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Weighted arithmetic–geometric operator mean inequalities

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Abstract

In this paper, we refine and generalize some weighted arithmetic–geometric operator mean inequalities due to Lin (Stud. Math. 215:187–194, 2013) and Zhang (Banach J. Math. Anal. 9:166–172, 2015) as follows: Let A and B be positive operators. If $0 < m \le A \le m' < M' \le B \le M$ or $0 < m \le B \le m' < M' \le A \le M$, then for a positive unital linear map Φ,

$$\Phi^2(A\nabla_\alpha B) \le \left[\frac{K(h)}{S(h'^r)}\right]^2 \Phi^2(A\sharp_\alpha B),$$

$$\Phi^2(A\nabla_\alpha B) \le \left[\frac{K(h)}{S(h'^r)}\right]^2 \left[\Phi(A)\sharp_\alpha \Phi(B)\right]^2,$$

$$\Phi^{2p}(A\nabla_\alpha B) \le \frac{1}{16}\left[\frac{K^2(h)(M^2+m^2)^2}{S^2(h'^r)M^2m^2}\right]^p \Phi^{2p}(A\sharp_\alpha B),$$

$$\Phi^{2p}(A\nabla_\alpha B) \le \frac{1}{16}\left[\frac{K^2(h)(M^2+m^2)^2}{S^2(h'^r)M^2m^2}\right]^p \left[\Phi(A)\sharp_\alpha \Phi(B)\right]^{2p},$$

where $\alpha \in [0,1]$, $K(h) = \frac{(h+1)^2}{4h}$, $S(h') = \frac{h'^{\frac{1}{h'-1}}}{e\log h'^{\frac{1}{h'-1}}}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$, $r = \min\{\alpha, 1-\alpha\}$ and $p \ge 2$.

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1 Introduction

Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and I be the identity operator. $\|\cdot\|$ is the operator norm. $A \ge 0$ ($A > 0$) implies that A is a positive (strictly positive) operator. A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is called positive if $A \ge 0$ implies $\Phi(A) \ge 0$. It is said to be unital if $\Phi(I) = I$. For $A, B > 0$, the α-weighted arithmetic mean and α-weighted geometric mean of A and B are defined, respectively, by

$$A\nabla_\alpha B = (1-\alpha)A + \alpha B, \qquad A\sharp_\alpha B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}},$$

where $\alpha \in [0,1]$. When $\alpha = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity for $A\nabla_{\frac{1}{2}} B$ and $A\sharp_{\frac{1}{2}} B$, respectively.

Let $0 < m \leq A, B \leq M$, and Φ be a positive unital linear map. Tominaga [3] showed that the following operator inequality holds:

$$\frac{A+B}{2} \leq S(h) A \sharp B, \tag{1.1}$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ is called Specht's radio and $h = \frac{M}{m}$. Indeed

$$S(h) \leq K(h) = \frac{(h+1)^2}{4h} \leq S^2(h) \quad (h \geq 1) \tag{1.2}$$

was observed by Lin [1, (3.3)].

By (1.1) and (1.2), it is easy to obtain the following inequality:

$$\Phi\left(\frac{A+B}{2}\right) \leq K(h) \Phi(A \sharp B). \tag{1.3}$$

Lin [1, Theorem 2.1] proved that (1.3) can be squared as follows:

$$\Phi^2\left(\frac{A+B}{2}\right) \leq K^2(h) \Phi^2(A \sharp B) \tag{1.4}$$

and

$$\Phi^2\left(\frac{A+B}{2}\right) \leq K^2(h) \left[\Phi(A) \sharp \Phi(B)\right]^2. \tag{1.5}$$

Zhang [2] generalized (1.4) and (1.5) when $p \geq 2$

$$\Phi^{2p}\left(\frac{A+B}{2}\right) \leq \frac{[K(h)(M^2+m^2)]^{2p}}{16 M^{2p} m^{2p}} \Phi^{2p}(A \sharp B) \tag{1.6}$$

and

$$\Phi^{2p}\left(\frac{A+B}{2}\right) \leq \frac{[K(h)(M^2+m^2)]^{2p}}{16 M^{2p} m^{2p}} \left[\Phi(A) \sharp \Phi(B)\right]^{2p}. \tag{1.7}$$

A great number of results on operator inequalities have been given in the literature, for example, see [4–6] and the references therein.

