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A two-point-Padé-approximant-based method for bounding some trigonometric functions

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Abstract

Inequalities are frequently used for solving practical engineering problem. There are two key issues of bounding inequalities; one is to find the bounds, and the other is to prove the bounds. This paper takes Wilker type inequalities as an example, presents a two-point-Padé-approximant-based method for finding the bounds, and it also provides a method to prove the bounds in a new way. It not only recovers the estimates in Mortici's method, but it also provides new improvements of estimates obtained from prevailing methods. In principle, it can be applied for other inequalities.

Keywords: Wilker's inequality; Trigonometric approximation; Padé approximant; Two-sided bounds; Becker–Stark's inequality

1 Introduction

The Wilker inequality, which involves the trigonometric function

$$f(x) = \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x}, \tag{1}$$

has been discussed in the recent past; see also [2, 3, 6–9, 11–15, 17–23] and the references therein, such as the following ones in [14, 18]:

$$2 + \frac{16}{\pi^4}x^3 \tan x < f(x) < 2 + \frac{8}{45}x^3 \tan x, \quad 0 < x < \pi/2, \tag{2}$$

$$2 + \left(\frac{8}{45} - a(x)\right)x^3 \tan x < f(x) < 2 + \left(\frac{8}{45} - b(x)\right)x^3 \tan x, \quad 0 < x < 1, \tag{3}$$

$$2 + \left(\frac{16}{\pi^4} + c(x)\right)x^3 \tan x < f(x), \quad (\pi - 1)/2 < x < \pi/2, \tag{4}$$

$$f(x) < 2 + \left(\frac{16}{\pi^4} + d(x)\right)x^3 \tan x, \quad \pi/3 - 1/2 < x < \pi/2, \tag{5}$$

where $a(x) = \frac{8}{945}x^2$, $b(x) = \frac{8}{945}x^2 - \frac{16}{14{,}175}x^4$, $c(x) = (\frac{160}{\pi^5} - \frac{16}{\pi^3})(\frac{\pi}{2} - x)$, $d(x) = (\frac{160}{\pi^5} - \frac{16}{\pi^3})(\frac{\pi}{2} - x) + (\frac{960}{\pi^6} - \frac{96}{\pi^4})(\frac{\pi}{2} - x)^2$.

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Recently, Nenezić, Malešević and Mortici provided inequalities within the extended interval $(0, \pi/2)$ [15], e.g., Eq. (7) extends both Eq. (4) and Eq. (5), while Eq. (6) extends the left side of Eq. (3). We have

$$2 + \left(\frac{8}{45} - a(x)\right)x^3 \tan x < f(x) < 2 + \left(\frac{8}{45} - b_1(x)\right)x^3 \tan x, \quad 0 < x < \pi/2, \tag{6}$$

$$2 + \left(\frac{16}{\pi^4} + c(x)\right)x^3 \tan x < f(x) < 2 + \left(\frac{16}{\pi^4} + d(x)\right)x^3 \tan x, \quad 0 < x < \pi/2, \tag{7}$$

where $b_1(x) = \frac{8}{945}x^2 - \frac{\alpha}{14{,}175}x^4$ with $\alpha = \frac{480\pi^6 - 40{,}320\pi^4 + 3{,}628{,}800}{\pi^8} \approx 17.15041$.

In this paper, we consider

$$F(x) = f(x) \cdot \cos(x) = \left(\frac{\sin x}{x}\right)^2 \cdot \cos(x) + \frac{\sin x}{x} \tag{8}$$

instead of $f(x)$, which is bounded for $x \in (0, \pi/2]$. Firstly, we present a two-point-Padé approximant-based method [1] to find the two bounding functions

$$L(x) = l_1(x) \cdot \cos(x) + l_2(x) \cdot \sin(x), \; R(x) = r_1(x) \cdot \cos(x) + r_2(x) \cdot \sin(x), \tag{9}$$

such that

$$L(x) \le F(x) \le R(x), \quad 0 \le x \le \pi/2, \tag{10}$$

where $l_i(x)$ and $r_i(x)$ are unknown polynomials to be determined. Note that $\cos(x) > 0, \forall x \in (0, \pi/2)$, from Eq. (10), we obtain

$$l_1(x) + l_2(x) \cdot \tan(x) \le f(x) \le r_1(x) + r_2(x) \cdot \tan(x), \quad 0 \le x \le \pi/2. \tag{11}$$

Secondly, we also provide a new way for proving it.

2 The two-point-Padé approximant-based method and examples

Given an interval $[a, b] \subseteq [0, \pi/2]$. From Eq. (9), let

$$l_i(x) = \sum_{j=0}^{p_i} \alpha_{i,j} x^j \quad \text{and} \quad r_i(x) = \sum_{j=0}^{q_i} \beta_{i,j} x^j, \tag{12}$$

where $p_i, q_i \ge 2$, $\alpha_{i,j}$ and $\beta_{i,j}$ are the unknowns to be determined, and $i = 1, 2$; so there are $n_p = p_1 + p_2 + 2$ and $n_q = q_1 + q_2 + 2$ unknowns in $L(x)$ and $R(x)$ in Eq. (9), respectively. Let $E_1(x) = F(x) - L(x)$ and $E_2(x) = F(x) - R(x)$. For the sake of convenience, we introduce Theorem 3.5.1 in Page 67, Chap. 3.5 of [4] as follows.

