_j\biggr]G_k\biggr\}e^{\int_{t_n}^t (1-\alpha)p(\xi)\,d\xi} \\
&+ (1-\alpha)\int_{t_n}^{t} q(s)e^{\int_s^t (1-\alpha)p(\xi)\,d\xi}\,ds.
\end{aligned}
\tag{2.14}
$$

Hence (2.9) is valid on $[t_n, t_{n+1}]$. Therefore, the inequalities (2.9) is valid for $t_0 \le t \le t_{n+1}$. We know that $\nu(t) = m^{1-\alpha}(t)$, this completes the proof of Theorem 2.1. □

If $d_k \equiv 1$ in Theorem 2.1, we obtain the following theorem.

Theorem 2.2 *Suppose that* (H_0) *and* (H_1) *hold*, $p, q \in C[\mathbb{R}_+, \mathbb{R}]$ *and for* $k = 1, 2, \ldots, t \ge t_0$,

$$
m'(t) \le p(t)m(t) + q(t)m^{\alpha}(t), \quad t \ne t_k, \tag{2.15}
$$

$$
\Delta m^{1-\alpha}(t_k) \le \frac{c_k}{\Gamma(\beta_k)}\int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} m^{1-\alpha}(s)\,ds + b_k, \tag{2.16}
$$

where $0 < \alpha < 1$, $c_k, d_k \ge 0$, $\beta_k > 0$ *and* b_k $(k = 1, 2, \ldots)$ *are constants*, $\Delta m^{1-\alpha}(t_k) = m^{1-\alpha}(t_k^+) - m^{1-\alpha}(t_k)$. *Then we have the following inequality*:

$$
\begin{aligned}
m(t)^{1-\alpha} \le{}& \biggl\{m^{1-\alpha}(t_0)\prod_{t_0<t_k<t} M_k + \sum_{t_0<t_k<t}\biggl[\prod_{t_k<t_j<t} M_j\biggr]N_k\biggr\}e^{\int_{t_l}^t (1-\alpha)p(\xi)\,d\xi} \\
&+ (1-\alpha)\int_{t_0}^{t} q(s)e^{\int_s^t (1-\alpha)p(\xi)\,d\xi},
\end{aligned}
\tag{2.17}
$$

where

$$
\begin{aligned}
M_k &= \frac{c_k}{\Gamma(\beta_k)}\int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} e^{\int_{t_{k-1}}^s (1-\alpha)p(\xi)\,d\xi}\,ds + e^{\int_{t_{k-1}}^{t_k} (1-\alpha)p(\xi)\,d\xi}, \\
N_k &= \frac{c_k}{\Gamma(\beta_k)}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} (t_k - s)^{\beta_k - 1}(1-\alpha)q(\nu)e^{\int_\nu^s p(\xi)\,d\xi}\,d\nu\,ds + b_k.
\end{aligned}
$$

Proof As the proof of Theorem 2.1, from (2.11) we have

$$
m^{1-\alpha}(t) \le m^{1-\alpha}(t_0)e^{\int_{t_0}^t (1-\alpha)p(\xi)\,d\xi} + (1-\alpha)\int_{t_0}^{t} q(s)e^{\int_s^t (1-\alpha)p(\xi)\,d\xi}\,ds, \tag{2.18}
$$

which means that (2.17) holds for $t \in [t_0, t_1]$.

Now we use the method of mathematical induction; suppose that (2.17) holds for $t \in [t_0, t_n]$, then

$$\begin{aligned} m^{1-\alpha}(t) \le{}& \left\{ m^{1-\alpha}(t_0) \prod_{t_0<t_k<t} M_k + \sum_{t_0<t_k<t} \left[\prod_{t_k<t_j<t} M_j \right] N_k \right\} e^{\int_{t_{n-1}}^{t_n} (1-\alpha)p(\xi)\,d\xi} \\ &+ (1-\alpha) \int_{t_0}^{t} q(s) e^{\int_s^{t_n} (1-\alpha)p(\xi)\,d\xi}\,ds. \end{aligned} \tag{2.19}$$

Then, by (2.16),

$$\begin{aligned} m^{1-\alpha}\left(t_n^+\right) \le{}& m^{1-\alpha}(t_n) + \frac{c_n}{\Gamma(\beta_n)} \int_{t_{n-1}}^{t_n} (t_n - s)^{\beta_n - 1} m^{1-\alpha}(s)\,ds + b_n \\ \le{}& \left\{ m^{1-\alpha}(t_0) \prod_{t_0<t_k<t_n} M_k + \sum_{t_0<t_k<t_n} \left[\prod_{t_k<t_j<t_n} M_j \right] N_k \right\} e^{\int_{t_{n-1}}^{t_n} (1-\alpha)p(\xi)\,d\xi} \\ &+ (1-\alpha) \int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha)p(\xi)\,d\xi}\,ds \\ &+ \frac{c_n}{\Gamma(\beta_n)} \int_{t_{n-1}}^{t_n} (t_n - s)^{\beta_n - 1} \left\{ m^{1-\alpha}(t_0) \prod_{t_0<t_k<s} M_k + \sum_{t_0<t_k<s} \left[\prod_{t_k<t_j<s} M_j \right] N_k \right\} \\ &\times e^{\int_{t_{n-1}}^{s} (1-\alpha)p(\xi)\,d\xi}\,ds \\ &+ \frac{c_n}{\Gamma(\beta_n)} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{s} (t_n - s)^{\beta_n - 1} (1-\alpha) q(\nu) e^{\int_\nu^{t_n} (1-\alpha)p(\xi)\,d\xi}\,d\nu\,ds + b_n \\ ={}& \left\{ m^{1-\alpha}(t_0) \prod_{t_0<t_k<t_n} M_k + \sum_{t_0<t_k<t_n} \left[\prod_{t_k<t_j<t_n} M_j \right] N_k \right\} \\ &\times \left[\frac{c_n}{\Gamma(\beta_n)} \int_{t_{n-1}}^{t_n} (t_n - s)^{\beta_n - 1} e^{\int_{t_{n-1}}^{s} (1-\alpha)p(\xi)\,d\xi}\,ds + e^{\int_{t_{n-1}}^{t_n} (1-\alpha)p(\xi)\,d\xi} \right] \\ &+ (1-\alpha) \int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha)p(\xi)\,d\xi}\,ds + N_n \\ ={}& \left\{ m^{1-\alpha}(t_0) \prod_{t_0<t_k<t_n} M_k + \sum_{t_0<t_k<t_n} \left[\prod_{t_k<t_j<t_n} M_j \right] N_k \right\} M_n \\ &+ (1-\alpha) \int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha)p(\xi)\,d\xi}\,ds + N_n. \end{aligned}$$

By Lemma 1.2, we have

$$\begin{aligned} m^{1-\alpha}\left(t_n^+\right) \le{}& \left\{ m^{1-\alpha}(t_0) \prod_{t_0<t_k<t_{n+1}} M_k + \sum_{t_0<t_k<t_{n+1}} \left[\prod_{t_k<t_j<t_{n+1}} M_j \right] N_k \right\} \\ &+ (1-\alpha) \int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha)p(\xi)\,d\xi}\,ds. \end{aligned}$$

For $t \in [t_n, t_{n+1}]$, (2.18) can be replaced by

$$\begin{aligned} m^{1-\alpha}(t) \le{}& m^{1-\alpha}\left(t_n^+\right) e^{\int_{t_n}^{t} (1-\alpha)p(\xi)\,d\xi} + (1-\alpha) \int_{t_n}^{t} q(s) e^{\int_s^{t} (1-\alpha)p(\xi)\,d\xi}\,ds \\ \le{}& \left\{ \left\{ m^{1-\alpha}(t_0) \prod_{t_0<t_k<t} M_k + \sum_{t_0<t_k<t} \left[\prod_{t_k<t_j<t} M_j \right] N_k \right\} \right. \end{aligned}$$

$$
\begin{aligned}
&+(1-\alpha)\int_{t_0}^{t_n} q(s)e^{\int_s^{t_n}(1-\alpha)p(\xi)\,d\xi}\,ds\biggr\}e^{\int_{t_n}^{t}(1-\alpha)p(\xi)\,d\xi}\\
&+(1-\alpha)\int_{t_n}^{t} q(s)e^{\int_s^{t}(1-\alpha)p(\xi)\,d\xi}\,ds\\
=&\biggl\{m^{1-\alpha}(t_0)\prod_{t_0<t_k<t}M_k+\sum_{t_0<t_k<t}\biggl[\prod_{t_k<t_j<t}M_j\biggr]N_k\biggr\}e^{\int_{t_n}^{t}(1-\alpha)p(\xi)\,d\xi}\\
&+(1-\alpha)\int_{t_0}^{t_n} q(s)e^{\int_s^{t_n}(1-\alpha)p(\xi)\,d\xi}\,ds e^{\int_{t_n}^{t}(1-\alpha)p(\xi)\,d\xi}\\
&+(1-\alpha)\int_{t_n}^{t} q(s)e^{\int_s^{t}(1-\alpha)p(\xi)\,d\xi}\,ds.
\end{aligned}
$$

Therefore, the estimate (2.17) holds for $t\in[t_0,t_{n+1}]$. This completes the proof. □

Using different estimating methods, we have the following results.