In this paper, motivated by the aforementioned discussion, we extend (1.4)–(1.7) to the weighted arithmetic–geometric mean. In order to prove our results, we show a new operator weighted arithmetic–geometric mean inequality. Manipulating this operator inequality enables us to refine and generalize (1.4)–(1.7). Furthermore, a numerical example is given to demonstrate the effectiveness of the theoretical results.

2 Main results

In this section, the main results of this paper will be given. To do this, the following lemmas are necessary.

Lemma 1 ([7]) *Let* $A, B > 0$. *Then the following norm inequality holds*:

$$\|AB\| \le \frac{1}{4}\|A+B\|^2. \tag{2.1}$$

Lemma 2 ([8]) *Let* $A > 0$. *Then for every positive unital linear map* Φ,

$$\Phi\bigl(A^{-1}\bigr) \ge \Phi^{-1}(A). \tag{2.2}$$

Lemma 3 ([9]) *Let* $A, B > 0$. *Then for* $1 \le r < \infty$,

$$\bigl\|A^r + B^r\bigr\| \le \bigl\|(A+B)^r\bigr\|. \tag{2.3}$$

Lemma 4 ([10]) *Let* $0 < m \le A \le m' < M' \le B \le M$ *or* $0 < m \le B \le m' < M' \le A \le M$. *Then for each* $\alpha \in [0,1]$,

$$A\nabla_\alpha B \ge S\bigl(h'^r\bigr)A\sharp_\alpha B, \tag{2.4}$$

where $S(h') = \frac{h'^{\frac{1}{h'-1}}}{e\log h'^{\frac{1}{h'-1}}}$, $h' = \frac{M'}{m'}$ *and* $r = \min\{\alpha, 1-\alpha\}$.

Theorem 1 *Let* $0 < m \le A \le m' < M' \le B \le M$ *or* $0 < m \le B \le m' < M' \le A \le M$. *Then for each* $\alpha \in [0,1]$,

$$A\nabla_\alpha B + MmS\bigl(h'^r\bigr)(A\sharp_\alpha B)^{-1} \le M + m, \tag{2.5}$$

where $S(h') = \frac{h'^{\frac{1}{h'-1}}}{e\log h'^{\frac{1}{h'-1}}}$, $h' = \frac{M'}{m'}$ *and* $r = \min\{\alpha, 1-\alpha\}$.

Proof Since

$$0 < m \le A \le M,$$

then

$$(1-\alpha)(M-A)(m-A)A^{-1} \le 0.$$

That is,

$$(1-\alpha)\bigl(A + MmA^{-1}\bigr) \le (1-\alpha)(M+m). \tag{2.6}$$

Similarly, we get

$$\alpha\bigl(B + MmB^{-1}\bigr) \le \alpha(M+m). \tag{2.7}$$

Summing up inequalities (2.6) and (2.7), we get

$$A\nabla_\alpha B + MmA^{-1}\nabla_\alpha B^{-1} \le M + m.$$

By $(A\sharp_\alpha B)^{-1} = A^{-1}\sharp_\alpha B^{-1}$ and (2.4), we have

$$\begin{aligned} A\nabla_\alpha B + MmS(h'^r)(A\sharp_\alpha B)^{-1} &\le A\nabla_\alpha B + MmA^{-1}\nabla_\alpha B^{-1} \\ &\le M + m. \end{aligned}$$