Theorem 1 *Let $w_0, w_1, \ldots, w_r$ be $r+1$ distinct points in $[a, b]$, and $n_0, \ldots, n_r$ be $r+1$ integers ≥ 0. Let $N = n_0 + \cdots + n_r + r$. Suppose that $g(t)$ is a polynomial of degree N such that $g^{(i)}(w_j) = f^{(i)}(w_j)$, $i = 0, \ldots, n_j$, $j = 0, \ldots, r$. Then there exists $\xi_1(t) \in [a, b]$ such that $f(t) - g(t) = \frac{f^{(N+1)}(\xi_1(t))}{(N+1)!} \prod_{i=0}^{r} (t - w_i)^{n_i+1}$.*

We introduce the following constraints:

$$\begin{cases} E_1^{(i)}(a) = 0, & E_1^{(j)}(b) = 0,\ i = 0,1,\ldots,k, \quad \text{and } j = 0,1,\ldots,N_1, \\ E_2^{(i)}(a) = 0, & E_2^{(j)}(b) = 0,\ i = 0,1,\ldots,l, \quad \text{and } j = 0,1,\ldots,N_2, \end{cases} \tag{13}$$

where $N_1 \geq n_p - k - 1$ and $N_2 \geq n_q - l - 1$. By selecting suitable k and N_1, we can find n_p constraints for determining $L(x)$; similarly, by selecting suitable l and N_2, we can find n_q constraints for determining $R(x)$. Combining Theorem 1 with Eq. (13), there exists $\xi_i(x) \in [a,b], i = 1,2$, such that

$$\begin{cases} E_1(x) = \frac{E_1^{(N_1+k+2)}(\xi_1(x))}{(N_1+k+2)!}(x-a)^{k+1}(x-b)^{N_1+1}, & x \in [a,b], \\ E_2(x) = \frac{E_2^{(N_2+l+2)}(\xi_2(x))}{(N_2+l+2)!}(x-a)^{l+1}(x-b)^{N_2+1}, & x \in [a,b]. \end{cases} \tag{14}$$

From Eq. (14), if $(-1)^d \cdot E_1^{(N_1+k+2)}(\xi_1(x)) \geq 0$, $\forall x \in [a,b]$, we have $E_1(x) \cdot (-1)^{N_1+1+d} \geq 0$, where $d = 0$ or $d = 1$; similarly, if $(-1)^d \cdot E_2^{(N_2+l+2)}(\xi_2(x)) \geq 0$, $\forall x \in [a,b]$, we have $E_2(x) \cdot (-1)^{N_2+1+d} \geq 0$. Based on the above observations, one may find the bounding functions in the above way.

We will show three examples which recover or refine previous Wilker type inequalities, including Eq. (2), Eq. (6) and Eq. (7), where c_j is a unknown coefficient to be determined by interpolation constraints.

Example 1 Let $L_1(x) = 2\cos(x) + c_1\sin(x)$ and $R_1(x) = 2\cos(x) + c_2\sin(x)$, $E_{1,l}(x) = F(x) - L_1(x)$ and $E_{1,r}(x) = F(x) - R_1(x)$, $x \in [0, \pi/2]$. It can be verified that $E_{1,i}^{(j)}(0) = 0$, where $j = 0,1,2,3$, $i = l,r$. By applying the constraints $L_1(\pi/2) = F(\pi/2)$ and $R_1^{(4)}(0) = F^{(4)}(0)$, we obtain $c_1 = \frac{16}{\pi^4}$ and $c_2 = \frac{8}{45}$, respectively, which recovers Eq. (2).

Example 2 Let $L_2(x) = 2\cos(x) + (c_3 + c_4x + c_5x^2)x^3\sin(x)$ and $R_2(x) = 2\cos(x) + (c_6 + c_7x^2 + c_8x^4)x^3\sin(x)$, $E_{2,l}(x) = F(x) - L_2(x)$ and $E_{2,r}(x) = F(x) - R_2(x)$, $x \in [0,\pi/2]$. It can be verified that $E_{2,i}^{(j)}(0) = 0$, where $j = 0,1,2,3$, $i = l,r$. By applying the constraints $L_2^{(j)}(0) = F^{(j)}(0)$, $j = 4,5,6$, we obtain $c_3 = \frac{8}{45}$, $c_4 = 0$ and $c_5 = -\frac{8}{945}$, which recovers the left side of Eq. (6). By applying the constraints $R_2^{(4)}(0) = F^{(4)}(0)$, $R_2^{(5)}(0) = F^{(5)}(0)$ and $R_2(\pi/2) = F(\pi/2)$, we obtain $c_6 = \frac{8}{45}$, $c_7 = -\frac{8}{945}$ and $c_8 = \frac{\alpha}{14{,}175}$, which recovers the right side of Eq. (6).