Theorem 2.3 *Suppose that all the hypotheses of Theorem 2.1 are fulfilled. Then, for $t\geq t_0$, we have*:

(i) *For $k=1,2,\ldots$, the following estimation holds*:

$$
\begin{aligned}
m(t)\leq&\Biggl\{\biggl\{m^{1-\alpha}(t_0)\prod_{t_0<t_k<t}\biggl[\frac{c_k}{\Gamma(\beta_k)}\biggl(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\biggr)^{\frac{1}{\mu_k}}\biggl(\int_{t_{k-1}}^{t_k}e^{\nu_k(\int_{t_{k-1}}^{s}(1-\alpha)p(\xi)\,d\xi-s)}\,ds\biggr)^{\frac{1}{\nu_k}}\\
&+d_k e^{\int_{t_{k-1}}^{t_k}(1-\alpha)p(\xi)\,d\xi}\biggr]\\
&+\sum_{t_0<t_k<t}\biggl[\prod_{t_k<t_j<t}\biggl[\frac{c_j}{\Gamma(\beta_j)}\biggl(\frac{e^{\mu_j t_j}\Gamma(\beta_j^2)}{\mu_j^{\beta_j^2}}\biggr)^{\frac{1}{\mu_j}}\biggl(\int_{t_{j-1}}^{t_j}e^{\nu_j(\int_{t_{j-1}}^{s}(1-\alpha)p(\xi)\,d\xi-s)}\,ds\biggr)^{\frac{1}{\nu_j}}\\
&+d_k e^{\int_{t_{j-1}}^{t_j}(1-\alpha)p(\xi)\,d\xi}\biggr]\biggr]\biggl(\frac{c_k}{\Gamma(\beta_k)}\biggl(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\biggr)^{\frac{1}{\mu_k}}\\
&\times\biggl(\int_{t_{k-1}}^{t_k}e^{-\nu_k s}\biggl\{\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)e^{\int_\nu^s(1-\alpha)p(\xi)\,d\xi}\,d\nu\biggr\}^{\nu_k}\,ds\biggr)^{\frac{1}{\nu_k}}\\
&+d_k\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)e^{\int_\nu^s(1-\alpha)p(\xi)\,d\xi}\,d\nu+b_k\biggr)\biggr\}e^{\int_{t_l}^{t}(1-\alpha)p(\xi)\,d\xi}\\
&+(1-\alpha)\int_{t_l}^{t}q(s)e^{\int_s^t(1-\alpha)p(\xi)\,d\xi}\,ds\Biggr\}^{\frac{1}{1-\alpha}},
\end{aligned}
\tag{2.20}
$$

where $\mu_k=\beta_k+1$ and $\nu_k=1+\frac{1}{\beta_k}$.

(ii) *If we assume further that $\beta_k>\frac{1}{2}$, then, for $k=1,2,\ldots$, we have the following estimation*:

$$
\begin{aligned}
m(t)\leq&\Biggl\{\biggl\{m^{1-\alpha}(t_0)\prod_{t_0\leq t_k\leq t}\biggl[\frac{c_k}{\Gamma(\beta_k)}\biggl(\frac{e^{2t_k}\Gamma(2\beta_k-1)}{2^{2\beta_k-1}}\biggr)^{\frac{1}{2}}\biggl(\int_{t_{k-1}}^{t_k}e^{2(\int_{t_{k-1}}^{s}(1-\alpha)p(\xi)\,d\xi-s)}\,ds\biggr)^{\frac{1}{2}}\\
&+d_k e^{\int_{t_{k-1}}^{t_k}(1-\alpha)p(\xi)\,d\xi}\biggr]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{t_0<t_k<t}\left[\prod_{t_k<t_j<t}\left[\frac{c_j}{\Gamma(\beta_j)}\left(\frac{e^{2t_j}\Gamma(2\beta_j-1)}{2^{2\beta_j-1}}\right)^{\frac{1}{2}}\left(\int_{t_{j-1}}^{t_j} e^{2(\int_{t_{j-1}}^{s}(1-\alpha)p(\xi)\,d\xi-s)}\,ds\right)^{\frac{1}{2}}\right.\right.\\
&\left.\left.+d_k e^{\int_{t_{j-1}}^{t_j}(1-\alpha)p(\xi)\,d\xi}\right]\right]\left(\frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{2t_k}\Gamma(2\beta_k-1)}{2^{2\beta_k-1}}\right)^{\frac{1}{2}}\right.\\
&\times\left(\int_{t_{k-1}}^{t_k} e^{-2s}\left\{\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)e^{\int_\nu^s(1-\alpha)p(\xi)\,d\xi}\,d\nu\right\}^2 ds\right)^{\frac{1}{2}}\\
&\left.\left.+d_k\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)e^{\int_\nu^s(1-\alpha)p(\xi)\,d\xi}\,d\nu+b_k\right)\right\}e^{\int_{t_l}^{t}(1-\alpha)p(\xi)\,d\xi}\\
&\left.+(1-\alpha)\int_{t_l}^{t}q(s)e^{\int_s^t(1-\alpha)p(\xi)\,d\xi}\,ds\right\}^{\frac{1}{1-\alpha}}.
\end{aligned}
\tag{2.21}
$$

Proof To prove (i), we use the Hölder inequality. For $k = 1, 2, \ldots$, we get

$$
\begin{aligned}
&\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta_k-1}e^{\int_{t_{k-1}}^{s}(1-\alpha)p(\xi)\,d\xi}\,ds\\
&\le\left(\int_{t_{k-1}}^{t_k}(t_k-s)^{(\beta_k-1)\mu_k}e^{\mu_k s}\,ds\right)^{\frac{1}{\mu_k}}\left(\int_{t_{k-1}}^{t_k}e^{-\nu_k s}e^{\nu_k\int_{t_{k-1}}^{s}(1-\alpha)p(\xi)\,d\xi}\,ds\right)^{\frac{1}{\nu_k}}\\
&<\left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}\left(\int_{t_{k-1}}^{t_k}e^{-\nu_k s}e^{\nu_k\int_{t_{k-1}}^{s}(1-\alpha)p(\xi)\,d\xi}\,ds\right)^{\frac{1}{\nu_k}}.
\end{aligned}
$$

In fact, by changing the variable, we get

$$
\begin{aligned}
\int_{t_{k-1}}^{t_k}(t_k-s)^{(\beta_k-1)\mu_k}e^{\mu_k s}\,ds&=e^{\mu_k t_k}\int_0^{t_k-t_{k-1}}\eta^{\mu_k(\beta_k-1)}e^{-\mu_k\eta}\,d\eta\\
&=\frac{e^{\mu_k t_k}}{\mu_k^{1-\mu_k(1-\beta_k)}}\int_0^{\mu_k(t_k-t_{k-1})}\lambda^{\mu_k(\beta_k-1)}e^{-\lambda}\,d\lambda\\
&<\frac{e^{\mu_k t_k}}{\mu_k^{1-\mu_k(1-\beta_k)}}\Gamma\bigl(1-\mu_k(1-\beta_k)\bigr)\\
&=\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(t_k-s)^{\beta_k-1}(1-\alpha)q(\nu)e^{\int_\nu^s(1-\alpha)p(\xi)\,d\xi}\,d\nu\,ds\\
&\le\left(\int_{t_{k-1}}^{t_k}(t_k-s)^{(\beta_k-1)\mu_k}e^{\mu_k s}\,ds\right)^{\frac{1}{\mu_k}}\\
&\quad\times\left(\int_{t_{k-1}}^{t_k}e^{-\nu_k s}\left\{\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)e^{\int_\nu^s(1-\alpha)p(\xi)\,d\xi}\,d\nu\right\}^{\nu_k}ds\right)^{\frac{1}{\nu_k}}\\
&<\left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}\left(\int_{t_{k-1}}^{t_k}e^{-\nu_k s}\left\{\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)e^{\int_\nu^s(1-\alpha)p(\xi)\,d\xi}\,d\nu\right\}^{\nu_k}ds\right)^{\frac{1}{\nu_k}}.
\end{aligned}
$$

Substituting the above inequalities in (2.3), we obtain the desired inequality in (2.20).

