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Inequalities and asymptotic expansions related to the generalized Somos quadratic recurrence constant

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Abstract

In this paper, we give asymptotic expansions and inequalities related to the generalized Somos quadratic recurrence constant, using its relation with the generalized Euler constant.

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Keywords: Somos' quadratic recurrence constant; Inequality; Asymptotic expansion; Generalized Euler constant

1 Introduction

Somos' quadratic recurrence constant is defined (see [1–3]) by

$$\begin{aligned}\sigma &= \sqrt{1\sqrt{2\sqrt{3\cdots}}} = \prod_{n=1}^{\infty} n^{1/2^n} = \prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{1/2^k} = \exp\left\{\sum_{k=1}^{\infty}\frac{\ln k}{2^k}\right\} \\ &= 1.66168794\ldots \end{aligned} \tag{1.1}$$

or

$$\sigma = \exp\left\{-\int_0^1 \frac{1-x}{(2-x)\ln x}\,\mathrm{d}x\right\} = \exp\left\{-\int_0^1\int_0^1 \frac{x}{(2-xy)\ln(xy)}\,\mathrm{d}x\,\mathrm{d}y\right\}. \tag{1.2}$$

The constant σ arises in the study of the asymptotic behavior of the sequence

$$g_0 = 1, \qquad g_n = n g_{n-1}^2, \qquad n \in \mathbb{N} := \{1, 2, 3, \ldots\}, \tag{1.3}$$

with the first few terms

$$g_0 = 1, \qquad g_1 = 1, \qquad g_2 = 2, \qquad g_3 = 12, \qquad g_4 = 576, \qquad g_5 = 1{,}658{,}880, \qquad \ldots.$$

This sequence behaves as follows (see [4, p. 446] and [3, 5]):

$$\begin{aligned} g_n \sim \frac{\sigma^{2^n}}{n}\biggl(&1 + \frac{2}{n} - \frac{1}{n^2} + \frac{4}{n^3} - \frac{21}{n^4} + \frac{138}{n^5} - \frac{1091}{n^6} + \frac{10{,}088}{n^7} - \frac{106{,}918}{n^8} \\ &+ \frac{1{,}279{,}220}{n^9} - \frac{17{,}070{,}418}{n^{10}} + \frac{251{,}560{,}472}{n^{11}} - \frac{4{,}059{,}954{,}946}{n^{12}} + \cdots\biggr)^{-1}. \end{aligned} \tag{1.4}$$

The constant σ appears in important problems from pure and applied analysis, and it is the motivation for a large number of research papers (see, for example, [1, 6–16]).

Sondow and Hadjicostas [15] introduced and studied the generalized-Euler-constant function $\gamma(z)$, defined by

$$\gamma(z)=\sum_{n=1}^{\infty}z^{n-1}\left(\frac{1}{n}-\ln\frac{n+1}{n}\right), \tag{1.5}$$

where the series converges when $|z|\le 1$. Pilehrood and Pilehrood [13] considered the function $z\gamma(z)$ ($|z|\le 1$). The function $\gamma(z)$ generalizes both Euler's constant $\gamma(1)$ and the alternating Euler constant $\ln\frac{4}{\pi}=\gamma(-1)$ [17, 18].

Sondow and Hadjicostas [15] defined the generalized Somos constant

$$\sigma_t=\sqrt[t]{1\sqrt[t]{2\sqrt[t]{3\sqrt[t]{4\cdots}}}}=\prod_{n=1}^{\infty}n^{1/t^n}=\left(\frac{t}{t-1}\right)^{1/(t-1)}\exp\left\{-\frac{1}{t(t-1)}\gamma\left(\frac{1}{t}\right)\right\},\quad t>1. \tag{1.6}$$

Coffey [19] gave the integral and series representations for $\ln\sigma_t$:

$$\ln\sigma_t=\int_0^{\infty}\left(\frac{e^{-x}}{t-1}+\frac{1}{1-te^x}\right)\frac{\mathrm{d}x}{x} \tag{1.7}$$

and

$$\ln\sigma_t=\frac{1}{t-1}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\operatorname{Li}_k\left(\frac{1}{t}\right)=\frac{1}{t-1}\sum_{k=1}^{\infty}\frac{1}{k}\left[t\operatorname{Li}_k\left(\frac{1}{t}\right)-1\right] \tag{1.8}$$

in terms of the polylogarithm function.

It is known (see [15]) that

$$\gamma\left(\frac{1}{2}\right)=2\ln\frac{2}{\sigma},\quad \text{equivalently,}\quad \sigma=2\exp\left\{-\frac{1}{2}\gamma\left(\frac{1}{2}\right)\right\}. \tag{1.9}$$

Thus, formula (1.5) is closely related to Somos' quadratic recurrence constant σ.