Example 3 Let $L_3(x) = 2\cos(x) + (c_9 + c_{10}(\pi/2 - x))x^3\sin(x)$, $R_3(x) = 2\cos(x) + (c_{11} + c_{12}(\pi/2 - x) + c_{13}(x - \pi/2)^2)x^3\sin(x)$, $E_{3,l}(x) = F(x) - L_3(x)$ and $E_{3,r}(x) = F(x) - R_3(x)$, $x \in [0,\pi/2]$. It can be verified that $E_{3,i}^{(j)}(0) = 0$, where $j = 0,1,2,3$, $i = l,r$. By applying the constraints $L_3(\pi/2) = F(\pi/2)$ and $L_3'(\pi/2) = F'(\pi/2)$, we obtain $c_9 = \frac{16}{\pi^4}$ and $c_{10} = \frac{160}{\pi^5} - \frac{16}{\pi^3}$, which recovers the left side of Eq. (7). By applying the constraints $R_3^{(j)}(\pi/2) = F^{(j)}(\pi/2)$, $j = 0,1,2$, we obtain $c_{11} = \frac{16}{\pi^4}$, $c_{12} = \frac{160}{\pi^5} - \frac{16}{\pi^3}$ and $c_{13} = \frac{960}{\pi^6} - \frac{96}{\pi^4}$, which recovers the right side of Eq. (7).

3 Results

This section finds other two bounding functions $L(x)$ and $R(x)$ to improve the bounds of Eq. (6) and Eq. (7). Combining Eq. (12) with Eq. (13), by setting $p_1 = q_1 = 4$, $p_2 = q_2 = 5$,

$k = 8$, $N_1 = 1$, $l = 7$ and $N_2 = 2$, we obtain $L(x)$ and $R(x)$ in Eq. (10) as

$$L(x) = l_1(x) \cdot \cos(x) + l_2(x) \cdot \sin(x) = \left(\sum_{j=0}^{4} \alpha_{1,j} x^j\right) \cdot \cos(x) + \left(\sum_{j=0}^{5} \alpha_{2,j} x^j\right) \cdot \sin(x),$$

$$R(x) = r_1(x) \cdot \cos(x) + r_2(x) \cdot \sin(x) = \left(\sum_{j=0}^{4} \beta_{1,j} x^j\right) \cdot \cos(x) + \left(\sum_{j=0}^{5} \beta_{2,j} x^j\right) \cdot \sin(x),$$

where

$$\lambda_1 = \frac{16(2\pi^{10} - 177\pi^8 + 4935\pi^6 - 85{,}050\pi^4 + 831{,}600\pi^2 - 3{,}175{,}200)}{(\pi^8 - 360\pi^6 + 35{,}760\pi^4 - 604{,}800\pi^2 + 2{,}822{,}400)\pi},$$

$$\alpha_{1,1} = \frac{\lambda_1}{3}, \qquad \alpha_{1,3} = \frac{2\lambda_1}{-63}, \qquad \alpha_{2,2} = \frac{\lambda_1}{7},$$

$$\lambda_2 = \frac{8(3\pi^{10} - 308\pi^8 + 9300\pi^6 - 132{,}720\pi^4 + 957{,}600\pi^2 - 2{,}822{,}400)}{(\pi^8 - 360\pi^6 + 35{,}760\pi^4 - 604{,}800\pi^2 + 2{,}822{,}400)\pi^2},$$

$$\alpha_{1,2} = -\lambda_2, \qquad \alpha_{2,1} = \lambda_2,$$

$$\lambda_3 = \frac{(11\pi^{10} - 1065\pi^8 + 25{,}935\pi^6 - 346{,}500\pi^4 + 2{,}885{,}400\pi^2 - 10{,}584{,}000)}{(\pi^8 - 360\pi^6 + 35{,}760\pi^4 - 604{,}800\pi^2 + 2{,}822{,}400)\pi^2},$$

$$\alpha_{1,4} = \frac{64\lambda_3}{315}, \qquad \alpha_{2,3} = \frac{32\lambda_3}{-35}, \qquad \alpha_{1,0} = 2, \qquad \alpha_{2,0} = -\alpha_{1,1}, \qquad \alpha_{2,4} = \frac{16\lambda_1}{-315},$$

$$\alpha_{2,5} = \frac{32(2\pi^{10} - 141\pi^8 - 1965\pi^6 + 51{,}660\pi^4 + 12{,}600\pi^2 - 2{,}116{,}800)}{315(\pi^8 - 360\pi^6 + 35{,}760\pi^4 - 604{,}800\pi^2 + 2{,}822{,}400)\pi^2};$$

$$\lambda_4 = \frac{16(7\pi^{10} - 90\pi^8 - 2445\pi^6 + 94{,}500\pi^4 - 1{,}134{,}000\pi^2 + 4{,}536{,}000)}{(5\pi^{10} - 558\pi^8 + 12{,}480\pi^6 - 177{,}120\pi^4 + 1{,}756{,}800\pi^2 - 7{,}257{,}600)\pi},$$