To prove (ii), since in this case, $2\beta_k - 1 > 0$, $\Gamma(2\beta_k - 1)$ are well defined for $k = 1, 2, \ldots$. We use the Cauchy–Schwartz inequality to get

$$\begin{aligned}
&\int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} e^{\int_{t_{k-1}}^{s} (1-\alpha)p(\xi)\,d\xi}\,ds \\
&\quad \le \left(\int_{t_{k-1}}^{t_k} (t_k - s)^{2(\beta_k - 1)} e^{2s}\,ds\right)^{\frac{1}{2}} \left(\int_{t_{k-1}}^{t_k} e^{2(\int_{t_{k-1}}^{s} (1-\alpha)p(\xi)\,d\xi - s)}\,ds\right)^{\frac{1}{2}} \\
&\quad < \left(\frac{e^{2t_k}\Gamma(2\beta_k - 1)}{2^{2\beta_k - 1}}\right)^{\frac{1}{2}} \left(\int_{t_{k-1}}^{t_k} e^{2(\int_{t_{k-1}}^{s} (1-\alpha)p(\xi)\,d\xi - s)}\,ds\right)^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} (t_k - s)^{\beta_k - 1} (1-\alpha) q(\nu) e^{\int_{\nu}^{s}(1-\alpha)p(\xi)\,d\xi}\,d\nu\,ds \\
&\quad \le \left(\int_{t_{k-1}}^{t_k} (t_k - s)^{2(\beta_k - 1)} e^{2s}\,ds\right)^{\frac{1}{2}} \left(\int_{t_{k-1}}^{t_k} e^{-2s} \left\{\int_{t_{k-1}}^{s} (1-\alpha) q(\nu) e^{\int_{\nu}^{s}(1-\alpha)p(\xi)\,d\xi}\,d\nu\right\}^2 ds\right)^{\frac{1}{2}} \\
&\quad < \left(\frac{e^{2t_k}\Gamma(2\beta_k - 1)}{2^{2\beta_k - 1}}\right)^{\frac{1}{2}} \left(\int_{t_{k-1}}^{t_k} e^{-2s} \left\{\int_{t_{k-1}}^{s} (1-\alpha) q(\nu) e^{\int_{\nu}^{s}(1-\alpha)p(\xi)\,d\xi}\,d\nu\right\}^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Substituting these two inequalities in (2.3), we get the desired results. The proof is completed. □

If $d_k \equiv 0$ and $p(t)$ is constant function in Theorem 2.2, we obtain the following corollary.

Corollary 2.4 *Suppose* (H_0) *and* (H_1) *hold, and for* $q \in C[\mathbb{R}_+, \mathbb{R}]$, $k = 1, 2, \ldots$, $t \ge t_0$,

$$m'(t) \le p m(t) + q(t) m^{\alpha}(t), \quad t \ne t_k, \tag{2.22}$$

$$m^{1-\alpha}(t_k^+) \le \frac{c_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} m^{1-\alpha}(s)\,ds + b_k, \tag{2.23}$$

where $0 < \alpha < 1$, $c_k, d_k \ge 0$, $\beta_k > 0$ *and* b_k $(k = 1, 2, \ldots)$ *are constants. Then, for* $t \ge t_0$, *we have the following estimates.*

Case I: $p \ne \frac{1}{1-\alpha}$:

(i) *For* $k = 1, 2, \ldots$, *the following estimation holds*:

$$\begin{aligned}
m(t) \le \Biggl\{ & m^{1-\alpha}(t_0) \left(\prod_{t_0 < t_k < t} A_k\right) e^{p(1-\alpha)(t - t_0)} + \sum_{t_0 < t_k < t} \left(\prod_{t_k < t_j < t} A_j\right) B_k e^{p(1-\alpha)(t - t_k)} \\
& + \int_{t_l}^{t} (1-\alpha) q(s) e^{p(1-\alpha)(t-s)}\,ds \Biggr\}^{\frac{1}{1-\alpha}},
\end{aligned} \tag{2.24}$$

where

$$A_k = \frac{c_k}{\Gamma(\beta_k)} \left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}} \left(\frac{1 - e^{\nu_k (t_k - t_{k-1})(1 - (1-\alpha)p)}}{\nu_k((1-\alpha)p - 1)}\right)^{\frac{1}{\nu_k}},$$

$$B_k = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}\left(\int_{t_{k-1}}^{t_k} e^{\nu_k((1-\alpha)p-1)s}\left\{\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)e^{-(1-\alpha)p\nu}\,d\nu\right\}^{\nu_k} ds\right)^{\frac{1}{\nu_k}}$$
$$+ b_k.$$

(ii) *If we assume further* $\beta_k > \frac{1}{2}$, *then*, *for* $k = 1,2,\ldots,$

$$m(t) \le \left\{m^{1-\alpha}(t_0)\left(\prod_{t_0<t_k<t} C_k\right)e^{p(1-\alpha)(t-t_0)} + \sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t} C_j\right)D_k e^{p(1-\alpha)(t-t_k)} + \int_{t_l}^{t}(1-\alpha)q(s)e^{p(1-\alpha)(t-s)}\,ds\right\}^{\frac{1}{1-\alpha}}, \tag{2.25}$$

where

$$C_k = \frac{c_k}{2^{\beta_k}\Gamma(\beta_k)}\left\{\frac{\Gamma(2\beta_k-1)}{(1-\alpha)p-1}\left[1-e^{2(t_k-t_{k-1})(1-(1-\alpha)p)}\right]\right\}^{\frac{1}{2}},$$
$$D_k = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{2t_k}\Gamma(2\beta_k-1)}{2^{2\beta_k-1}}\right)^{\frac{1}{2}}\left(\int_{t_{k-1}}^{t_k} e^{2((1-\alpha)p-1)s}\left\{\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)e^{-(1-\alpha)p\nu}\,d\nu\right\}^{2} ds\right)^{\frac{1}{2}}$$
$$+ b_k.$$

Case II: $p = \frac{1}{1-\alpha}$:
(i) *For* $k = 1,2,\ldots$, *the following estimation holds*:

$$m(t) \le \left\{m^{1-\alpha}(t_0)\left(\prod_{t_0<t_k<t} E_k\right)e^{(t-t_0)} + \sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t} E_j\right)F_k e^{(t-t_k)} + \int_{t_l}^{t}(1-\alpha)q(s)e^{(t-s)}\,ds\right\}^{\frac{1}{1-\alpha}}, \tag{2.26}$$

where

$$E_k = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}(t_k - t_{k-1})^{\frac{1}{\nu_k}},$$
$$F_k = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}\left(\int_{t_{k-1}}^{t_k}\left\{\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)e^{-\nu}\,d\nu\right\}^{\nu_k} ds\right)^{\frac{1}{\nu_k}} + b_k.$$

(ii) *If we assume further* $\beta_k > \frac{1}{2}$, *then*, *for* $k = 1,2,\ldots,$

$$m(t) \le \left\{m^{1-\alpha}(t_0)\left(\prod_{t_0<t_k<t} G_k\right)e^{(t-t_0)} + \sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t} G_j\right)H_k e^{(t-t_k)} + \int_{t_l}^{t}(1-\alpha)q(s)e^{(t-s)}\,ds\right\}^{\frac{1}{1-\alpha}}, \tag{2.27}$$

where

$$G_k = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(2\beta_k-1)(t_k-t_{k-1})}{2^{2\beta_k-1}}\right)^{\frac{1}{2}},$$

$$H_k = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{2t_k}\Gamma(2\beta_k-1)}{2^{2\beta_k-1}}\right)^{\frac{1}{2}}\left(\int_{t_{k-1}}^{t_k}\left\{\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)e^{-\nu}\,d\nu\right\}^2 ds\right)^{\frac{1}{2}} + b_k.$$