Define

$$\gamma_n(z)=\sum_{k=1}^{n}z^{k-1}\left(\frac{1}{k}-\ln\frac{k+1}{k}\right),\quad |z|\le 1.$$

Mortici [11] proved that for $n\in\mathbb{N}$,

$$\frac{270(n+1)}{2^n(270n^3+1530n^2+1065n+6293)}<\gamma\left(\frac{1}{2}\right)-\gamma_n\left(\frac{1}{2}\right)<\frac{18}{2^n(18n^2+84n-13)} \tag{1.10}$$

and

$$\begin{aligned}&\sum_{k=n+1}^{\infty}\frac{1}{2k^2\cdot 3^{k-1}}-\frac{22{,}400(n+1)}{3^n(44{,}800n^4+280{,}000n^3+435{,}120n^2+744{,}380n-2{,}477{,}677)}\\&\quad<\gamma\left(\frac{1}{3}\right)-\gamma_n\left(\frac{1}{3}\right)<\sum_{k=n+1}^{\infty}\frac{1}{2k^2\cdot 3^{k-1}}-\frac{160}{3^n(320n^3+1680n^2+1428n+3889)}.\end{aligned} \tag{1.11}$$

Lu and Song [10] improved Mortici's results and obtained the inequalities:

$$\begin{aligned}\frac{690n^2+3524n+145}{6(2^n)(n+1)^2(115n^2+894n+779)} &< \gamma\left(\frac{1}{2}\right)-\gamma_n\left(\frac{1}{2}\right)\\ &< \frac{48n+127}{3(2^n)(16n+85)(n+1)^2}\end{aligned} \tag{1.12}$$

and

$$\begin{aligned}&\sum_{k=n+1}^{\infty}\frac{1}{2k^2\cdot 3^{k-1}}-\frac{987840n^2+8444340n+10946779}{40(3^n)(n+1)^2(49392n^3+582741n^2+1769516n+1236167)}\\ &\quad< \gamma\left(\frac{1}{3}\right)-\gamma_n\left(\frac{1}{3}\right)<\sum_{k=n+1}^{\infty}\frac{1}{2k^2\cdot 3^{k-1}}-\frac{1620n^2+6995n+1847}{40(3^n)(81n^2+532n+451)(n+1)^3}\end{aligned} \tag{1.13}$$

for $n\in\mathbb{N}$.

You and Chen [16] further improved inequalities (1.10)–(1.13). Recently, Chen and Han [7] gave new bounds for $\gamma(1/2)-\gamma_n(1/2)$:

$$\begin{aligned}&\frac{1}{2^n}\left(\frac{1}{(n+1)^2}-\frac{8}{3(n+1)^3}+\frac{23}{2(n+1)^4}-\frac{332}{5(n+1)^5}+\frac{479}{(n+1)^6}-\frac{29{,}024}{7(n+1)^7}\right)\\ &\quad<\gamma\left(\frac{1}{2}\right)-\gamma_n\left(\frac{1}{2}\right)\\ &\quad<\frac{1}{2^n}\left(\frac{1}{(n+1)^2}-\frac{8}{3(n+1)^3}+\frac{23}{2(n+1)^4}-\frac{332}{5(n+1)^5}+\frac{479}{(n+1)^6}\right)\end{aligned} \tag{1.14}$$

for $n\in\mathbb{N}$, and presented the following asymptotic expansion:

$$\begin{aligned}&\gamma\left(\frac{1}{2}\right)-\gamma_n\left(\frac{1}{2}\right)\\ &\quad\sim\frac{1}{2^n(n+1)^2}\left\{1-\frac{8}{3(n+1)}+\frac{23}{2(n+1)^2}-\frac{332}{5(n+1)^3}+\frac{479}{(n+1)^4}-\cdots\right\}\end{aligned} \tag{1.15}$$

as $n\to\infty$. Moreover, these authors gave a formula for successively determining the coefficients in (1.15).

Chen and Han [7] pointed out that the lower bound in (1.14) is for $n\geq 24$ sharper than the one in (1.12), and the upper bound in (1.14) is for $n\geq 18$ sharper than the one in (1.12),

For any positive integer $m\geq 2$, in this paper we give the asymptotic expansion of $\gamma(1/m)-\gamma_n(1/m)$ as $n\to\infty$. Based on the result obtained, we establish the inequality for $\gamma(1/4)-\gamma_n(1/4)$. We also consider the asymptotic expansion for $\gamma(-1)-\gamma_n(-1)$.

2 Lemmas

Lemma 2.1 *As $x\to\infty$,*

$$\frac{1}{x}-\ln\left(1+\frac{1}{x}\right)-\sum_{j=2}^{m-1}\frac{(-1)^j}{j}\frac{1}{x^j}\sim A(x)-\frac{1}{m}A(x+1), \tag{2.1}$$

where $A(x)$ is defined by

$$A(x) = \sum_{j=m}^{\infty} \frac{a_j}{x^j} \tag{2.2}$$

with the coefficients a_j given by the recurrence relation

$$a_j = \frac{(-1)^j}{m-1}\left\{\frac{m}{j} + \sum_{k=m}^{j-1}(-1)^k a_k \binom{j-1}{j-k}\right\}, \quad j \geq m. \tag{2.3}$$

Here, and throughout this paper, an empty sum is understood to be zero.

Proof Using the Maclaurin series of $\ln(1+t)$,

$$\ln(1+t) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} t^j, \quad -1 < t \leq 1,$$

we obtain

$$\frac{1}{x} - \ln\left(1+\frac{1}{x}\right) - \sum_{j=2}^{m-1} \frac{(-1)^j}{j}\frac{1}{x^j} = \sum_{j=m}^{\infty} \frac{(-1)^j}{j}\frac{1}{x^j}. \tag{2.4}$$

In view of (2.4), we can let

$$\frac{1}{x} - \ln\left(1+\frac{1}{x}\right) - \sum_{j=2}^{m-1} \frac{(-1)^j}{j}\frac{1}{x^j} \sim \sum_{j=m}^{\infty} \frac{a_j}{x^j} - \frac{1}{m}\sum_{j=m}^{\infty} \frac{a_j}{(x+1)^j}, \tag{2.5}$$

where a_j are real numbers to be determined.