$$\beta_{1,1} = \lambda_4, \qquad \beta_{2,0} = -\lambda_4, \qquad \beta_{1,3} = \frac{2\lambda_4}{-21}, \qquad \beta_{1,0} = 2,$$

$$\beta_{1,2} = \frac{40(\pi^{12} - 234\pi^{10} + 6180\pi^8 - 8568\pi^6 - 1{,}572{,}480\pi^4 + 20{,}260{,}800\pi^2 - 76{,}204{,}800)}{21(5\pi^{10} - 558\pi^8 + 12{,}480\pi^6 - 177{,}120\pi^4 + 1{,}756{,}800\pi^2 - 7{,}257{,}600)\pi^2},$$

$$\beta_{2,1} = -\beta_{1,2},$$

$$\beta_{1,4} = \frac{32(12\pi^{12} + 4615\pi^{10} - 188{,}175\pi^8 + 2{,}650{,}200\pi^6 - 11{,}692{,}800\pi^4 - 45{,}360{,}000\pi^2 + 381{,}024{,}000)}{105(5\pi^{10} - 558\pi^8 + 12{,}480\pi^6 - 177{,}120\pi^4 + 1{,}756{,}800\pi^2 - 7{,}257{,}600)\pi^4},$$

$$\beta_{2,2} = \frac{3\lambda_4}{7},$$

$$\beta_{2,3} = \frac{32(\pi^{14} - 165\pi^{12} + 3108\pi^{10} + 13{,}401\pi^8 - 980{,}280\pi^6 + 9{,}933{,}840\pi^4 - 22{,}680{,}000\pi^2 - 76{,}204{,}800)}{21(5\pi^{10} - 558\pi^8 + 12{,}480\pi^6 - 177{,}120\pi^4 + 1{,}756{,}800\pi^2 - 7{,}257{,}600)\pi^4},$$

$$\beta_{2,4} = \frac{16\lambda_4}{-105},$$

$$\beta_{2,5} = \frac{32(13\pi^{12} - 2050\pi^{10} + 58{,}995\pi^8 - 616{,}200\pi^6 + 882{,}000\pi^4 + 25{,}704{,}000\pi^2 - 127{,}008{,}000)}{-105(5\pi^{10} - 558\pi^8 + 12{,}480\pi^6 - 177{,}120\pi^4 + 1{,}756{,}800\pi^2 - 7{,}257{,}600)\pi^4}.$$

In principle, more bounds can be found by setting different parameters in Eq. (12) and Eq. (13). The main result is as follows.

Theorem 2 *We have* $L(x) \le F(x) \le R(x)$, $\forall x \in [0, \pi/2]$.

Proof (1) Firstly, we give the bounds of $\sin(x)$, $\cos(x)$ and $\sin(2x)$. Let $\Delta_{1,1}(x) = \sin(x) - P_1(x)$, $\Delta_{1,2}(x) = \sin(x) - Q_1(x)$, $\Delta_{2,1}(x) = \cos(x) - P_2(x)$, $\Delta_{2,2}(x) = \cos(x) - Q_2(x)$, $\Delta_{3,1}(x) = \sin(2x)/2 - P_3(x)$, $\Delta_{3,2}(x) = \sin(2x)/2 - Q_3(x)$, where $P_1(x)$, $Q_1(x)$, $P_2(x)$, $Q_2(x)$, $P_3(x)$ and $Q_3(x)$ are polynomials of degree 12, 12, 13, 13, 15 and 15, respectively. By introducing the following constraints:

$$\begin{aligned}
&\Delta_{1,1}^{(i)}(0)=0, &&\Delta_{1,1}^{(j)}(\pi/2)=0, &&i=0,1,\dots,10, j=0,1;\\
&\Delta_{1,2}^{(i)}(0)=0, &&\Delta_{1,2}^{(j)}(\pi/2)=0, &&i=0,1,\dots,9, j=0,1,2;\\
&\Delta_{2,1}^{(i)}(0)=0, &&\Delta_{2,1}^{(j)}(\pi/2)=0, &&i=0,1,\dots,10, j=0,1,2;\\
&\Delta_{2,2}^{(i)}(0)=0, &&\Delta_{2,2}^{(j)}(\pi/2)=0, &&i=0,1,\dots,11, j=0,1;\\
&\Delta_{3,1}^{(i)}(0)=0, &&\Delta_{3,1}^{(j)}(\pi/2)=0, &&i=0,1,\dots,13, j=0,1;\\
&\Delta_{3,2}^{(i)}(0)=0, &&\Delta_{3,2}^{(j)}(\pi/2)=0, &&i=0,1,\dots,12, j=0,1,2;
\end{aligned} \tag{15}$$