If $p(t)\equiv 0$ and $d_k\equiv 1$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.5 *Let* (H_0) *and* (H_1) *hold, and for* $q\in C[\mathbb{R}_+,\mathbb{R}]$, $k=1,2,\ldots$, $t\ge t_0$

$$m'(t)\le q(t)m^{\alpha}(t),\quad t\ne t_k, \tag{2.28}$$

$$\Delta m^{1-\alpha}(t_k^+)\le \frac{c_k}{\Gamma(\beta_k)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta_k-1}m^{1-\alpha}(s)\,ds + b_k, \tag{2.29}$$

where $0<\alpha<1$, $c_k, d_k\ge 0$, $\beta_k>0$ *and* b_k $(k=1,2,\ldots)$ *are constants,* $\Delta m^{1-\alpha}(t_k)=m^{1-\alpha}(t_k^+)-m^{1-\alpha}(t_k)$. *Then, for* $t\ge t_0$, *we have*:

(i) *For* $k=1,2,\ldots$, *the following estimation holds*:

$$m(t)\le\left\{m^{1-\alpha}(t_0)\left(\prod_{t_0<t_k<t}I_k\right)+\sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t}I_j\right)L_k + \int_{t_l}^{t}(1-\alpha)q(s)\,ds\right\}^{\frac{1}{1-\alpha}}, \tag{2.30}$$

where

$$I_k = 1+\frac{c_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}\left(\frac{e^{\nu_k(t_k-t_{k-1})-1}}{\nu_k}\right)^{\frac{1}{\nu_k}},$$

$$L_k = \int_{t_{k-1}}^{t_k}q(s)\,ds + \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}} \times\left(\int_{t_{k-1}}^{t_k}e^{-\nu_k s}\left\{\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)e^{-(1-\alpha)p\nu}\,d\nu\right\}^{\nu_k}ds\right)^{\frac{1}{\nu_k}} + b_k.$$

(ii) *If we assume further* $\beta_k>\frac{1}{2}$, *then, for* $k=1,2,\ldots$,

$$m(t)\le\left\{m^{1-\alpha}(t_0)\left(\prod_{t_0<t_k<t}M_k\right)+\sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t}M_j\right)N_k + \int_{t_l}^{t}(1-\alpha)q(s)\,ds\right\}^{\frac{1}{1-\alpha}}, \tag{2.31}$$

where

$$M_k = 1+\frac{c_k}{2^{\beta_k}\Gamma(\beta_k)}\left\{\frac{\Gamma(2\beta_k-1)}{2^{2\beta_k}}\left[e^{2(t_k-t_{k-1})}\right]\right\}^{\frac{1}{2}},$$

$$N_k = \int_{t_{k-1}}^{t_k} q(s)\,ds + \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{2t_k}\Gamma(2\beta_k-1)}{2^{2\beta_k-1}}\right)^{\frac{1}{2}}\left(\int_{t_{k-1}}^{t_k} e^{-2s}\left\{\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)\,d\nu\right\}^2 ds\right)^{\frac{1}{2}} + b_k.$$

Next, we give another kind of nonlinear impulsive differential inequalities.

Theorem 2.6 *Suppose that* (H_0) *and* (H_1) *hold*, $p,q \in C[\mathbb{R}_+,\mathbb{R}]$ *and for* $k = 1,2,\ldots$, $t \geq t_0$,

$$m'(t) \leq p(t)m(t) + q(t)m^{\alpha}(t), \quad t \neq t_k, \tag{2.32}$$

$$\Delta m(t_k) \leq \frac{c_k}{\Gamma(\beta_k)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta_k-1}m(s)\,ds + b_k, \tag{2.33}$$

where $0<\alpha<1$, $c_k, d_k \geq 0$, $\beta_k > 0$ *and* b_k $(k = 1,2,\ldots)$ *are constants*, $\Delta m(t_k) = m(t_k^+) - m(t_k)$. *Then we have the following estimation*:

$$m(t) \leq \left\{\left\{m(t_0)\prod_{t_0<t_k<t}P_k + \sum_{t_0<t_k<t}\left[\prod_{t_k<t_j<t}P_j\right]Q_k\right\}^{1-\alpha} e^{\int_{t_l}^{t}(1-\alpha)p(\xi)\,d\xi} + (1-\alpha)2^{(k-1)\alpha}\int_{t_0}^{t} q(s)e^{\int_s^t(1-\alpha)p(\xi)\,d\xi}\,ds\right\}^{\frac{1}{1-\alpha}}, \quad t \geq t_0, \tag{2.34}$$

where

$$P_k = 2^{\frac{\alpha}{1-\alpha}}\left[e^{\int_{t_{k-1}}^{t_k}p(\xi)\,d\xi} + \frac{c_k}{\Gamma(\beta_k)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta_k-1}e^{\int_{t_{k-1}}^{s}p(\xi)\,d\xi}\,ds\right],$$

$$Q_k = \frac{c_k}{\Gamma(\beta_k)}2^{\frac{(k-1)\alpha}{1-\alpha}}(1-\alpha)^{\frac{1}{1-\alpha}}\int_{t_{k-1}}^{t_k}\left(\int_{t_0}^{s} q(\nu)e^{\int_\nu^s(1-\alpha)p(\xi)\,d\xi}\,d\nu\right)^{\frac{1}{1-\alpha}} + b_k.$$

Proof Obviously, using (2.11), we have (2.34) holds for $t \in [t_0, t_1]$. Suppose (2.34) holds for $t \in [t_0, t_n]$, then, by mathematical induction, we see that

$$m^{1-\alpha}(t_n) \leq \left\{\left\{m(t_0)\prod_{t_0<t_k<t_n}P_k + \sum_{t_0<t_k<t_n}\left[\prod_{t_k<t_j<t_n}P_j\right]Q_k\right\}^{1-\alpha} e^{\int_{t_{n-1}}^{t_n}(1-\alpha)p(\xi)\,d\xi} + (1-\alpha)2^{(n-1)\alpha}\int_{t_0}^{t_n} q(s)e^{\int_s^{t_n}(1-\alpha)p(\xi)\,d\xi}\,ds\right\}.$$

Since $\frac{1}{1-\alpha} > 1$, by Lemma 1.1,

$$\begin{aligned} m(t_n) &\leq 2^{\frac{\alpha}{1-\alpha}}\left\{m(t_0)\prod_{t_0<t_k<t_n}P_k + \sum_{t_0<t_k<t_n}\left[\prod_{t_k<t_j<t_n}P_j\right]Q_k\right\}e^{\int_{t_{n-1}}^{t_n}p(\xi)\,d\xi} \\ &\quad + 2^{\frac{\alpha}{1-\alpha}}2^{\frac{(n-1)\alpha}{1-\alpha}}(1-\alpha)^{\frac{1}{1-\alpha}}\left(\int_{t_0}^{t_n} q(s)e^{\int_s^{t_n}(1-\alpha)p(\xi)\,d\xi}\,ds\right)^{\frac{1}{1-\alpha}} \\ &= 2^{\frac{\alpha}{1-\alpha}}\left\{m(t_0)\prod_{t_0<t_k<t_n}P_k + \sum_{t_0<t_k<t_n}\left[\prod_{t_k<t_j<t_n}P_j\right]Q_k\right\}e^{\int_{t_{n-1}}^{t_n}p(\xi)\,d\xi} \\ &\quad + 2^{\frac{n\alpha}{1-\alpha}}(1-\alpha)^{\frac{1}{1-\alpha}}\left(\int_{t_0}^{t_n} q(s)e^{\int_s^{t_n}(1-\alpha)p(\xi)\,d\xi}\,ds\right)^{\frac{1}{1-\alpha}}, \end{aligned}$$