Write (2.5) as

$$\frac{1}{x} - \ln\left(1+\frac{1}{x}\right) - \sum_{j=2}^{m-1} \frac{(-1)^j}{j}\frac{1}{x^j} \sim \sum_{j=m}^{\infty} \frac{a_j}{x^j} - \frac{1}{m}\sum_{j=m}^{\infty} \frac{a_j}{x^j}\left(1+\frac{1}{x}\right)^{-j}. \tag{2.6}$$

Direct computation yields

$$\begin{aligned}
\sum_{j=m}^{\infty} \frac{a_j}{x^j}\left(1+\frac{1}{x}\right)^{-j} &= \sum_{j=m}^{\infty} \frac{a_j}{x^j}\sum_{k=0}^{\infty}\binom{-j}{k}\frac{1}{x^k} \\
&= \sum_{j=m}^{\infty} \frac{a_j}{x^j}\sum_{k=0}^{\infty}(-1)^k\binom{k+j-1}{k}\frac{1}{x^k} \\
&= \sum_{j=m}^{\infty}\sum_{k=m}^{j} a_k(-1)^{j-k}\binom{j-1}{j-k}\frac{1}{x^j}.
\end{aligned} \tag{2.7}$$

It follows from (2.4), (2.6), and (2.7) that

$$\sum_{j=m}^{\infty} \frac{(-1)^j}{j}\frac{1}{x^j} \sim \sum_{j=m}^{\infty}\left\{a_j - \frac{1}{m}\sum_{k=m}^{j} a_k(-1)^{j-k}\binom{j-1}{j-k}\right\}\frac{1}{x^j}. \tag{2.8}$$

Equating coefficients of the term x^{-j} on both sides of (2.8) yields

$$\frac{(-1)^j}{j} = a_j - \frac{1}{m}\sum_{k=m}^{j} a_k(-1)^{j-k}\binom{j-1}{j-k}, \quad j \geq m. \tag{2.9}$$

For $j = m$, we obtain $a_m = \frac{(-1)^m}{m-1}$, and for $j \geq m+1$, we have

$$\frac{(-1)^j}{j} = a_j - \frac{1}{m}\left[\sum_{k=m}^{j-1} a_k(-1)^{j-k}\binom{j-1}{j-k} + a_j\right], \quad j \geq m+1.$$

We then obtain the recursive formula

$$a_m = \frac{(-1)^m}{m-1}, \qquad a_j = \frac{(-1)^j m}{(m-1)j} + \frac{1}{m-1}\sum_{k=m}^{j-1} a_k(-1)^{j-k}\binom{j-1}{j-k}, \quad j \geq m+1,$$

which can be written as (2.3). The proof of Lemma 2.1 is complete. □

Lemma 2.2 *Let*

$$a(x) = \frac{1}{3x^4} - \frac{32}{45x^5} \quad \text{and} \quad b(x) = \frac{1}{3x^4} - \frac{32}{45x^5} + \frac{68}{27x^6}. \tag{2.10}$$

Then, for $x \geq 1$,

$$a(x) - \frac{1}{4}a(x+1) < \frac{1}{x} - \ln\left(1 + \frac{1}{x}\right) - \frac{1}{2x^2} + \frac{1}{3x^3} < b(x) - \frac{1}{4}b(x+1). \tag{2.11}$$

Proof It is well known that for $-1 < t \leq 1$ and $m \in \mathbb{N}$,

$$\sum_{j=1}^{2m} \frac{(-1)^{j-1}}{j} t^j < \ln(1+t) < \sum_{j=1}^{2m-1} \frac{(-1)^{j-1}}{j} t^j,$$

which implies that for $x \geq 1$ and $m \geq 2$,

$$\sum_{j=4}^{2m+1} \frac{(-1)^j}{jx^j} < \frac{1}{x} - \ln\left(1 + \frac{1}{x}\right) - \frac{1}{2x^2} + \frac{1}{3x^3} < \sum_{j=4}^{2m} \frac{(-1)^j}{jx^j}. \tag{2.12}$$

Using (2.12), we find that

$$\begin{aligned}
&\frac{1}{x} - \ln\left(1 + \frac{1}{x}\right) - \frac{1}{2x^2} + \frac{1}{3x^3} - a(x) + \frac{1}{4}a(x+1) \\
&\quad > \frac{1}{4x^4} - \frac{1}{5x^5} - a(x) + \frac{1}{4}a(x+1) = \frac{310x^4 + 770x^3 + 845x^2 + 445x + 92}{180x^5(x+1)^5} > 0
\end{aligned}$$

and

$$\frac{1}{x} - \ln\left(1 + \frac{1}{x}\right) - \frac{1}{2x^2} + \frac{1}{3x^3} - b(x) + \frac{1}{4}b(x+1)$$

$$< \frac{1}{4x^4} - \frac{1}{5x^5} + \frac{1}{6x^6} - b(x) + \frac{1}{4}b(x+1)$$
$$= -\frac{4380x^5 + 14{,}205x^4 + 21{,}530x^3 + 17{,}439x^2 + 7344x + 1270}{540x^6(x+1)^6} < 0.$$

The proof of Lemma 2.2 is complete. □

Remark 2.1 Using the methods from [20–22] it is possible to get estimations (based on the power series expansions) of the logarithm function that can be used, for example, in the analysis of parameterized Euler-constant function, which will be an item for further work.

