Hong et al. *Journal of Inequalities and Applications* (2018) 2018:155
https://doi.org/10.1186/s13660-018-1749-0

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Some inequalities for generalized eigenvalues of perturbation problems on Hermitian matrices

Yan Hong[1], Dongkyu Lim[2] and Feng Qi[3,4,5*]

*Correspondence:
qifeng618@hotmail.com
[3]Institute of Mathematics, Henan Polytechnic University, Jiaozuo, China
[4]Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin, China
Full list of author information is available at the end of the article

Abstract

In the paper, the authors establish some inequalities for generalized eigenvalues of perturbation problems on Hermitian matrices and modify shortcomings of some known inequalities for generalized eigenvalues in the related literature.

MSC: Primary 15B33; Secondary 11R52, 15A42, 15A48, 16H05, 20G20

Keywords: Generalized eigenvalue; Hermitian matrix; Inequality; Perturbation problem

1 Introduction

Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian matrices with B being positive definite. We now consider a perturbation problem for $A\boldsymbol{x} = \lambda B\boldsymbol{x}$. It is known that the n generalized eigenvalues of the matrix pencil $\langle A, B\rangle$ are real numbers and that the generalized eigenvalues of $\langle A, B\rangle$ and the eigenvalues of AB^{-1} are the same. Without loss of generality, we can line up the eigenvalues of a Hermitian matrix A as

$$\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A)$$

and order the generalized eigenvalues of $\langle A, B\rangle$ by

$$\lambda_1\big(AB^{-1}\big) \ge \lambda_2\big(AB^{-1}\big) \ge \cdots \ge \lambda_n\big(AB^{-1}\big).$$

For a standard Hermitian eigenvalue problem $A\boldsymbol{x} = \lambda\boldsymbol{x}$, Weyl's theorem [2] is perhaps the best-known perturbation result. We denote the spectral norm of a matrix by $\|\cdot\|_2$ which is also called the largest singular value or the matrix 2-norm.

We now recall several known conclusions in the literature.

Theorem 1.1 ([2, Weyl's theorem]) *Let $A, E \in \mathbb{C}^{n\times n}$ be Hermitian matrices, and let $\widetilde{A} = A + E$ be a perturbation of A, then*

$$\max_{1\le i\le n} \big|\lambda_i(A) - \lambda_i(\widetilde{A})\big| \le \|E\|_2.$$

Theorem 1.2 ([3]) *Let $A, E \in \mathbb{C}^{n\times n}$ be Hermitian matrices, and let $\widetilde{A} = A + E$ be a perturbation of A, then*

$$\bigl|\lambda(\widetilde{A}) - \lambda(A)\bigr| \le \bigl(\|A\|_2 + \|E\|_2\bigr)^{1-1/n}\|E\|_2^{1/n}.$$

Theorem 1.3 ([1, 4]) *Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian matrices, and let B be a positive definite Hermitian matrix. Then the equalities*

$$\lambda_i\bigl(AB^{-1}\bigr) = \max_{\substack{S\subseteq\mathbb{C}^n\\ \dim S=i}} \min_{0\neq \boldsymbol{x}\in S}\left\{\frac{\boldsymbol{x}^*A\boldsymbol{x}}{\boldsymbol{x}^*B\boldsymbol{x}}\right\} = \min_{\substack{T\subseteq\mathbb{C}^n\\ \dim T=n-i+1}} \max_{0\neq \boldsymbol{x}\in T}\left\{\frac{\boldsymbol{x}^*A\boldsymbol{x}}{\boldsymbol{x}^*B\boldsymbol{x}}\right\}$$

hold for $1 \le i \le n$. In particular, if $B = I_n$, we have

$$\lambda_i(A) = \max_{\substack{S\subseteq\mathbb{C}^n\\ \dim S=i}} \min_{0\neq \boldsymbol{x}\in S} \boldsymbol{x}^*A\boldsymbol{x} = \min_{\substack{T\subseteq\mathbb{C}^n\\ \dim T=n-i+1}} \max_{0\neq \boldsymbol{x}\in T} \boldsymbol{x}^*A\boldsymbol{x}, \quad 1 \le i \le n.$$