thus

$$\begin{aligned}
m\bigl(t_n^+\bigr) &\le m(t_n) + \frac{c_n}{\Gamma(\beta_n)} \int_{t_{n-1}}^{t_n} (t_n - s)^{\beta_n - 1} m(s)\,ds + b_n \\
&\le 2^{\frac{\alpha}{1-\alpha}} \biggl\{ m(t_0) \prod_{t_0<t_k<t_n} P_k + \sum_{t_0<t_k<t_n} \biggl[\prod_{t_k<t_j<t_n} P_j \biggr] Q_k \biggr\} e^{\int_{t_{n-1}}^{t_n} p(\xi)\,d\xi} \\
&\quad + 2^{\frac{n\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \biggl(\int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha) p(\xi)\,d\xi}\,ds \biggr)^{\frac{1}{1-\alpha}} \\
&\quad + \frac{c_n}{\Gamma(\beta_n)} 2^{\frac{\alpha}{1-\alpha}} \int_{t_{n-1}}^{t_n} (t_n - s)^{\beta_n - 1} \biggl\{ m(t_0) \prod_{t_0<t_k<t_n} P_k + \sum_{t_0<t_k<t_n} \biggl[\prod_{t_k<t_j<t_n} P_j \biggr] Q_k \biggr\} \\
&\quad \times e^{\int_{t_{n-1}}^{t_n} p(\xi)\,d\xi}\,ds \\
&\quad + \frac{c_n}{\Gamma(\beta_n)} 2^{\frac{n\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \int_{t_{n-1}}^{t_n} (t_n - s)^{\beta_n - 1} \biggl(\int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha) p(\xi)\,d\xi}\,ds \biggr)^{\frac{1}{1-\alpha}}\,d\nu \\
&= \biggl\{ m(t_0) \prod_{t_0<t_k<t_n} P_k + \sum_{t_0<t_k<t_n} \biggl[\prod_{t_k<t_j<t_n} P_j \biggr] Q_k \biggr\} \\
&\quad \times \biggl[2^{\frac{\alpha}{1-\alpha}} \biggl(e^{\int_{t_{n-1}}^{t_n} p(\xi)\,d\xi} + \frac{c_n}{\Gamma(\beta_n)} \int_{t_{n-1}}^{t_n} (t_n - s)^{\beta_n - 1} e^{\int_s^{t_n} (1-\alpha) p(\xi)\,d\xi}\,ds \biggr) \biggr] + R_n \\
&\quad + 2^{\frac{n\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \biggl(\int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha) p(\xi)\,d\xi}\,ds \biggr)^{\frac{1}{1-\alpha}} \\
&= \biggl\{ m(t_0) \prod_{t_0<t_k<t_n} P_k + \sum_{t_0<t_k<t_n} \biggl[\prod_{t_k<t_j<t_n} P_j \biggr] Q_k \biggr\} P_n + R_n \\
&\quad + 2^{\frac{n\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \biggl(\int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha) p(\xi)\,d\xi}\,ds \biggr)^{\frac{1}{1-\alpha}} \\
&= \biggl\{ m(t_0) \prod_{t_0<t_k<t_{n+1}} P_k + \sum_{t_0<t_k<t_{n+1}} \biggl[\prod_{t_k<t_j<t_{n+1}} P_j \biggr] Q_k \biggr\} \\
&\quad + 2^{\frac{n\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \biggl(\int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha) p(\xi)\,d\xi}\,ds \biggr)^{\frac{1}{1-\alpha}}.
\end{aligned}$$

Then, for $t \in [t_n, t_{n+1}]$, since $0 < 1 - \alpha < 1$, by Lemma 1.1 and (2.11), we obtain

$$\begin{aligned}
m^{1-\alpha}(t) &\le m^{1-\alpha}\bigl(t_n^+\bigr) e^{\int_{t_n}^{t} (1-\alpha) p(\xi)\,d\xi} + (1-\alpha) \int_{t_n}^{t} q(s) e^{\int_s^{t} (1-\alpha) p(\xi)\,d\xi}\,ds \\
&\le \biggl\{ \biggl\{ m(t_0) \prod_{t_0<t_k<t} P_k + \sum_{t_0<t_k<t} \biggl[\prod_{t_k<t_j<t} P_j \biggr] Q_k \biggr\} + 2^{\frac{n\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \\
&\quad \times \biggl(\int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha) p(\xi)\,d\xi}\,ds \biggr)^{\frac{1}{1-\alpha}} \biggr\}^{1-\alpha} \\
&\quad \times e^{\int_{t_n}^{t} (1-\alpha) p(\xi)\,d\xi} + (1-\alpha) \int_{t_n}^{t} q(s) e^{\int_s^{t} (1-\alpha) p(\xi)\,d\xi}\,ds \\
&\le \biggl\{ \biggl\{ m(t_0) \prod_{t_0<t_k<t} P_k + \sum_{t_0<t_k<t} \biggl[\prod_{t_k<t_j<t} P_j \biggr] Q_k \biggr\}^{1-\alpha}
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2^{n\alpha}(1-\alpha)\int_{t_0}^{t_n} q(s)e^{\int_s^{t_n}(1-\alpha)p(\xi)\,d\xi}\,ds\Bigg\} \\
&\quad \times e^{\int_{t_n}^{t}(1-\alpha)p(\xi)\,d\xi} + (1-\alpha)\int_{t_n}^{t} q(s)e^{\int_s^{t}(1-\alpha)p(\xi)\,d\xi}\,ds \\
&\leq \Bigg\{m(t_0)\prod_{t_0<t_k<t} P_k + \sum_{t_0<t_k<t}\Bigg[\prod_{t_k<t_j<t} P_j\Bigg]Q_k\Bigg\}^{1-\alpha} e^{\int_{t_n}^{t}(1-\alpha)p(\xi)\,d\xi} \\
&\quad + 2^{n\alpha}(1-\alpha)\int_{t_0}^{t} q(s)e^{\int_s^{t}(1-\alpha)p(\xi)\,d\xi}\,ds.
\end{aligned}
$$

Therefore, the estimation (2.34) is valid on $[t_n, t_{n+1}]$. This completes the proof. □

Now, we present and prove a bound for the solutions of nonlinear impulsive integral inequalities with nonlocal jump conditions.

Theorem 2.7 *Assume that* (H_0) *and* (H_1) *hold*, $p, q, m \in C[\mathbb{R}_+, \mathbb{R}_+]$, *and*, *for* $k = 1, 2, \ldots$, $t \geq t_0$,

$$
\begin{aligned}
m(t) &\leq c + \int_{t_0}^{t} p(s)m(s)\,ds + \int_{t_0}^{t} q(s)m^{\alpha}(s)\,ds \\
&\quad + \sum_{t_0<t_k<t} \frac{\gamma_k}{\Gamma(\beta_k)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta_k-1}m(s)\,ds,
\end{aligned}
\tag{2.35}
$$

where $0 < \alpha < 1$, $\gamma_k \geq 0$, $\beta_k > 0$ $(k = 1, 2, \ldots)$ *and* c *are constants*. *Then*, *for* $t \geq t_0$, *the following assertions hold*:

$$
\begin{aligned}
m(t) &\leq \Bigg\{\Bigg\{c\prod_{t_0<t_k<t} R_k + \sum_{t_0<t_k<t}\Bigg[\prod_{t_k<t_j<t} R_j\Bigg]S_k\Bigg\}^{1-\alpha} e^{\int_{t_l}^{t}(1-\alpha)p(\xi)\,d\xi} \\
&\quad + (1-\alpha)2^{(k-1)\alpha}\int_{t_0}^{t} q(s)e^{\int_s^{t}(1-\alpha)p(\xi)\,d\xi}\,ds\Bigg\}^{\frac{1}{1-\alpha}},
\end{aligned}
\tag{2.36}
$$

where

$$
\begin{aligned}
R_k &= 2^{\frac{\alpha}{1-\alpha}}\Bigg[e^{\int_{t_{k-1}}^{t_k} p(\xi)\,d\xi} + \frac{\gamma_k}{\Gamma(\beta_k)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta_k-1}e^{\int_s^{t_k} p(\xi)\,d\xi}\,ds\Bigg], \\
S_k &= \frac{\gamma_k}{\Gamma(\beta_k)}2^{\frac{(k-1)\alpha}{1-\alpha}}(1-\alpha)^{\frac{1}{1-\alpha}}\int_{t_{k-1}}^{t_k}\left(\int_{t_0}^{s} q(\nu)e^{\int_\nu^{s}(1-\alpha)p(\xi)\,d\xi}\,d\nu\right)^{\frac{1}{1-\alpha}} + b_k.
\end{aligned}
$$

Proof We denote by $g(t)$ the right-side function of (2.35), and $g(t_0) = c$. Then we get

$$
\begin{cases}
g'(t) = p(t)m(t) + q(t)m^{\alpha}(t), & t \neq t_k, \\
g(t_k^+) = \frac{\gamma_k}{\Gamma(\beta_k)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta_k-1}m(s)\,ds + g(t_k),
\end{cases}
$$

Since $m(t) \leq g(t)$, we have

$$
\begin{cases}
g'(t) \leq p(t)g(t) + q(t)m^{\alpha}(t), & t \neq t_k, \\
g(t_k^+) \leq \frac{\gamma_k}{\Gamma(\beta_k)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta_k-1}g(s)\,ds + g(t_k).
\end{cases}
$$

Applying Theorem 2.6, we deduce the estimation of $g(t)$ as

$$g(t) \le \left\{ \left\{ c \prod_{t_0<t_k<t} R_k + \sum_{t_0<t_k<t} \left[\prod_{t_k<t_j<t} R_j \right] S_k \right\}^{1-\alpha} e^{\int_{t_l}^{t}(1-\alpha)p(\xi)\,d\xi} \right.$$
$$\left. + (1-\alpha)2^{(k-1)\alpha} \int_{t_0}^{t} q(s) e^{\int_s^t (1-\alpha)p(\xi)\,d\xi}\,ds \right\}^{\frac{1}{1-\alpha}}. \tag{2.37}$$

Moreover, $m(t) \le g(t)$, this completes the proof. □

3 Impulsive fractional differential and integral equations with integral jump conditions

In this section, we give some examples about impulsive nonlinear differential and integral inequalities with Riemann–Liouville fractional integral jump conditions.