This completes the proof. □

Theorem 2 *Let Φ be a positive unital linear map and let A and B be positive operators. If $0 < m \le A \le m' < M' \le B \le M$ or $0 < m \le B \le m' < M' \le A \le M$, then for each $\alpha \in [0,1]$,*

$$\Phi^2(A\nabla_\alpha B) \le \left[\frac{K(h)}{S(h'^r)}\right]^2 \Phi^2(A\sharp_\alpha B), \tag{2.8}$$

$$\Phi^2(A\nabla_\alpha B) \le \left[\frac{K(h)}{S(h'^r)}\right]^2 \left[\Phi(A)\sharp_\alpha \Phi(B)\right]^2, \tag{2.9}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $S(h') = \frac{h'^{\frac{1}{h'-1}}}{e\log h'^{\frac{1}{h'-1}}}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ and $r = \min\{\alpha, 1-\alpha\}$.

Proof Inequality (2.8) is equivalent to

$$\left\| \Phi(A\nabla_\alpha B)\Phi^{-1}(A\sharp_\alpha B)\right\| \le \frac{K(h)}{S(h'^r)}.$$

By (2.1), (2.2) and (2.5), we have

$$\begin{aligned} \left\| \Phi(A\nabla_\alpha B)MmS(h'^r)\Phi^{-1}(A\sharp_\alpha B)\right\| &\le \frac{1}{4}\left\| \Phi(A\nabla_\alpha B) + MmS(h'^r)\Phi^{-1}(A\sharp_\alpha B)\right\|^2 \\ &\le \frac{1}{4}\left\| \Phi(A\nabla_\alpha B) + MmS(h'^r)\Phi\left[(A\sharp_\alpha B)^{-1}\right]\right\|^2 \\ &= \frac{1}{4}\left\| \Phi\left[(A\nabla_\alpha B) + MmS(h'^r)(A\sharp_\alpha B)^{-1}\right]\right\|^2 \\ &\le \frac{1}{4}\left\| \Phi(M+m)\right\|^2 \\ &= \frac{1}{4}(M+m)^2. \end{aligned}$$

That is,

$$\left\| \Phi(A\nabla_\alpha B)\Phi^{-1}(A\sharp_\alpha B)\right\| \le \frac{(M+m)^2}{4MmS(h'^r)} = \frac{K(h)}{S(h'^r)}.$$

Thus, (2.8) holds.

Inequality (2.9) is equivalent to

$$\left\| \Phi(A\nabla_\alpha B)\left[\Phi(A)\sharp_\alpha \Phi(B)\right]^{-1}\right\| \le \frac{K(h)}{S(h'^r)}.$$

By (2.1) and (2.5), we have

$$
\begin{aligned}
&\left\| \Phi(A\nabla_\alpha B) MmS(h'^r)\left[\Phi(A)\sharp_\alpha\Phi(B)\right]^{-1}\right\| \\
&\le \frac{1}{4}\left\|\Phi(A\nabla_\alpha B) + MmS(h'^r)\left[\Phi(A)\sharp_\alpha\Phi(B)\right]^{-1}\right\|^2 \\
&= \frac{1}{4}\left\|\Phi(A)\nabla_\alpha\Phi(B) + MmS(h'^r)\left[\Phi(A)\sharp_\alpha\Phi(B)\right]^{-1}\right\|^2 \\
&\le \frac{1}{4}(M+m)^2.
\end{aligned}
$$

That is,

$$
\left\|\Phi(A\nabla_\alpha B)\left[\Phi(A)\sharp_\alpha\Phi(B)\right]^{-1}\right\| \le \frac{K(h)}{S(h'^r)}.
$$