Lemma 2.3 *As $x \to \infty$, we have*

$$\frac{1}{x} - \ln\left(1 + \frac{1}{x}\right) \sim C(x) + C(x+1), \tag{2.13}$$

where $C(x)$ is defined by

$$C(x) = \sum_{j=2}^{\infty} \frac{c_j}{x^j} \tag{2.14}$$

with the coefficients c_j given by the recurrence relation

$$c_2 = \frac{1}{4}, \qquad c_j = \frac{(-1)^j}{2j} - \frac{1}{2}\sum_{k=2}^{j-1} c_k(-1)^{j-k}\binom{j-1}{j-k}, \quad j \geq 3. \tag{2.15}$$

Proof In view of (2.4), we can let

$$\frac{1}{x} - \ln\left(1 + \frac{1}{x}\right) \sim \sum_{j=2}^{\infty} \frac{c_j}{x^j} + \sum_{j=2}^{\infty} \frac{c_j}{(x+1)^j}, \tag{2.16}$$

where c_j are real numbers to be determined. Write (2.16) as

$$\frac{1}{x} - \ln\left(1 + \frac{1}{x}\right) \sim \sum_{j=2}^{\infty} \frac{c_j}{x^j} + \sum_{j=2}^{\infty} \frac{c_j}{x^j}\left(1 + \frac{1}{x}\right)^{-j}.$$

Noting that (2.7) holds, we have

$$\sum_{j=2}^{\infty} \frac{(-1)^j}{j}\frac{1}{x^j} \sim \sum_{j=2}^{\infty}\left\{c_j + \sum_{k=2}^{j} c_k(-1)^{j-k}\binom{j-1}{j-k}\right\}\frac{1}{x^j}. \tag{2.17}$$

Equating coefficients of the term x^{-j} on both sides of (2.17) yields

$$\frac{(-1)^j}{j} = c_j + \sum_{k=2}^{j} c_k(-1)^{j-k}\binom{j-1}{j-k}, \quad j \geq 2.$$

For $j = 2$, we obtain $c_2 = 1/4$, and for $j \geq 3$ we have

$$\frac{(-1)^j}{j} = 2c_j + \sum_{k=2}^{j-1} c_k (-1)^{j-k} \binom{j-1}{j-k}, \quad j \geq 3.$$

We then obtain the recursive formula (2.15). The proof of Lemma 2.3 is complete. □

The first few coefficients c_j are

$$c_2 = \frac{1}{4}, \qquad c_3 = \frac{1}{12}, \qquad c_4 = -\frac{1}{8}, \qquad c_5 = -\frac{1}{10}, \qquad c_6 = \frac{1}{4},$$
$$c_7 = \frac{17}{56}, \qquad c_8 = -\frac{17}{16}, \qquad c_9 = -\frac{31}{18}.$$

3 Main results

For any positive integer $m \geq 2$, Theorem 3.1 gives the asymptotic expansion of $\gamma(1/m) - \gamma_n(1/m)$ as $n \to \infty$.

Theorem 3.1 *For any positive integer $m \geq 2$, we have*

$$\gamma\left(\frac{1}{m}\right) - \gamma_n\left(\frac{1}{m}\right) \sim \sum_{k=n+1}^{\infty} \sum_{j=2}^{m-1} \frac{(-1)^j}{j \cdot m^{k-1}} \frac{1}{k^j} + \frac{A(n+1)}{m^n}, \quad n \to \infty, \tag{3.1}$$

where $A(x)$ is given in (2.2). Namely,

$$\begin{aligned}
&\gamma\left(\frac{1}{m}\right) - \gamma_n\left(\frac{1}{m}\right) \\
&\quad\sim \sum_{k=n+1}^{\infty} \sum_{j=2}^{m-1} \frac{(-1)^j}{j \cdot m^{k-1}} \frac{1}{k^j} \\
&\qquad + \frac{(-1)^m}{m^n} \left\{ \frac{1}{(m-1)(n+1)^m} - \frac{2m^2}{(m+1)(m-1)^2(n+1)^{m+1}} \right. \\
&\qquad + \frac{m^2(m^2+8m+3)}{2(m+2)(m-1)^3(n+1)^{m+2}} - \frac{(m+1)(m^3+12m^2+51m+8)m^2}{6(m-1)^4(m+3)(n+1)^{m+3}} \\
&\qquad \left. + \frac{m^2(m^6+25m^5+216m^4+866m^3+1241m^2+501m+30)}{24(m-1)^5(m+4)(n+1)^{m+4}} - \cdots \right\}.
\end{aligned} \tag{3.2}$$

Proof Write (2.1) as

$$\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right) - \sum_{j=2}^{m-1} \frac{(-1)^j}{j} \frac{1}{k^j} = A_N(k) - \frac{1}{m} A_N(k+1) + O\left(\frac{1}{k^{N+1}}\right), \tag{3.3}$$

where

$$A_N(k) = \sum_{j=m}^{N} \frac{a_j}{k^j} \tag{3.4}$$

with the coefficients a_j given by the recurrence relation (2.3). From (3.3), we have

$$\begin{aligned}
&\frac{1}{m^{k-1}}\left(\frac{1}{k}-\ln\left(1+\frac{1}{k}\right)\right)-\frac{1}{m^{k-1}}\sum_{j=2}^{m-1}\frac{(-1)^j}{j}\frac{1}{k^j}\\
&\quad=\frac{A_N(k)}{m^{k-1}}-\frac{A_N(k+1)}{m^k}+O\left(\frac{1}{m^{k-1}k^{N+1}}\right).
\end{aligned} \tag{3.5}$$