we can obtain $P_1(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362{,}880}x^9 + \frac{\gamma_{1,1}}{30{,}240\pi^{11}}x^{11} + \frac{\gamma_{1,2}}{22{,}680\pi^{12}}x^{12}$, $Q_1(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362{,}880}x^9 - \frac{\gamma_{1,3}}{60{,}480\pi^{10}}x^{10} + \frac{\gamma_{1,4}}{30{,}240\pi^{11}}x^{11} - \frac{\gamma_{1,5}}{45{,}360\pi^{12}}x^{12}$, $P_2(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40{,}320}x^8 - \frac{1}{3{,}628{,}800}x^{10} + \frac{\gamma_{2,1}}{604{,}800\pi^{11}}x^{11} - \frac{\gamma_{2,2}}{302{,}400\pi^{12}}x^{12} + \frac{\gamma_{2,3}}{453{,}600\pi^{13}}x^{13}$, $Q_2(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40{,}320}x^8 - \frac{1}{3{,}628{,}800}x^{10} + \frac{\gamma_{2,4}}{302{,}400\pi^{12}}x^{12} - \frac{\gamma_{2,5}}{226{,}800\pi^{13}}x^{13}$, $P_3(x) = x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \frac{4}{315}x^7 + \frac{2}{2835}x^9 - \frac{4}{155{,}925}x^{11} + \frac{4}{6{,}081{,}075}x^{13} - \frac{\gamma_{3,1}}{6{,}081{,}075\pi^{13}}x^{14} + \frac{\gamma_{3,2}}{6{,}081{,}075\pi^{14}}x^{15}$, $Q_3(x) = x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \frac{4}{315}x^7 + \frac{2}{2835}x^9 - \frac{4}{155{,}925}x^{11} + \frac{\gamma_{3,3}}{51{,}975\pi^{12}}x^{13} - \frac{\gamma_{3,4}}{155{,}925\pi^{13}}x^{14} + \frac{\gamma_{3,5}}{155{,}925\pi^{14}}x^{15}$, where $\gamma_{1,1} = -743{,}178{,}240 + 340{,}623{,}360\pi - 11{,}612{,}160\pi^3 + 112{,}896\pi^5 - 480\pi^7 + \pi^9$, $\gamma_{1,2} = -1{,}021{,}870{,}080 + 464{,}486{,}400\pi - 15{,}482{,}880\pi^3 + 145{,}152\pi^5 - 576\pi^7 + \pi^9$, $\gamma_{1,3} = \pi^9 - 960\pi^7 + 338{,}688\pi^5 - 46{,}448{,}640\pi^3 + 7{,}741{,}440\pi^2 + 1{,}703{,}116{,}800\pi - 4{,}087{,}480{,}320$, $\gamma_{1,4} = \pi^9 - 1440\pi^7 + 564{,}480\pi^5 - 81{,}285{,}120\pi^3 + 15{,}482{,}880\pi^2 + 3{,}065{,}610{,}240\pi - 7{,}431{,}782{,}400$, $\gamma_{1,5} = \pi^9 - 1728\pi^7 + 725{,}760\pi^5 - 108{,}380{,}160\pi^3 + 23{,}224{,}320\pi^2 + 4{,}180{,}377{,}600\pi - 10{,}218{,}700{,}800$, $\gamma_{2,1} = \pi^{10} - 1200\pi^8 + 564{,}480\pi^6 - 116{,}121{,}600\pi^4 + 8{,}515{,}584{,}000\pi^2 + 7{,}431{,}782{,}400\pi - 96{,}613{,}171{,}200$, $\gamma_{2,2} = \pi^{10} - 1800\pi^8 + 940{,}800\pi^6 - 203{,}212{,}800\pi^4 + 15{,}328{,}051{,}200\pi^2 + 14{,}244{,}249{,}600\pi - 177{,}124{,}147{,}200$, $\gamma_{2,3} = \pi^{10} - 2160\pi^8 + 1{,}209{,}600\pi^6 - 270{,}950{,}400\pi^4 + 20{,}901{,}888{,}000\pi^2 + 20{,}437{,}401{,}600\pi - 245{,}248{,}819{,}200$, $\gamma_{2,4} = \pi^1 0 - 600\pi^8 + 188{,}160\pi^6 - 29{,}030{,}400\pi^4 + 1{,}703{,}116{,}800\pi^2 + 619{,}315{,}200\pi - 16{,}102{,}195{,}200$, $\gamma_{2,5} = \pi^1 0 - 720\pi^8 + 241{,}920\pi^6 - 38{,}707{,}200\pi^4 + 2{,}322{,}432{,}000\pi^2 + 928{,}972{,}800\pi - 22{,}295{,}347{,}200$, $\gamma_{3,1} = 16(\pi^{12} - 312\pi^{10} + 51{,}480\pi^8 - 4{,}942{,}080\pi^6 + 259{,}459{,}200\pi^4 - 6{,}227{,}020{,}800\pi^2 + 40{,}475{,}635{,}200)$, $\gamma_{3,2} = 16(\pi^{12} - 468\pi^{10} + 85{,}800\pi^8 - 8{,}648{,}640\pi^6 + 467{,}026{,}560\pi^4 - 11{,}416{,}204{,}800\pi^2 + 74{,}724{,}249{,}600)$, $\gamma_{3,3} = 32(\pi^{10} - 275\pi^8 + 36{,}960\pi^6 - 2{,}494{,}800\pi^4 + 73{,}180{,}800\pi^2 - 512{,}265{,}600)$, $\gamma_{3,4} = 256(\pi^{10} - 330\pi^8 + 47{,}520\pi^6 - 3{,}326{,}400\pi^4 + 99{,}792{,}000\pi^2 - 703{,}533{,}600)$, $\gamma_{3,5} = 64(3\pi^{10} - 1100\pi^8 + 166{,}320\pi^6 - 11{,}975{,}040\pi^4 + 365{,}904{,}000\pi^2 - 2{,}594{,}592{,}000)$.