Theorem 1.4 ([5, p. 336]) *Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian matrices and $i, j, k, \ell, \hbar \in \mathbb{N}$ with $j + k - 1 \le i \le \ell + \hbar - n - 1$. Then*

$$\lambda_\ell(A) + \lambda_\hbar(A) \le \lambda_i(A + B) \le \lambda_j(A) + \lambda_k(B).$$

In particular, we have

$$\lambda_i(A) + \lambda_n(B) \le \lambda_i(A + B) \le \lambda_i(A) + \lambda_1(B).$$

Let $A, E \in \mathbb{C}^{n\times n}$ be Hermitian matrices, B be a positive definite Hermitian matrix,

$$\widetilde{B} = B + E, \qquad \beta_n = \min_{1\le i\le n} \lambda_i(B), \qquad \mu = \frac{\|E\|_2}{\beta_n} = \frac{\|E\|_2}{\lambda_n(B)}.$$

Then μ is a sufficient condition for $\widetilde{B}$ to be a Hermitian positive definite matrix.

Theorem 1.5 ([4]) *Let $A, B, H, E \in \mathbb{C}^{n\times n}$ be Hermitian matrices, B be a positive definite Hermitian matrix, and $\widetilde{B} = B + E$. If $\mu = \frac{\|E\|_2}{\lambda_n(B)} < 1$, then the double inequality*

$$(1 - \mu)\lambda_i\bigl(AB^{-1}\bigr) + \lambda_n\bigl(HB^{-1}\bigr) \le \lambda_i\bigl((A + H)\widetilde{B}^{-1}\bigr) \le \frac{\lambda_i(AB^{-1}) + \lambda_1(HB^{-1})}{1 - \mu}$$

is valid for all $1 \le i \le n$.

Theorem 1.6 ([4]) *Let $A, B, H, E \in \mathbb{C}^{n\times n}$ be Hermitian matrices, B be a positive definite Hermitian matrix, and $\widetilde{B} = B + E$. If $\varepsilon \triangleq \max_{1\le i\le n} |\lambda_i(EB^{-1})| < 1$, then the double inequality*

$$(1 - \varepsilon)\lambda_i\bigl(AB^{-1}\bigr) + \lambda_n\bigl(HB^{-1}\bigr) \le \lambda_i\bigl((A + H)\widetilde{B}^{-1}\bigr) \le \frac{\lambda_i(AB^{-1}) + \lambda_1(HB^{-1})}{1 - \varepsilon}$$

is valid for all $1 \le i \le n$.

Remark 1.1 Let

$$A = \operatorname{diag}(-3,-2), \qquad B = \operatorname{diag}(3,4), \qquad H = I_2, \qquad E = \operatorname{diag}(2,1).$$

Then

$$\lambda_2\bigl(HB^{-1}\bigr) + (1-\mu)\lambda_2\bigl(AB^{-1}\bigr) = \frac{1}{3} > 0 = \lambda_2\bigl((A+H)\widetilde{B}^{-1}\bigr).$$

Let

$$A = \operatorname{diag}(-3,-2), \qquad B = \operatorname{diag}(3,4), \qquad H = -2I_n, \qquad E = \operatorname{diag}(2,1).$$

Then

$$\lambda_1\bigl((A+H)\widetilde{B}^{-1}\bigr) = -\frac{4}{5} > -3 = \frac{\lambda_1(AB^{-1}) + \lambda_1(HB^{-1})}{1-\varepsilon}.$$

These two examples demonstrate that Theorems 1.5 and 1.6 are not necessarily true.

In this paper, we will establish some inequalities of perturbation problems for generalized eigenvalues.

2 Main results

We are now in a position to state and prove our main results in this paper.