Adding (3.5) from $k=n+1$ to $k=\infty$, we have

$$\gamma\left(\frac{1}{m}\right)-\gamma_n\left(\frac{1}{m}\right)-\sum_{k=n+1}^{\infty}\sum_{j=2}^{m-1}\frac{(-1)^j}{j\cdot m^{k-1}}\frac{1}{k^j}=\frac{1}{m^n}\left\{A_N(n+1)+O\left(\frac{1}{(n+1)^{N+1}}\right)\right\},$$

which can be written as (3.1). The proof of Theorem 3.1 is complete. □

Remark 3.1 For $m=2$ in (3.2), we obtain (1.15). For $m=3$ in (3.2), we find

$$\begin{aligned}
&\gamma\left(\frac{1}{3}\right)-\gamma_n\left(\frac{1}{3}\right)\\
&\quad\sim\sum_{k=n+1}^{\infty}\frac{1}{2k^2 3^{k-1}}\\
&\qquad+\frac{1}{3^n(n+1)^3}\left\{-\frac{1}{2}+\frac{9}{8(n+1)}-\frac{81}{20(n+1)^2}+\frac{37}{2(n+1)^3}-\frac{5661}{56(n+1)^4}+\cdots\right\}.
\end{aligned} \tag{3.6}$$

For $m=4$ in (3.2), we find

$$\begin{aligned}
&\gamma\left(\frac{1}{4}\right)-\gamma_n\left(\frac{1}{4}\right)\\
&\quad\sim\sum_{k=n+1}^{\infty}\left(\frac{1}{2}-\frac{1}{3k}\right)\frac{1}{k^2 4^{k-1}}\\
&\qquad+\frac{1}{4^n(n+1)^4}\left\{\frac{1}{3}-\frac{32}{45(n+1)}+\frac{68}{27(n+1)^2}-\frac{2080}{189(n+1)^3}\right.\\
&\qquad\left.+\frac{9017}{162(n+1)^4}-\cdots\right\}.
\end{aligned} \tag{3.7}$$

Formula (3.7) motivated us to establish Theorem 3.2.

Theorem 3.2 *For* $n\in\mathbb{N}$,

$$\begin{aligned}
&\sum_{k=n+1}^{\infty}\left(\frac{1}{2}-\frac{1}{3k}\right)\frac{1}{k^2 4^{k-1}}+\frac{1}{4^n}\left\{\frac{1}{3(n+1)^4}-\frac{32}{45(n+1)^5}\right\}\\
&\quad<\gamma\left(\frac{1}{4}\right)-\gamma_n\left(\frac{1}{4}\right)\\
&\quad<\sum_{k=n+1}^{\infty}\left(\frac{1}{2}-\frac{1}{3k}\right)\frac{1}{k^2 4^{k-1}}+\frac{1}{4^n}\left\{\frac{1}{3(n+1)^4}-\frac{32}{45(n+1)^5}+\frac{68}{27(n+1)^6}\right\}.
\end{aligned} \tag{3.8}$$

Proof From the double inequality (2.11), we have

$$\begin{aligned}\frac{a(k)}{4^{k-1}}-\frac{a(k+1)}{4^k}&<\frac{1}{4^{k-1}}\left(\frac{1}{k}-\ln\left(1+\frac{1}{k}\right)\right)-\frac{1}{4^{k-1}}\left(\frac{1}{2k^2}-\frac{1}{3k^3}\right)\\&<\frac{b(k)}{4^{k-1}}-\frac{b(k+1)}{4^k},\end{aligned}\tag{3.9}$$

where $a(x)$ and $b(x)$ are given in (2.10). Adding inequalities (3.9) from $k=n+1$ to $k=\infty$, we have

$$\frac{a(n+1)}{4^n}<\sum_{k=n+1}^{\infty}\frac{1}{4^{k-1}}\left(\frac{1}{k}-\ln\left(1+\frac{1}{k}\right)\right)-\sum_{k=n+1}^{\infty}\frac{1}{4^{k-1}}\left(\frac{1}{2k^2}-\frac{1}{3k^3}\right)<\frac{b(n+1)}{4^n},$$

which can be written as (3.8). The proof of Theorem 3.2 is complete. □

Remark 3.2 Inequality (3.8) can be further refined by inserting additional terms on both sides of the inequality. For example, for $n\in\mathbb{N}$, we have

$$\begin{aligned}&\sum_{k=n+1}^{\infty}\left(\frac{1}{2}-\frac{1}{3k}\right)\frac{1}{k^2 4^{k-1}}+\frac{1}{4^n}\left\{\frac{1}{3(n+1)^4}-\frac{32}{45(n+1)^5}+\frac{68}{27(n+1)^6}-\frac{2080}{189(n+1)^7}\right\}\\&\quad<\gamma\left(\frac{1}{4}\right)-\gamma_n\left(\frac{1}{4}\right)\\&\quad<\sum_{k=n+1}^{\infty}\left(\frac{1}{2}-\frac{1}{3k}\right)\frac{1}{k^2 4^{k-1}}\\&\qquad+\frac{1}{4^n}\left\{\frac{1}{3(n+1)^4}-\frac{32}{45(n+1)^5}+\frac{68}{27(n+1)^6}-\frac{2080}{189(n+1)^7}+\frac{9017}{162(n+1)^8}\right\}.\end{aligned}\tag{3.10}$$

Remark 3.3 Following the same method as the one used in the proof of Theorem 3.2, we can prove the following inequality:

$$\begin{aligned}&\sum_{k=n+1}^{\infty}\frac{1}{2k^2 3^{k-1}}+\frac{1}{3^n}\left\{-\frac{1}{2(n+1)^3}+\frac{9}{8(n+1)^4}-\frac{81}{20(n+1)^5}+\frac{37}{2(n+1)^6}-\frac{5661}{56(n+1)^7}\right\}\\&\quad<\gamma\left(\frac{1}{3}\right)-\gamma_n\left(\frac{1}{3}\right)\\&\quad<\sum_{k=n+1}^{\infty}\frac{1}{2k^2 3^{k-1}}+\frac{1}{3^n}\left\{-\frac{1}{2(n+1)^3}+\frac{9}{8(n+1)^4}-\frac{81}{20(n+1)^5}+\frac{37}{2(n+1)^6}\right\}\end{aligned}\tag{3.11}$$

for $n\in\mathbb{N}$. We omit the proof.