Thus, (2.9) holds.
This completes the proof. □

Theorem 3 *Let Φ be a positive unital linear map and let A and B be positive operators. If $0 < m \le A \le m' < M' \le B \le M$ or $0 < m \le B \le m' < M' \le A \le M$ and $2 \le p < \infty$, then for each $\alpha \in [0,1]$,*

$$
\Phi^{2p}(A\nabla_\alpha B) \le \frac{1}{16}\left[\frac{K^2(h)(M^2+m^2)^2}{S^2(h'^r)M^2m^2}\right]^p \Phi^{2p}(A\sharp_\alpha B), \tag{2.10}
$$

$$
\Phi^{2p}(A\nabla_\alpha B) \le \frac{1}{16}\left[\frac{K^2(h)(M^2+m^2)^2}{S^2(h'^r)M^2m^2}\right]^p \left[\Phi(A)\sharp_\alpha\Phi(B)\right]^{2p}, \tag{2.11}
$$

where $K(h) = \frac{(h+1)^2}{4h}$, $S(h') = \frac{h'^{\frac{1}{h'-1}}}{e\log h'^{\frac{1}{h'-1}}}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ and $r = \min\{\alpha, 1-\alpha\}$.

Proof By (2.8), we have

$$
\Phi^{-2}(A\sharp_\alpha B) \le L^2\Phi^{-2}(A\nabla_\alpha B), \tag{2.12}
$$

where $L = \frac{K(h)}{S(h'^r)}$.

Inequality (2.10) is equivalent to

$$
\left\|\Phi^p(A\nabla_\alpha B)\Phi^{-p}(A\sharp_\alpha B)\right\| \le \frac{1}{4}\left[\frac{K^2(h)(M^2+m^2)^2}{S^2(h'^r)M^2m^2}\right]^{\frac{p}{2}}.
$$

By (2.1), (2.3) and (2.12), we have

$$
\begin{aligned}
&\left\|\Phi^p(A\nabla_\alpha B)M^pm^p\Phi^{-p}(A\sharp_\alpha B)\right\| \\
&\le \frac{1}{4}\left\|L^{\frac{p}{2}}\Phi^p(A\nabla_\alpha B) + \left(\frac{M^2m^2}{L}\right)^{\frac{p}{2}}\Phi^{-p}(A\sharp_\alpha B)\right\|^2 \\
&\le \frac{1}{4}\left\|L\Phi^2(A\nabla_\alpha B) + \frac{M^2m^2}{L}\Phi^{-2}(A\sharp_\alpha B)\right\|^p
\end{aligned}
$$

$$\leq \frac{1}{4}\left\|L\Phi^2(A\nabla_\alpha B) + LM^2m^2\Phi^{-2}(A\nabla_\alpha B)\right\|^p$$
$$\leq \frac{1}{4}\left[L\left(M^2 + m^2\right)\right]^p.$$

That is,

$$\left\|\Phi^p(A\nabla_\alpha B)\Phi^{-p}(A\sharp_\alpha B)\right\| \leq \frac{1}{4}\left[\frac{L(M^2+m^2)}{Mm}\right]^p = \frac{1}{4}\left[\frac{K^2(h)(M^2+m^2)^2}{S^2(h'^r)M^2m^2}\right]^{\frac{p}{2}}.$$

Thus, (2.10) holds.

Similarly, (2.11) holds by inequality (2.9).

This completes the proof. □

Remark 1 When $\alpha = \frac{1}{2}$, because of $\frac{K(h)}{S(\sqrt{h'})} < K(h)$, inequalities (2.8), (2.9), (2.10) and (2.11) are sharper than (1.4), (1.5), (1.6) and (1.7), respectively.

In what follows, when $\alpha = \frac{1}{2}$, we present an example showing that inequalities (2.8)–(2.11) are sharper than (1.4)–(1.7), respectively.

Example 1 Take $A = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{5}{7} \end{bmatrix}$ and $B = \begin{bmatrix} \frac{10}{3} & 0 \\ 0 & \frac{23}{7} \end{bmatrix}$. We find $\frac{1}{2} < A < \frac{3}{4} < 3 < B < 4$. A calculation shows $\frac{K(8)}{S(2)} \approx 2.3847 < K(8) \approx 2.5313$.

3 Conclusions

In this paper, we have presented some new weighted arithmetic–geometric operator mean inequalities. These inequalities are refinements and generalizations of some corresponding results of [1, 2].

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Competing interests
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Authors' contributions
The author read and approved the final manuscript.

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