Theorem 2.1 *Let $A, B, H, E \in \mathbb{C}^{n\times n}$ be Hermitian matrices, B be a positive definite Hermitian matrix, and $\widetilde{B} = B + E$. If $\mu = \frac{\|E\|_2}{\lambda_n(B)} < 1$ and $i, j, k, \ell, \hbar \in \mathbb{N}$ with $j + k - 1 \le i \le \ell + \hbar - n - 1$, then*

1. *when $\lambda_i(A+H) \ge 0$, we have*

$$\frac{\lambda_\ell(AB^{-1}) + \lambda_\hbar(HB^{-1})}{1+\mu} \le \lambda_i\bigl((A+H)\widetilde{B}^{-1}\bigr) \le \frac{\lambda_j(AB^{-1}) + \lambda_k(HB^{-1})}{1-\mu};$$

2. *when $\lambda_i(A+H) \le 0$, we have*

$$\frac{\lambda_j(AB^{-1}) + \lambda_k(HB^{-1})}{1-\mu} \le \lambda_i\bigl((A+H)\widetilde{B}^{-1}\bigr) \le \frac{\lambda_\ell(AB^{-1}) + \lambda_\hbar(HB^{-1})}{1+\mu}.$$

Proof Since $B^{-1/2}(A+H)B^{-1/2}$ is a Hermitian matrix, then there exists an orthogonal matrix $U = (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n) \in \mathbb{C}^{n\times n}$ such that

$$B^{-1/2}(A+H)B^{-1/2} = U^* \operatorname{diag}\bigl(\lambda_1\bigl((A+H)B^{-1}\bigr), \ldots, \lambda_n\bigl((A+H)B^{-1}\bigr)\bigr)U.$$

Let

$$T_i = \operatorname{Span}(\boldsymbol{u}_i, \boldsymbol{u}_{i+1}, \ldots, \boldsymbol{u}_n), \quad 1 \le i \le n.$$

By virtue of Theorems 1.3 and 1.4, if $j+k-1 \le i \le \ell + \hbar - n - 1$, we have

$$\begin{aligned}
\lambda_i\big((A+H)\widetilde{B}^{-1}\big) &\le \max_{0\neq \boldsymbol{x}\in T}\left\{\frac{\boldsymbol{x}^*B^{-1/2}(A+H)B^{-1/2}\boldsymbol{x}}{\boldsymbol{x}^*(I_n+B^{-1/2}EB^{-1/2})\boldsymbol{x}}\right\} \\
&\le \begin{cases} \frac{1}{1-\mu}\max_{0\neq \boldsymbol{x}\in T}\{\frac{\boldsymbol{x}^*B^{-1/2}(A+H)B^{-1/2}\boldsymbol{x}}{\boldsymbol{x}^*\boldsymbol{x}}\}, & \lambda_i(A+H)\ge 0; \\ \frac{1}{1+\mu}\max_{0\neq \boldsymbol{x}\in T}\{\frac{\boldsymbol{x}^*B^{-1/2}(A+H)B^{-1/2}\boldsymbol{x}}{\boldsymbol{x}^*\boldsymbol{x}}\}, & \lambda_i(A+H)< 0 \end{cases} \\
&= \begin{cases} \frac{1}{1-\mu}\lambda_i((A+H)B^{-1}), & \lambda_i(A+H)\ge 0; \\ \frac{1}{1+\mu}\lambda_i((A+H)B^{-1}), & \lambda_i(A+H)< 0 \end{cases} \\
&\le \begin{cases} \frac{\lambda_j(AB^{-1})+\lambda_k(HB^{-1})}{1-\mu}, & \lambda_i(A+H)\ge 0; \\ \frac{\lambda_\ell(AB^{-1})+\lambda_\hbar(HB^{-1})}{1+\mu}, & \lambda_i(A+H)< 0. \end{cases}
\end{aligned} \tag{2.1}$$

Similarly, we have

$$\lambda_i\big((A+H)\widetilde{B}^{-1}\big) \ge \begin{cases} \frac{\lambda_\ell(AB^{-1})+\lambda_\hbar(HB^{-1})}{1+\mu}, & \lambda_i(A+H)\ge 0; \\ \frac{\lambda_j(AB^{-1})+\lambda_k(HB^{-1})}{1-\mu}, & \lambda_i(A+H)< 0. \end{cases} \tag{2.2}$$