In view of (1.14), (3.11), (3.8), and (3.10), we pose the following conjecture.

Conjecture 3.1 For any positive integer $m\geq 2$, we have

$$\begin{aligned}\frac{(-1)^m}{m^n}\sum_{j=m}^{2N}\frac{a_j}{(n+1)^j}&<(-1)^m\left\{\gamma\left(\frac{1}{m}\right)-\gamma_n\left(\frac{1}{m}\right)-\sum_{k=n+1}^{\infty}\sum_{j=2}^{m-1}\frac{(-1)^j}{j\cdot m^{k-1}k^j}\right\}\\&<\frac{(-1)^m}{m^n}\sum_{j=m}^{2N+1}\frac{a_j}{(n+1)^j},\end{aligned}\tag{3.12}$$

with the coefficients a_j given in (2.3).

By using the Maple software, we find, as $n \to \infty$,

$$\gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) \sim \frac{1}{2^n(n+1)^2}\left(1 + \frac{-\frac{8}{3}}{n+\frac{85}{16}} + \frac{-\frac{2689}{160}}{(n+\frac{807,797}{129,072})^3} + \cdots\right), \tag{3.13}$$

$$\begin{aligned} &\gamma\left(\frac{1}{3}\right) - \gamma_n\left(\frac{1}{3}\right) \\ &\quad \sim \sum_{k=n+1}^{\infty} \frac{1}{2k^2 3^{k-1}} + \frac{1}{3^n(n+1)^3}\left(-\frac{1}{2} + \frac{\frac{9}{8}}{n+\frac{23}{5}} + \frac{\frac{98}{25}}{(n+\frac{140,843}{27,440})^3} + \cdots\right) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \gamma\left(\frac{1}{4}\right) - \gamma_n\left(\frac{1}{4}\right) \sim{}& \sum_{k=n+1}^{\infty}\left(\frac{1}{2} - \frac{1}{3k}\right)\frac{1}{k^2 4^{k-1}} \\ &+ \frac{1}{4^n(n+1)^4}\left\{\frac{1}{3} + \frac{-\frac{32}{45}}{n+\frac{109}{24}} + \frac{-\frac{2365}{1134}}{(n+\frac{825,361}{170,280})^3} + \cdots\right\}. \end{aligned} \tag{3.15}$$

From a computational viewpoint, formulas (3.13), (3.14), and (3.15) improve formulas (1.15), (3.6), and (3.7), respectively.

For any positive integer $m \geq 2$, we here provide a pair of recurrence relations for determining the constants $p_\ell \equiv p_\ell(m)$ and $q_\ell \equiv q_\ell(m)$ (see Remark 3.4) such that

$$\begin{aligned} &\gamma\left(\frac{1}{m}\right) - \gamma_n\left(\frac{1}{m}\right) \\ &\quad \sim \sum_{k=n+1}^{\infty}\sum_{j=2}^{m-1} \frac{(-1)^j}{j \cdot m^{k-1}}\frac{1}{k^j} + \frac{1}{m^n(n+1)^m}\left(a_m + \sum_{\ell=1}^{\infty}\frac{p_\ell}{(n+q_\ell)^{2\ell-1}}\right) \end{aligned} \tag{3.16}$$

as $n \to \infty$. This develops formulas (3.13), (3.14), and (3.15) to produce a general result given by Theorem 3.3.

Theorem 3.3 *For any positive integer $m \geq 2$, we have*

$$\gamma\left(\frac{1}{m}\right) - \gamma_{n-1}\left(\frac{1}{m}\right) \sim \sum_{k=n}^{\infty}\sum_{j=2}^{m-1} \frac{(-1)^j}{j \cdot m^{k-1}}\frac{1}{k^j} + \frac{1}{m^{n-1}n^m}\left(a_m + \sum_{\ell=1}^{\infty}\frac{\lambda_\ell}{(n+\mu_\ell)^{2\ell-1}}\right) \tag{3.17}$$

as $n \to \infty$, where $\lambda_\ell \equiv \lambda_\ell(m)$ and $\mu_\ell \equiv \mu_\ell(m)$ are given by a pair of recurrence relations

$$\lambda_\ell = a_{m+2\ell-1} - \sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k}\binom{2\ell-2}{2\ell-2k}, \quad \ell \geq 2, \tag{3.18}$$

and

$$\mu_\ell = -\frac{1}{(2\ell-1)\lambda_\ell}\left\{a_{m+2\ell} + \sum_{k=1}^{\ell-1}\lambda_k \mu_k^{2\ell-2k+1}\binom{2\ell-1}{2\ell-2k+1}\right\}, \quad \ell \geq 2, \tag{3.19}$$

with

$$\lambda_1 = a_{m+1} = \frac{(-1)^{m+1}2m^2}{(m+1)(m-1)^2} \quad \textit{and} \quad \mu_1 = -\frac{a_{m+2}}{\lambda_1} = \frac{(m+1)(m^2+8m+3)}{4(m+2)(m-1)}.$$

Here a_j are given in (2.3).