Definition 3.1 The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function f: $[t_0, \infty) \to \mathbb{R}$ is defined by

$$\left(I_{t_0}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-x)^{\alpha-1} f(x)\,dx,$$

where $\Gamma(\cdot)$ is the Gamma function.

Proposition 3.2 *Suppose that $y \in PC^1[J, \mathbb{R}]$ which satisfies*

$$\begin{cases} y'(t) - Ry(t) + a(t)y^{\alpha}(t) \le 0, & t \ne t_k, t \in J = [0, T], \\ y^{1-\alpha}(t_k^+) \le c_k (I_{t_{k-1}}^{\beta_k} y^{1-\alpha})(t_k) - b_k, & k = 1, 2, \ldots, n, \\ y^{1-\alpha}(0) = y^{1-\alpha}(T) + \theta, \end{cases}$$

where $R > 0$, $a \in C[\mathbb{R}_+, \mathbb{R}_+]$, $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$, $0 < \alpha < 1$, $c_k, b_k \ge 0$, $\beta_k > 0$ $(k = 1, 2, \ldots, n)$ and θ are constants. If either of the following four cases fulfilled:

(i) *$R \ne \frac{1}{1-\alpha}$ and for $k = 1, 2, \ldots, n$, the following hypotheses hold*:

$$(\mathrm{P}_1) \quad \prod_{k=1}^{n} \frac{c_k}{\Gamma(\beta_k)} \left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}} \right)^{\frac{1}{\mu_k}} \left(\frac{1 - e^{\nu_k(t_k - t_{k-1})(1-(1-\alpha)p)}}{\nu_k((1-\alpha)p - 1)} \right)^{\frac{1}{\nu_k}} < e^{-(1-\alpha)RT},$$

$$(\mathrm{P}_2) \quad \frac{c_k}{\Gamma(\beta_k)} \left(\frac{e^{\mu_k t_k} \Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}} \right)^{\frac{1}{\mu_k}}$$
$$\times \left(\int_{t_{k-1}}^{t_k} e^{\nu_k((1-\alpha)R-1)s} \left\{ - \int_{t_{k-1}}^{s} (1-\alpha)a(\nu)e^{-(1-\alpha)R\nu}\,d\nu \right\}^{\nu_k} ds \right)^{\frac{1}{\nu_k}} \le b_k,$$

$$(\mathrm{P}_3) \quad \theta \le (1-\alpha) \int_{t_n}^{t} a(s) e^{(1-\alpha)R(T-s)}\,ds,$$

where $\mu_k = \beta_k + 1$ and $\nu_k = 1 + \frac{1}{\beta_k}$;

(ii) $R \neq \frac{1}{1-\alpha}$, $\beta_k > \frac{1}{2}$, *and for* $k = 1, 2, \dots, n$,

$$(\mathrm{P}_4)\quad \prod_{k=1}^{n} \frac{c_k}{2^{\beta_k}\Gamma(\beta_k)} \left\{ \frac{\Gamma(2\beta_k - 1)}{(1-\alpha)R - 1} \left[1 - e^{2(t_k - t_{k-1})(1-(1-\alpha)R)}\right] \right\}^{\frac{1}{2}} < e^{-(1-\alpha)RT},$$

$$(\mathrm{P}_5)\quad \frac{c_k}{\Gamma(\beta_k)} \left(\frac{e^{2t_k}\Gamma(2\beta_k - 1)}{2^{2\beta_k - 1}} \right)^{\frac{1}{2}} \times \left(\int_{t_{k-1}}^{t_k} e^{2((1-\alpha)R-1)s} \left\{ \int_{t_{k-1}}^{s} (1-\alpha)a(\nu)e^{-(1-\alpha)R\nu}\, d\nu \right\}^2 ds \right)^{\frac{1}{2}} \leq b_k,$$

and (P_3) *holds*;

(iii) $R = \frac{1}{1-\alpha}$ *and for* $k = 1, 2, \dots, n$,

$$(\mathrm{P}_6)\quad \prod_{k=1}^{n} \frac{c_k}{\Gamma(\beta_k)} \left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}} \right)^{\frac{1}{\mu_k}} (t_k - t_{k-1})^{\frac{1}{\nu_k}} < e^{-T},$$

$$(\mathrm{P}_7)\quad \frac{c_k}{\Gamma(\beta_k)} \left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}} \right)^{\frac{1}{\mu_k}} \left(\int_{t_{k-1}}^{t_k} \left\{ \int_{t_{k-1}}^{s} (1-\alpha)a(\nu)e^{-\nu}\, d\nu \right\}^{\nu_k} ds \right)^{\frac{1}{\nu_k}} \leq b_k,$$

$$(\mathrm{P}_8)\quad \theta \leq (1-\alpha) \int_{t_n}^{T} a(s)e^{(T-s)}\, ds;$$

(iv) $R = \frac{1}{1-\alpha}$, $\beta_k > \frac{1}{2}$, *and for* $k = 1, 2, \dots, n$,

$$(\mathrm{P}_9)\quad \prod_{k=1}^{n} \frac{c_k}{\Gamma(\beta_k)} \left(\frac{\Gamma(2\beta_k - 1)(t_k - t_{k-1})}{2^{2\beta_k - 1}} \right)^{\frac{1}{2}} < e^{-T},$$

$$(\mathrm{P}_{10})\quad \frac{c_k}{\Gamma(\beta_k)} \left(\frac{e^{2t_k}\Gamma(2\beta_k - 1)}{2^{2\beta_k - 1}} \right)^{\frac{1}{2}} \left(\int_{t_{k-1}}^{t_k} \left\{ \int_{t_{k-1}}^{s} (1-\alpha)a(\nu)e^{-\nu}\, d\nu \right\}^2 ds \right)^{\frac{1}{2}} \leq b_k,$$

and (P_8) *holds. Then we get* $y(t) \leq 0$ *for* $t \in [0, T]$.

Proof Firstly, we use Case I(i) of Corollary 2.4 to prove (i). For $t \in [0, T]$, we have

$$y(t) \leq \left\{ y^{1-\alpha}(0) \left(\prod_{t_0 < t_k < t} \widetilde{A_k} \right) e^{R(1-\alpha)t} + \sum_{t_0 < t_k < t} \left(\prod_{t_k < t_j < t} \widetilde{A_j} \right) \widetilde{B_k} e^{R(1-\alpha)(t-t_k)} - \int_{t_l}^{t} (1-\alpha)a(s)e^{R(1-\alpha)(t-s)}\, ds \right\}^{\frac{1}{1-\alpha}},$$

where

$$\widetilde{A_k} = \frac{c_k}{\Gamma(\beta_k)} \left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}} \right)^{\frac{1}{\mu_k}} \left(\frac{1 - e^{\nu_k(t_k - t_{k-1})(1-(1-\alpha)R)}}{\nu_k((1-\alpha)R - 1)} \right)^{\frac{1}{\nu_k}},$$

$$\widetilde{B}_k = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}} \times \left(\int_{t_{k-1}}^{t_k} e^{\nu_k((1-\alpha)R-1)s}\left\{-\int_{t_{k-1}}^{s}(1-\alpha)a(\nu)e^{-(1-\alpha)R\nu}\,d\nu\right\}^{\nu_k} ds\right)^{\frac{1}{\nu_k}} - b_k.$$