Combining Theorem 1 with Eq. (15), there exists $\eta_i(x) \in [0, \pi/2]$, $i = 1, 2, \dots, 6$, such that

$$\Delta_{1,1}(x) = \frac{\Delta_{1,1}^{(13)}(\eta_1(x))}{13!}x^{11}(x-\pi/2)^2 = \frac{\cos(\eta_1(x))}{13!}x^{11}(x-\pi/2)^2 \geq 0, \quad \forall x \in [0, \pi/2],$$

$$\Delta_{1,2}(x) = \frac{\Delta_{1,2}^{(13)}(\eta_2(x))}{13!}x^{10}(x-\pi/2)^3 = \frac{\cos(\eta_2(x))}{13!}x^{10}(x-\pi/2)^3 \leq 0, \quad \forall x \in [0, \pi/2],$$

$$\Delta_{2,1}(x) = \frac{\Delta_{2,1}^{(14)}(\eta_3(x))}{14!}x^{11}(x-\pi/2)^3 = \frac{-\cos(\eta_3(x))}{14!}x^{11}(x-\pi/2)^3 \geq 0, \quad \forall x \in [0, \pi/2],$$

$$\Delta_{2,2}(x) = \frac{\Delta_{2,2}^{(13)}(\eta_4(x))}{13!}x^{11}(x-\pi/2)^2 = \frac{-\sin(\eta_4(x))}{13!}x^{11}(x-\pi/2)^2 \leq 0, \quad \forall x \in [0, \pi/2],$$

$$\begin{aligned}
\Delta_{3,1}(x) &= \frac{\Delta_{3,1}^{(16)}(2\eta_5(x))}{16!}x^{14}(x-\pi/2)^2 \\
&= \frac{2^{15}\sin(2\eta_5(x))}{16!}x^{14}(x-\pi/2)^2 \geq 0, \quad \forall x\in[0,\pi/2], \\
\Delta_{3,2}(x) &= \frac{\Delta_{3,2}^{(16)}(2\eta_6(x))}{16!}x^{13}(x-\pi/2)^3 \\
&= \frac{2^{15}\sin(2\eta_6(x))}{16!}x^{13}(x-\pi/2)^3 \leq 0, \quad \forall x\in[0,\pi/2].
\end{aligned}$$

So for $\forall x\in[0,\pi/2]$, we have

$$\Delta_{i,1}(x)\geq 0 \quad \text{and} \quad \Delta_{i,2}(x)\leq 0, \quad i=1,2,3, \tag{16}$$

i.e., $Q_1(x)\geq\sin(x)\geq P_1(x)$, $Q_2(x)\geq\cos(x)\geq P_2(x)$ and $Q_3(x)\geq\frac{\sin(2x)}{2}\geq P_3(x)$.

(2) Secondly, we prove that $\Delta_4(x)=(F(x)-L(x))\cdot x^2\geq 0$, $\forall x\in[0,\pi/2]$, which means that $F(x)\geq L(x)$.

Note that $l_i(x)$ and $r_i(x)$ are polynomials of degree $3+i$, $i=1,2$, polynomials $P_1(x)$, $Q_1(x)$, $P_2(x)$, $Q_2(x)$, $P_3(x)$ and $Q_3(x)$ are of degree 12, 12, 13, 13, 15 and 15, respectively, by using Maple software, $\forall x\in(0,\pi/2)$, we obtain

$$\begin{aligned}
&P_i(x)>0 \quad \text{and} \quad Q_i(x)>0, \quad i=1,2,3, \\
&l_1(x)\cdot x^2>0 \quad \text{and} \quad x-l_2(x)\cdot x^2>0, \\
&r_1(x)\cdot x^2>0 \quad \text{and} \quad x-r_2(x)\cdot x^2>0.
\end{aligned} \tag{17}$$