Proof In view of (3.13), (3.14), and (3.15), we let

$$\gamma\left(\frac{1}{m}\right) - \gamma_{n-1}\left(\frac{1}{m}\right) \sim \sum_{k=n}^{\infty}\sum_{j=2}^{m-1}\frac{(-1)^j}{j\cdot m^{k-1}}\frac{1}{k^j} + \frac{1}{m^{n-1}n^m}\left(a_m + \sum_{\ell=1}^{\infty}\frac{\lambda_\ell}{(n+\mu_\ell)^{2\ell-1}}\right),$$

where λ_ℓ and μ_ℓ are real numbers to be determined. This can be written as

$$\begin{aligned} &m^{n-1}n^m\left\{\gamma\left(\frac{1}{m}\right) - \gamma_{n-1}\left(\frac{1}{m}\right) - \sum_{k=n}^{\infty}\sum_{j=2}^{m-1}\frac{(-1)^j}{j\cdot m^{k-1}}\frac{1}{k^j}\right\} \\ &\quad \sim a_m + \sum_{j=1}^{\infty}\frac{\lambda_j}{n^{2j-1}}\left(1+\frac{\mu_j}{n}\right)^{-2j+1}. \end{aligned} \tag{3.20}$$

Direct computation yields

$$\begin{aligned} \sum_{j=1}^{\infty}\frac{\lambda_j}{n^{2j-1}}\left(1+\frac{\mu_j}{n}\right)^{-2j+1} &= \sum_{j=1}^{\infty}\frac{\lambda_j}{n^{2j-1}}\sum_{k=0}^{\infty}\binom{-2j+1}{k}\frac{\mu_j^k}{n^k} \\ &= \sum_{j=1}^{\infty}\frac{\lambda_j}{n^{2j-1}}\sum_{k=0}^{\infty}(-1)^k\binom{k+2j-2}{k}\frac{\mu_j^k}{n^k} \\ &= \sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\lambda_{k+1}\mu_{k+1}^{j-k-1}(-1)^{j-k-1}\binom{j+k-1}{j-k-1}\frac{1}{n^{j+k}}, \end{aligned}$$

which can be written as

$$\sum_{j=1}^{\infty}\frac{\lambda_j}{n^{2j-1}}\left(1+\frac{\mu_j}{n}\right)^{-2j+1} \sim \sum_{j=1}^{\infty}\left\{\sum_{k=1}^{\lfloor\frac{j+2}{2}\rfloor}\lambda_k\mu_k^{j-2k+1}(-1)^{j-1}\binom{j-1}{j-2k+1}\right\}\frac{1}{n^j}. \tag{3.21}$$

Substituting (3.21) into (3.20) we have

$$\begin{aligned} &m^{n-1}n^m\left\{\gamma\left(\frac{1}{m}\right) - \gamma_{n-1}\left(\frac{1}{m}\right) - \sum_{k=n}^{\infty}\sum_{j=2}^{m-1}\frac{(-1)^j}{j\cdot m^{k-1}}\frac{1}{k^j}\right\} \\ &\quad \sim a_m + \sum_{j=1}^{\infty}\left\{\sum_{k=1}^{\lfloor\frac{j+2}{2}\rfloor}\lambda_k\mu_k^{j-2k+1}(-1)^{j-1}\binom{j-1}{j-2k+1}\right\}\frac{1}{n^j}. \end{aligned} \tag{3.22}$$

On the other hand, it follows from (3.1) that

$$m^{n-1}n^m\left\{\gamma\left(\frac{1}{m}\right) - \gamma_{n-1}\left(\frac{1}{m}\right) - \sum_{k=n}^{\infty}\sum_{j=2}^{m-1}\frac{(-1)^j}{j\cdot m^{k-1}}\frac{1}{k^j}\right\} \sim \sum_{j=0}^{\infty}\frac{a_{m+j}}{n^j}. \tag{3.23}$$

Equating coefficients of the term n^{-j} on the right-hand sides of (3.22) and (3.23), we obtain

$$a_{m+j} = \sum_{k=1}^{\lfloor\frac{j+2}{2}\rfloor} \lambda_k \mu_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1}, \quad j \in \mathbb{N}. \tag{3.24}$$

Setting $j = 2\ell - 1$ and $j = 2\ell$ in (3.24), respectively, yields

$$a_{m+2\ell-1} = \sum_{k=1}^{\ell} \lambda_k \mu_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} \tag{3.25}$$

and

$$\begin{aligned} a_{m+2\ell} &= -\sum_{k=1}^{\ell+1} \lambda_k \mu_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} \\ &= -\sum_{k=1}^{\ell} \lambda_k \mu_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} - \lambda_{\ell+1} \mu_{\ell+1}^{-1} \binom{2\ell-1}{-1} \\ &= -\sum_{k=1}^{\ell} \lambda_k \mu_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1}. \end{aligned} \tag{3.26}$$

For $\ell = 1$, from (3.25) and (3.26) we obtain

$$\lambda_1 = a_{m+1} = \frac{(-1)^{m+1} 2m^2}{(m+1)(m-1)^2} \quad \text{and} \quad \mu_1 = -\frac{a_{m+2}}{\lambda_1} = \frac{(m+1)(m^2+8m+3)}{4(m+2)(m-1)},$$

and for $\ell \geq 2$ we have

$$a_{m+2\ell-1} = \sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} + \lambda_\ell$$

and

$$a_{m+2\ell} = -\sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} - (2\ell-1)\lambda_\ell \mu_\ell.$$