It is obvious that $\widetilde{A_k} \geq 0$ for all $k = 1,2,\ldots,n$. In fact, for the case of $(1-\alpha)R > 1$ and $(1-\alpha)R \leq 1$, both the denominator and the numerator in $\widetilde{A_k}$ have the same sign, hence we get $\widetilde{A_k} \geq 0$. The condition (P_2) implies that $\widetilde{B_k} \geq 0$ for all $k = 1,2,\ldots n$. Then it is easy to show that $y(0) \leq 0$. In fact, for $t = T$, we have

$$y(T) \leq \left\{y^{1-\alpha}(0)\left(\prod_{t_0<t_k<t}\widetilde{A_k}\right)e^{R(1-\alpha)T} + \sum_{t_0<t_k<T}\left(\prod_{t_k<t_j<T}\widetilde{A_j}\right)\widetilde{B_k}e^{R(1-\alpha)(T-t_k)} - \int_{t_n}^{T}(1-\alpha)a(s)e^{R(1-\alpha)(T-s)}\,ds\right\}^{\frac{1}{1-\alpha}},$$

that is to say,

$$y(T)^{1-\alpha} \leq y^{1-\alpha}(0)\left(\prod_{k=1}^{n}\widetilde{A_k}\right)e^{R(1-\alpha)T} + \sum_{t_0<t_k<T}\left(\prod_{t_k<t_j<T}\widetilde{A_j}\right)\widetilde{B_k}e^{R(1-\alpha)(T-t_k)} - \int_{t_n}^{T}(1-\alpha)a(s)e^{R(1-\alpha)(T-s)}\,ds.$$

Using the hypothesis $y^{1-\alpha}(0) = y^{1-\alpha}(T) + \theta$ and (P_1), (P_2), we get

$$\begin{aligned} &y^{1-\alpha}(0)\left[1 - \left(\prod_{k=1}^{n}\widetilde{A_k}\right)e^{R(1-\alpha)T}\right] \\ &\leq \theta + \sum_{t_0<t_k<T}\left(\prod_{t_k<t_j<T}\widetilde{A_j}\right)\widetilde{B_k}e^{R(1-\alpha)(T-t_k)} - \int_{t_n}^{T}(1-\alpha)a(s)e^{R(1-\alpha)(T-s)}\,ds \\ &\leq 0, \end{aligned}$$

which implies that $y^{1-\alpha}(0) \leq 0$, since $0 < \alpha < 1$, we get $y(0) \leq 0$.

To prove (ii), applying Case I(ii) of Corollary 2.4 for $t \in [0,T]$, we have

$$y(t) \leq \left\{y^{1-\alpha}(0)\left(\prod_{t_0<t_k<t}\widetilde{C_k}\right)e^{R(1-\alpha)t} + \sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t}\widetilde{C_j}\right)\widetilde{D_k}e^{R(1-\alpha)(t-t_k)} - \int_{t_l}^{t}(1-\alpha)a(s)e^{R(1-\alpha)(t-s)}\,ds\right\}^{\frac{1}{1-\alpha}},$$

where

$$\widetilde{C_k} = \frac{c_k}{2^{\beta_k}\Gamma(\beta_k)}\left\{\frac{\Gamma(2\beta_k-1)}{(1-\alpha)p-1}\left[1 - e^{2(t_k-t_{k-1})(1-(1-\alpha)p)}\right]\right\}^{\frac{1}{2}},$$

$$\widetilde{D_k} = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{2t_k}\Gamma(2\beta_k-1)}{2^{2\beta_k-1}}\right)^{\frac{1}{2}}\left(\int_{t_{k-1}}^{t_k} e^{2((1-\alpha)p-1)s}\left\{\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)e^{-(1-\alpha)p\nu}\,d\nu\right\}^2 ds\right)^{\frac{1}{2}} - b_k.$$

Then using a similar method to proof of Case I(i) with conditions (P_1) and (P_3), we deduce $y(0) \le 0$.

Next, we prove (iii). Applying Case II(i) of Corollary 2.4 for $t \in [0, T]$, we have

$$y(t) \le \left\{y^{1-\alpha}(0)\left(\prod_{t_0<t_k<t}\widetilde{E_k}\right)e^t + \sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t}\widetilde{E_j}\right)\widetilde{F_k}e^{(t-t_k)} - \int_{t_l}^{t}(1-\alpha)a(s)e^{(t-s)}\,ds\right\}^{\frac{1}{1-\alpha}},$$

where

$$\widetilde{E_k} = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}(t_k-t_{k-1})^{\frac{1}{\nu_k}},$$
$$\widetilde{F_k} = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{\mu_k t_k}\Gamma(\beta_k^2)}{\mu_k^{\beta_k^2}}\right)^{\frac{1}{\mu_k}}\left(\int_{t_{k-1}}^{t_k}\left\{\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)e^{-\nu}\,d\nu\right\}^{\nu_k} ds\right)^{\frac{1}{\nu_k}} - b_k.$$

It is easy to see that $\widetilde{E_k} \ge 0$ and $\widetilde{F_k} \le 0$ for all $k = 1, 2, \ldots, n$. Then using a similar method to proof (i) with conditions (P_6) and (P_7), it is easy to show that $y(0) \le 0$.

Similarly, to prove (iv), we apply Case II(ii) of Corollary 2.4 for $t \in [0, T]$, we get

$$m(t) \le \left\{m^{1-\alpha}(t_0)\left(\prod_{t_0<t_k<t}\widetilde{G_k}\right)e^{(t-t_0)} + \sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t}\widetilde{G_j}\right)\widetilde{H_k}e^{(t-t_k)} + \int_{t_l}^{t}(1-\alpha)q(s)e^{(t-s)}\,ds\right\}^{\frac{1}{1-\alpha}}, \tag{3.1}$$

where

$$\widetilde{G_k} = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{\Gamma(2\beta_k-1)(t_k-t_{k-1})}{2^{2\beta_k-1}}\right)^{\frac{1}{2}},$$
$$\widetilde{H_k} = \frac{c_k}{\Gamma(\beta_k)}\left(\frac{e^{2t_k}\Gamma(2\beta_k-1)}{2^{2\beta_k-1}}\right)^{\frac{1}{2}}\left(\int_{t_{k-1}}^{t_k}\left\{\int_{t_{k-1}}^{s}(1-\alpha)q(\nu)e^{-\nu}\,d\nu\right\}^2 ds\right)^{\frac{1}{2}} + b_k.$$

Then using a similar method as that of (iii), it is easy to show that $y(0) \le 0$. This completes the proof. □

Example 3.3 Let $x \in PC^1[\mathbb{R}_+, \mathbb{R}]$, and for $k = 1, 2, \ldots$, we suppose

$$\begin{cases} x'(t) = f(t, x(t)), & t \neq t_k, t \in [t_0, \infty), \\ \Delta x(t_k) = X_k((I_{t_{k-1}}^{\beta_k}x)(t_k)), \\ x(t_0) = x_0, \end{cases} \tag{3.2}$$

where $f \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $X_k \in C(\mathbb{R}, \mathbb{R})$, $0 \le t_0 < t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = \infty$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $\beta_k > 0$ $(k = 1, 2, \ldots)$ and x_0 are constants. Assume there exists a constant $L > 0$, such that

$$\bigl|f\bigl(t, x(t)\bigr)\bigr| \le L\bigl|x(t)\bigr|^{\alpha} \quad \text{for } t \ge t_0, \tag{3.3}$$

and there exists a constant $M_k > 0$, such that

$$\bigl|X_k(x)\bigr| \le M_k|x|, \quad x \in \mathbb{R}, k = 1, 2, \ldots. \tag{3.4}$$

Then, for $t \ge t_0$, the following inequalities hold:

$$\begin{aligned} \bigl|x(t)\bigr| \le {}& \Biggl\{ \Biggl\{ x_0 \prod_{t_0<t_k<t} \widetilde{R}_k + \sum_{t_0<t_k<t} \Biggl[\prod_{t_k<t_j<t} \widetilde{R}_j \Biggr] \widetilde{S}_k \Biggr\}^{1-\alpha} \\ & + (1-\alpha)2^{(k-1)\alpha} L(t-t_0) \Biggr\}^{\frac{1}{1-\alpha}}, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \widetilde{R}_k &= 2^{\frac{\alpha}{1-\alpha}} \biggl[1 + \frac{M_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1}\,ds \biggr], \\ \widetilde{S}_k &= \frac{M_k}{\Gamma(\beta_k)} \frac{1-\alpha}{2-\alpha} 2^{\frac{(k-1)\alpha}{1-\alpha}} \bigl(L(1-\alpha)\bigr)^{\frac{1}{1-\alpha}} \bigl[(t_k - t_0)^{\frac{2-\alpha}{1-\alpha}} - (t_{k-1} - t_0)^{\frac{2-\alpha}{1-\alpha}}\bigr]. \end{aligned}$$