Combining Eq. (17) with Eq. (16), we have

$$\begin{aligned}
\Delta_4(x) &= \sin(x)^2\cos(x)-l_1(x)x^2\cos(x)+\big(x-l_2(x)x^2\big)\sin(x) \\
&\geq P_3(x)P_1(x)-l_1(x)x^2Q_2(x)+\big(x-l_2(x)x^2\big)P_1(x) \\
&= \frac{(\pi-2x)^2x^{11}}{2{,}206{,}700{,}496{,}000(\pi^4-180\pi^2+1680)^2\pi^{26}}H_1(x),
\end{aligned} \tag{18}$$

where

$$H_1(x)=\sum_{i=0}^{14}\rho_{1,i}x^i,$$

and

$$\begin{aligned}
\rho_{1,0} &= 118{,}609{,}920\big(2\pi^{10}-177\pi^8+4935\pi^6-85{,}050\pi^4+831{,}600\pi^2-3{,}175{,}200\big)\pi^{23} \\
&\quad >0, \\
\rho_{1,1} &= 5265\big(40{,}981\pi^{19}+8{,}062{,}512\pi^{17}-1{,}200{,}402{,}000\pi^{15}+10{,}812{,}049{,}920\pi^{13} \\
&\quad +1{,}876{,}776{,}249{,}600\pi^{11}-245{,}548{,}461{,}312{,}000\pi^9+20{,}600{,}900{,}812{,}800\pi^8 \\
&\quad +16{,}840{,}163{,}450{,}880{,}000\pi^7-7{,}416{,}324{,}292{,}608{,}000\pi^6 \\
&\quad -541{,}159{,}913{,}226{,}240{,}000\pi^5+736{,}688{,}213{,}065{,}728{,}000\pi^4 \\
&\quad +6{,}619{,}069{,}431{,}152{,}640{,}000\pi^3-12{,}459{,}424{,}811{,}581{,}440{,}000\pi^2 \\
&\quad -26{,}649{,}325{,}291{,}438{,}080{,}000\pi+58{,}143{,}982{,}454{,}046{,}720{,}000\big)\pi^{13}>0,
\end{aligned}$$

$$\begin{aligned}
\rho_{1,2} = {} & -21{,}060\bigl(484\pi^{21} - 1799\pi^{19} - 31{,}876{,}698\pi^{17} + 7{,}133{,}539{,}980\pi^{15} \\
& - 859{,}925{,}324{,}160\pi^{13} + 60{,}601{,}122{,}187{,}200\pi^{11} - 27{,}467{,}867{,}750{,}400\pi^{10} \\
& - 2{,}219{,}580{,}715{,}968{,}000\pi^{9} + 2{,}419{,}747{,}474{,}636{,}800\pi^{8} \\
& + 41{,}725{,}676{,}095{,}488{,}000\pi^{7} - 63{,}759{,}788{,}015{,}616{,}000\pi^{6} \\
& - 423{,}071{,}687{,}098{,}368{,}000\pi^{5} + 769{,}031{,}627{,}341{,}824{,}000\pi^{4} \\
& + 2{,}296{,}485{,}418{,}106{,}880{,}000\pi^{3} - 4{,}672{,}284{,}304{,}343{,}040{,}000\pi^{2} \\
& - 5{,}450{,}998{,}355{,}066{,}880{,}000\pi + 12{,}113{,}329{,}677{,}926{,}400{,}000\bigr)\pi^{12} < 0, \\
\rho_{1,3} = {} & 810\bigl(3287\pi^{21} - 9{,}411{,}072\pi^{19} + 1{,}953{,}992{,}280\pi^{17} - 104{,}047{,}433{,}280\pi^{15} \\
& - 4{,}486{,}871{,}592{,}000\pi^{13} + 792{,}548{,}506{,}713{,}600\pi^{11} - 115{,}307{,}819{,}827{,}200\pi^{10} \\
& - 41{,}557{,}678{,}312{,}550{,}400\pi^{9} + 48{,}741{,}731{,}323{,}084{,}800\pi^{8} \\
& + 730{,}819{,}102{,}261{,}248{,}000\pi^{7} - 3{,}383{,}354{,}610{,}155{,}520{,}000\pi^{6} \\
& - 538{,}971{,}067{,}514{,}880{,}000\pi^{5} + 61{,}377{,}087{,}827{,}607{,}552{,}000\pi^{4} \\
& - 77{,}611{,}833{,}722{,}142{,}720{,}000\pi^{3} - 404{,}931{,}306{,}376{,}396{,}800{,}000\pi^{2} \\
& + 440{,}925{,}200{,}276{,}520{,}960{,}000\pi + 818{,}861{,}086{,}227{,}824{,}640{,}000\bigr)\pi^{11} < 0, \\
\rho_{1,4} = {} & 42{,}120 \times \bigl(7155\pi^{21} - 3{,}921{,}584\pi^{19} + 897{,}324{,}984\pi^{17} - 94{,}307{,}498{,}880\pi^{15} \\
& + 3{,}633{,}527{,}540{,}160\pi^{13} - 7{,}030{,}466{,}150{,}400\pi^{12} + 28{,}926{,}516{,}326{,}400\pi^{11} \\
& + 617{,}046{,}029{,}107{,}200\pi^{10} - 5{,}729{,}151{,}646{,}310{,}400\pi^{9} \\
& - 15{,}296{,}168{,}853{,}504{,}000\pi^{8} + 163{,}845{,}831{,}131{,}136{,}000\pi^{7} \\
& + 158{,}438{,}094{,}667{,}776{,}000\pi^{6} - 2{,}245{,}772{,}867{,}272{,}704{,}000\pi^{5} \\
& - 381{,}528{,}683{,}053{,}056{,}000\pi^{4} + 15{,}718{,}487{,}320{,}166{,}400{,}000\pi^{3} \\
& - 5{,}595{,}204{,}660{,}756{,}480{,}000\pi^{2} - 44{,}819{,}319{,}808{,}327{,}680{,}000\pi \\
& + 33{,}917{,}323{,}098{,}193{,}920{,}000\bigr)\pi^{10} > 0, \\
\rho_{1,5} = {} & -324\bigl(3013\pi^{23} - 1{,}983{,}240\pi^{21} + 462{,}480{,}560\pi^{19} - 40{,}847{,}734{,}080\pi^{17} \\
& - 668{,}523{,}878{,}400\pi^{15} + 303{,}913{,}241{,}049{,}600\pi^{13} - 85{,}019{,}590{,}656{,}000\pi^{12} \\
& - 14{,}955{,}232{,}900{,}608{,}000\pi^{11} + 33{,}853{,}738{,}254{,}336{,}000\pi^{10} \\
& + 452{,}685{,}482{,}016{,}768{,}000\pi^{9} - 1{,}559{,}110{,}508{,}347{,}392{,}000\pi^{8} \\
& - 12{,}135{,}218{,}135{,}040{,}000{,}000\pi^{7} + 37{,}422{,}223{,}023{,}144{,}960{,}000\pi^{6} \\
& + 196{,}234{,}567{,}389{,}020{,}160{,}000\pi^{5} - 506{,}271{,}257{,}654{,}722{,}560{,}000\pi^{4} \\
& - 1{,}522{,}241{,}762{,}859{,}417{,}600{,}000\pi^{3} + 3{,}494{,}407{,}199{,}470{,}387{,}200{,}000\pi^{2} \\
& + 4{,}409{,}252{,}002{,}765{,}209{,}600{,}000\pi - 9{,}448{,}397{,}148{,}782{,}592{,}000{,}000\bigr)\pi^{9} > 0,
\end{aligned}$$