We then obtain the recurrence relations (3.18) and (3.19). The proof of Theorem 3.3 is complete. □

Here we give explicit numerical values of some first terms of λ_ℓ and μ_ℓ by using formulas (3.18) and (3.19). This shows how easily we can determine the constants a_ℓ and b_ℓ in (3.17).

$$\lambda_1 = a_{m+1} = \frac{(-1)^{m+1} 2m^2}{(m+1)(m-1)^2},$$

$$\mu_1 = -\frac{a_{m+2}}{\lambda_1} = \frac{(m+1)(m^2+8m+3)}{4(m+2)(m-1)},$$

$$\begin{aligned}
\lambda_2 &= a_{m+3} - \lambda_1\mu_1^2 \\
&= \frac{(-1)^{m+1} m^2(m+1)(m^3+12m^2+51m+8)}{6(m-1)^4(m+3)} \\
&\quad - \frac{(-1)^{m+1}2m^2}{(m+1)(m-1)^2}\left(\frac{(m+1)(m^2+8m+3)}{4(m+2)(m-1)}\right)^2 \\
&= (-1)^{m+1}\frac{m^2(m+1)(m^5+7m^4+58m^3+266m^2+485m+47)}{24(m-1)^4(m+3)(m+2)^2},
\end{aligned}$$

$$\begin{aligned}
\mu_2 &= -\frac{a_{m+4}+\lambda_1\mu_1^3}{3\lambda_2} \\
&= -\frac{\frac{(-1)^m m^2(m^6+25m^5+216m^4+866m^3+1241m^2+501m+30)}{24(m-1)^5(m+4)} + \lambda_1\mu_1^3}{3\lambda_2} \\
&= \big((m+3)\big(m^9+34m^8+450m^7+3634m^6+17{,}584m^5+48{,}642m^4+71{,}302m^3 \\
&\quad + 50{,}926m^2+14{,}151m+636\big)\big)/\big(12(m+2)(m+4)\big(m^5+7m^4+58m^3+266m^2 \\
&\quad + 485m+47\big)\big(m^2-1\big)\big).
\end{aligned}$$

Remark 3.4 The constants p_ℓ and q_ℓ in (3.16) are given by

$$p_\ell := \lambda_\ell \quad \text{and} \quad q_\ell := 1 + \mu_\ell.$$

Setting $m = 2, 3$, and 4 in (3.16), respectively, yields (3.13), (3.14), and (3.15).

Noting that $\ln\frac{4}{\pi} = \gamma(-1)$ holds, Theorem 3.4 presents the asymptotic expansion for $\ln\frac{4}{\pi}$.

Theorem 3.4 *As $n \to \infty$, we have*

$$\gamma(-1) - \gamma_n(-1) \sim (-1)^n C(n+1), \tag{3.27}$$

where $C(x)$ is given in (2.14). *Namely,*

$$\begin{aligned}
&\gamma(-1) - \gamma_n(-1) \\
&\quad \sim (-1)^n\left\{\frac{1}{4(n+1)^2} + \frac{1}{12(n+1)^3} - \frac{1}{8(n+1)^4} - \frac{1}{10(n+1)^5} + \cdots\right\}.
\end{aligned} \tag{3.28}$$

Proof Write (2.13) as

$$\frac{1}{k} - \ln\left(1+\frac{1}{k}\right) = C_N(k) + C_N(k+1) + O\left(\frac{1}{k^{N+1}}\right), \tag{3.29}$$

where

$$C_N(x) = \sum_{j=2}^{N}\frac{c_j}{x^j} \tag{3.30}$$

with the coefficients c_j given by the recurrence relation (2.15).

From (3.29), we have

$$(-1)^{k-1}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right) = (-1)^{k-1}C_N(k) - (-1)^k C_N(k+1) + O\left(\frac{(-1)^{k-1}}{k^{N+1}}\right). \tag{3.31}$$

Adding (3.31) from $k = n + 1$ to $k = \infty$, we have

$$\begin{aligned}\gamma(-1) - \gamma_n(-1) &= \sum_{k=n+1}^{\infty} (-1)^{k-1}\left(\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right)\right) \\ &= (-1)^n C_N(n+1) + O\left(\frac{1}{(n+1)^{N+1}}\right),\end{aligned} \tag{3.32}$$

which can be written as (3.27). The proof of Theorem 3.4 is complete. □

Remark 3.5 We see from (3.28) that the alternating Euler constant $\ln \frac{4}{\pi}$ has the following expansion:

$$\begin{aligned}\ln \frac{4}{\pi} &\sim \sum_{k=1}^{n} (-1)^{k-1}\left(\frac{1}{k} - \ln \frac{k+1}{k}\right) \\ &\quad + (-1)^n \left\{\frac{1}{4(n+1)^2} + \frac{1}{12(n+1)^3} - \frac{1}{8(n+1)^4} - \frac{1}{10(n+1)^5} + \cdots\right\}.\end{aligned} \tag{3.33}$$

4 Conclusions

In this paper, we give asymptotic expansions related to the generalized Somos quadratic recurrence constant (Theorems 3.1 and 3.3). We present the inequalities for $\gamma(\frac{1}{4}) - \gamma_n(\frac{1}{4})$ and $\gamma(\frac{1}{3}) - \gamma_n(\frac{1}{3})$ (see (3.8), (3.10), and (3.11)). The expansion of the alternating Euler constant $\ln \frac{4}{\pi}$ is also obtained (see (3.33)).

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