The proof of Theorem 2.1 is complete. □

Corollary 2.1 *Let $A,B,H,E\in\mathbb{C}^{n\times n}$ be Hermitian matrices, B be a positive definite Hermitian matrix, and $\widetilde{B}=B+E$. If $\mu=\frac{\|E\|_2}{\lambda_n(B)}<1$, then*

1. *when $\lambda_i(A+H)\ge 0$ for $1\le i\le n$,*

$$\frac{\lambda_i(AB^{-1})+\lambda_n(HB^{-1})}{1+\mu} \le \lambda_i\big((A+H)\widetilde{B}^{-1}\big) \le \frac{\lambda_i(AB^{-1})+\lambda_1(HB^{-1})}{1-\mu};$$

2. *when $\lambda_i(A+H)\le 0$ for $1\le i\le n$,*

$$\frac{\lambda_i(AB^{-1})+\lambda_1(HB^{-1})}{1-\mu} \le \lambda_i\big((A+H)\widetilde{B}^{-1}\big) \le \frac{\lambda_i(AB^{-1})+\lambda_n(HB^{-1})}{1+\mu}.$$

Corollary 2.2 *Let $A,B,H,E\in\mathbb{C}^{n\times n}$ be Hermitian matrices, B be a positive definite Hermitian matrix, and $\widetilde{B}=B+E$. If $\mu=\frac{\|E\|_2}{\lambda_n(B)}<1$, then*

1. *when $\lambda_i(A+H)\ge 0$ for $1\le i\le n$, then*

$$\frac{1}{1+\mu}\left[\lambda_i\big(AB^{-1}\big)-\frac{\|H\|}{\lambda_n(B)}\right] \le \lambda_i\big((A+H)\widetilde{B}^{-1}\big) \le \frac{1}{1-\mu}\left[\lambda_i\big(AB^{-1}\big)+\frac{\|H\|}{\lambda_n(B)}\right];$$

2. *when $\lambda_i(A+H)\le 0$ for $1\le i\le n$, then*

$$\frac{1}{1-\mu}\left[\lambda_i\big(AB^{-1}\big)-\frac{\|H\|}{\lambda_n(B)}\right] \le \lambda_i\big((A+H)\widetilde{B}^{-1}\big) \le \frac{1}{1+\mu}\left[\lambda_i\big(AB^{-1}\big)+\frac{\|H\|}{\lambda_n(B)}\right].$$

Theorem 2.2 *Let $A,B,H,E\in\mathbb{C}^{n\times n}$ be Hermitian matrices, B be a positive definite Hermitian matrix, and $\widetilde{B}=B+E$. If $\varepsilon=\max_{1\le i\le n}|\lambda_i(EB^{-1})|<1$, then*

1. *when* $\lambda_i(A+H) \geq 0$ *for* $1 \leq i \leq n$,

$$\frac{\lambda_i(AB^{-1}) + \lambda_n(HB^{-1})}{1+\varepsilon} \leq \lambda_i\big((A+H)\widetilde{B}^{-1}\big) \leq \frac{\lambda_i(AB^{-1}) + \lambda_1(HB^{-1})}{1-\varepsilon};$$

2. *when* $\lambda_i(A+H) \leq 0$ *for* $1 \leq i \leq n$,

$$\frac{\lambda_i(AB^{-1}) + \lambda_1(HB^{-1})}{1-\varepsilon} \leq \lambda_i\big((A+H)\widetilde{B}^{-1}\big) \leq \frac{\lambda_i(AB^{-1}) + \lambda_n(HB^{-1})}{1+\varepsilon}.$$

Proof Using inequalities (2.1) and (2.2), we obtain the required results. The proof of Theorem 2.2 is thus complete. □