Proof Suppose $x = x(t)$ is a solution of (3.2), we integrate this equation to obtain

$$x(t) = x(t_0) + \int_{t_0}^{t} f\bigl(s, x(s)\bigr)\,ds + \sum_{t_0<t_k<t} X_k\bigl(\bigl(I_{t_{k-1}}^{\beta_k} x\bigr)(t_k)\bigr),$$

then (3.3) and (3.4) imply that

$$\begin{aligned} \bigl|x(t)\bigr| &\le \bigl|x(t_0)\bigr| + \int_{t_0}^{t} \bigl|f\bigl(s, x(s)\bigr)\bigr|\,ds + \sum_{t_0<t_k<t} \bigl|X_k\bigl(\bigl(I_{t_{k-1}}^{\beta_k} x\bigr)(t_k)\bigr)\bigr| \\ &\le |x_0| + \int_{t_0}^{t} L\bigl|x(s)\bigr|^{\alpha}\,ds + \sum_{t_0<t_k<t} M_k\bigl|\bigl(I_{t_{k-1}}^{\beta_k} x\bigr)(t_k)\bigr|. \end{aligned}$$

Then Theorem 2.7 yields the estimate of (3.5), and

$$\begin{aligned} \widetilde{S}_k &= M_k \Gamma(\beta_k) 2^{\frac{(k-1)\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \int_{t_{k-1}}^{t_k} \biggl(\int_{t_0}^{s} L\,dv \biggr)^{\frac{1}{1-\alpha}} ds \\ &= \frac{M_k}{\Gamma(\beta_k)} \frac{1-\alpha}{2-\alpha} 2^{\frac{(k-1)\alpha}{1-\alpha}} \bigl(L(1-\alpha)\bigr)^{\frac{1}{1-\alpha}} \bigl[(t_k - t_0)^{\frac{2-\alpha}{1-\alpha}} - (t_{k-1} - t_0)^{\frac{2-\alpha}{1-\alpha}}\bigr]. \end{aligned}$$ □

As a special case, we consider the following initial value problem of impulsive differential equation with finite discontinuous points.

Example 3.4 Consider the initial value problem of the form

$$\begin{cases} x'(t) = f(x(t)), \quad t \in [0,\infty), t \neq t_k, 1 \leq k \leq 10, \\ \Delta x(t_k) = (I_{k-1}^{\beta_k} x)(t_k), \\ x(0) = 0, \end{cases} \tag{3.6}$$

where $f(x) = \begin{cases} 2\sqrt{|x|}, & |x| < 1, \\ \sqrt[3]{x} + 1, & |x| \geq 1, \end{cases}$ $t_k = k$, $\beta_k = \frac{1}{k+1}$ for $1 \leq k \leq 10$. In this case, we see that $|f(x)| \leq 2\sqrt{|x|}$ with $L = 2$ and $\alpha = \frac{1}{2}$, and $|X_k(x)| = |x|$ with $M_k = 1$. By direct calculation, we get

$$\begin{aligned} \widetilde{R}_k &= 2\left[1 + \frac{1}{\Gamma(\frac{1}{k+1})} \int_{k-1}^{k} (k-s)^{\frac{1}{k+1}-1}\, ds\right] = 2\left[1 + \frac{k+1}{\Gamma(\frac{1}{k+1})}\right] = 2 + \frac{2}{\Gamma(\frac{k+2}{k+1})}, \\ \widetilde{S}_k &= \frac{1}{\Gamma(\frac{1}{k+1})} \frac{1}{3} 2^{k-1}\left[k^3 - (k-1)^3\right] = \frac{2^{k-1}}{\Gamma(\frac{1}{k+1})}\left[k^2 - k + \frac{1}{3}\right]. \end{aligned}$$

Then the solution of the initial value problem (3.6) can be estimated as

$$\left|x(t)\right| \leq \left\{\left(\sum_{0<k<t}\left[\prod_{k<j<t}\left(2 + \frac{2}{\Gamma(\frac{j+2}{j+1})}\right)\right] \frac{2^{k-1}}{\Gamma(\frac{1}{k+1})}\left[k^2 - k + \frac{1}{3}\right]\right)^{1/2} + 2^{\frac{k-1}{2}} t\right\}^2. \tag{3.7}$$

Moreover, for $t \geq 10$, we have

$$\left|x(t)\right| \leq \left(c^* + 16\sqrt{2}t\right)^2$$

for some constant calculated through (3.7) with $k = 10$.

Acknowledgements

The authors sincerely thank the referees for constructive suggestions and corrections, which have significantly improved the contents and the exposition of the paper.

Funding

This project is supported by the NNSF of China (Grants 11671227 and 11271225), NSF of Shandong (Grant No. ZR2018LA004), and Science and Technology Project of High Schools of Shandong Province (Grant Nos. J18KA220, J18KB107).

Competing interests

The authors declare that there are no competing interests.

Authors' contributions

ZZ came with the main thoughts and helped to draft the manuscript, YZ proved the main theorems, JS revised the paper. All authors read and approved the final manuscript.

Author details

[1]School of Mathematical Sciences, Qufu Normal University, Qufu, P.R. China. [2]Department of Mathematics, Jining University, Qufu, P.R. China. [3]Institute of Applied Physics and Computational Mathematics, Beijing, P.R. China.

Publisher's Note

Received: 11 July 2017 Accepted: 23 May 2018 Published online: 13 July 2018

References

1. Lakshmikantham, V., Bainov, D.D., Simeonov, P.S.: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
2. Bainov, D.D., Simeonov, P.S.: Theory of Impulsive Differential Equations: Periodic Solutions and Applications. Longman, Harlow (1993)

3. Bainov, D.D., Simeonov, P.S.: Impulsive Differential Equations: Asymptotic Properties of the Solutions. World Scientific, Singapore (1995)
4. Borysenko, D.S.: About one integral inequality for piece-wise continuous functions. In: Proceedings of the X International Kravchuk Conference, Kyiv, p. 323 (2004)
5. Iovane, G.: Some new integer integral inequalities of Bellman–Bihari type with delay for discontinuous functions. Nonlinear Anal. TMA **66**(2), 498–508 (2007)
6. Gallo, A., Piccirillo, A.M.: Multidimensional impulse inequalities and general Bahari type inequalities for discontinuous functions with delay. Nonlinear Stud. **19**(1), 13–24 (2012)
7. Hristova, S.G.: Nonlinear delay integral inequalities for piecewise continuous functions and applications. J. Inequal. Pure Appl. Math. **5**(4), Article ID 88 (2004)
8. Mi, Y.: Some generalized Gronwall–Bellman type impulsive integral inequalities and their applications. J. Appl. Math. **2014**, Article ID 353210 (2014)
9. Zheng, Z., Gao, X., Shao, J.: Some new generalized retarded inequalities for discontinuous functions and their applications. J. Inequal. Appl. **2016**, 7 (2016)
10. Ye, H.P., Gao, J.M., Ding, Y.S.: A generalized Gronwall inequality and its application to a fractional differential equation. J. Math. Anal. Appl. **328**, 1075–1081 (2007)
11. Wang, X.: A generalized Halanay inequality with impulse and delay. Adv. Dev. Technol. Inter. **1**(2), 9–23 (2012)
12. Thiramanus, P., Tariboon, J.: Impulsive differential and impulsive integral inequalities with integral jump conditions. J. Inequal. Appl. **2012**, 25 (2012)
13. Shao, J., Meng, F.: Nonlinear impulsive differential and integral inequalities with integral jump conditions. Adv. Differ. Equ. **2016**, 112 (2016)
14. Feng, Q., Meng, F., Zheng, B.: Gronwall–Bellman type nonlinear delay integral inequalities on times scales. J. Math. Anal. Appl. **382**(2), 772–784 (2011)
15. Xu, R.: Some new nonlinear weakly singular integral inequalities and their applications. J. Math. Inequal. **11**(4), 1007–1018 (2017)
16. Xu, R., Ma, X.: Some new retarded nonlinear Volterra–Fredholm type integral inequalities with maxima in two variables and their applications. J. Inequal. Appl. **2017**, 187 (2017)
17. Liu, H.: A class of retarded Volterra–Fredholm type integral inequalities on time scales and their applications. J. Inequal. Appl. **2017**, 293 (2017)
18. Gu, J., Meng, F.: Some new nonlinear Volterra–Fredholm type dynamic integral inequalities on time scales. Appl. Math. Comput. **245**, 235–242 (2014)