Theorem 2.3 *Let* $A, B, H, E \in \mathbb{C}^{n\times n}$ *be Hermitian matrices,* B *be a positive definite Hermitian matrix, and* $\widetilde{B} = B + E$. *If* $\mu = \frac{\|E\|_2}{\lambda_n(B)} < 1$, *then*

$$\begin{aligned}\beta_i(A)\lambda_i\big(AB^{-1}\big) + \beta_n(H)\lambda_n\big(HB^{-1}\big) &\leq \lambda_i\big((A+H)\widetilde{B}^{-1}\big) \\ &\leq \alpha_i(A)\lambda_i\big(AB^{-1}\big) + \alpha_1(H)\lambda_1\big(HB^{-1}\big)\end{aligned}$$

for $1 \leq i \leq n$, *where*

$$\alpha_i(A) = \begin{cases} \frac{1}{1-\mu}, & \lambda_i(A) \geq 0; \\ \frac{1}{1+\mu}, & \lambda_i(A) < 0 \end{cases} \quad \text{and} \quad \beta_i(A) = \begin{cases} \frac{1}{1-\mu}, & \lambda_i(A) < 0; \\ \frac{1}{1+\mu}, & \lambda_i(A) \geq 0. \end{cases}$$

Proof Since

$$\begin{aligned}\lambda_i\big(\widetilde{B}^{-1/2}A\widetilde{B}^{-1/2}\big) + \lambda_n\big(\widetilde{B}^{-1/2}H\widetilde{B}^{-1/2}\big) &\leq \lambda_i\big((A+H)\widetilde{B}^{-1}\big) \\ &\leq \lambda_i\big(\widetilde{B}^{-1/2}A\widetilde{B}^{-1/2}\big) + \lambda_1\big(\widetilde{B}^{-1/2}H\widetilde{B}^{-1/2}\big)\end{aligned}$$

for $1 \leq i \leq n$. From inequalities in (2.1) and (2.2), it follows that

$$\begin{aligned}&\beta_i(A)\lambda_i\big(AB^{-1}\big) \leq \lambda_i\big(\widetilde{B}^{-1/2}A\widetilde{B}^{-1/2}\big) = \lambda_i\big(A\widetilde{B}^{-1}\big) \leq \alpha_i(A)\lambda_i\big(AB^{-1}\big), \\ &\beta_n(H)\lambda_n\big(HB^{-1}\big) \leq \lambda_n\big(H\widetilde{B}^{-1}\big), \qquad \lambda_1\big(H\widetilde{B}^{-1}\big) \leq \alpha_1(H)\lambda_1\big(AB^{-1}\big)\end{aligned}$$

for $1 \leq i \leq n$. The proof of Theorem 2.3 is complete. □

Acknowledgements
The authors appreciate the anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

Funding
The first and third authors were partially supported by the National Natural Science Foundation of China under Grant No. 11361038, by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region under Grant No. NJZZ18154, and by the Science Research Fund of Inner Mongolia University for Nationalities under Grant No. NMDYB15019, China. The second author was partially supported by the National Research Foundation of Korea (NRF) under Grant Nos. NRF-2016R1A5A1008055 and NRF-2018R1D1A1B07041846, South Korea.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Author details
[1]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, China. [2]Department of Mathematics, Sungkyunkwan University, Suwon, South Korea. [3]Institute of Mathematics, Henan Polytechnic University, Jiaozuo, China. [4]Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin, China. [5]Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, China.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 7 March 2018 Accepted: 8 June 2018 Published online: 03 July 2018

References

1. Amir-Moéz, A.R.: Extreme properties of eigenvalues of a Hermitian transformation and singular values of the sum and product of linear transformations. Duke Math. J. **23**, 463–476 (1956). https://doi.org/10.1215/S0012-7094-56-02343-2
2. Bai, Z., Demmel, J., Dongarra, J., Ruhe, A., van der Vorst, H.: Templates for the Solution of Algebraic Eigenvalue Problems: A Practical Guide. Software, Environments and Tools, vol. 11. Society for Industrial and Applied Mathematics, Philadelphia (2000). https://doi.org/10.1137/1.9780898719581
3. Elsner, L.: An optimal bound for the spectral variation of two matrices. Linear Algebra Appl. **71**, 77–80 (1985). https://doi.org/10.1016/0024-3795(85)90236-8
4. Huang, L.: Some perturbation problems for the generalized eigenvalues. Acta Sci. Nat. Univ. Pekin. **1978** 22–27 (1978)
5. Marshall, A.W., Olkin, I., Arnold, B.C.: Inequalities: Theory of Majorization and Its Applications, 2nd edn. Springer, New York (2011). https://doi.org/10.1007/978-0-387-68276-1