$$
\begin{aligned}
\rho_{1,6} = {} & -5616\bigl(523\pi^{23} - 103{,}800\pi^{21} - 73{,}486{,}320\pi^{19} + 32{,}685{,}822{,}720\pi^{17} \\
& - 4{,}913{,}000{,}467{,}200\pi^{15} - 1{,}798{,}491{,}340{,}800\pi^{14} + 336{,}334{,}858{,}675{,}200\pi^{13} \\
& + 147{,}670{,}445{,}260{,}800\pi^{12} - 11{,}847{,}491{,}453{,}952{,}000\pi^{11} \\
& - 167{,}995{,}441{,}152{,}000\pi^{10} + 267{,}060{,}636{,}057{,}600{,}000\pi^{9} \\
& - 118{,}249{,}170{,}665{,}472{,}000\pi^{8} - 4{,}235{,}673{,}962{,}741{,}760{,}000\pi^{7} \\
& + 3{,}896{,}145{,}366{,}220{,}800{,}000\pi^{6} + 43{,}348{,}415{,}490{,}293{,}760{,}000\pi^{5} \\
& - 59{,}726{,}131{,}636{,}469{,}760{,}000\pi^{4} - 242{,}050{,}284{,}099{,}993{,}600{,}000\pi^{3} \\
& + 430{,}888{,}441{,}400{,}524{,}800{,}000\pi^{2} + 545{,}099{,}835{,}506{,}688{,}000{,}000\pi \\
& - 1{,}162{,}879{,}649{,}080{,}934{,}400{,}000\bigr)\pi^{8} < 0, \\
\rho_{1,7} = {} & 6\bigl(4603\pi^{17} - 1{,}561{,}248\pi^{15} + 172{,}972{,}800\pi^{13} - 1{,}793{,}381{,}990{,}400\pi^{9} \\
& + 144{,}666{,}147{,}225{,}600\pi^{7} - 114{,}776{,}447{,}385{,}600\pi^{6} \\
& - 4{,}787{,}134{,}326{,}374{,}400\pi^{5} + 5{,}624{,}045{,}921{,}894{,}400\pi^{4} \\
& + 74{,}317{,}749{,}682{,}176{,}000\pi^{3} - 128{,}549{,}621{,}071{,}872{,}000\pi^{2} \\
& - 385{,}648{,}863{,}215{,}616{,}000\pi + 819{,}503{,}834{,}333{,}184{,}000\bigr) \\
& \times \bigl(\pi^{4} - 180\pi^{2} + 1680\bigr)^{2}\pi^{7} < 0, \\
\rho_{1,8} = {} & 312\bigl(199\pi^{17} - 31{,}680\pi^{15} + 766{,}402{,}560\pi^{11} - 116{,}876{,}390{,}400\pi^{9} \\
& + 7{,}035{,}575{,}500{,}800\pi^{7} - 4{,}782{,}351{,}974{,}400\pi^{6} - 204{,}560{,}507{,}289{,}600\pi^{5} \\
& + 231{,}760{,}134{,}144{,}000\pi^{4} + 3{,}051{,}508{,}432{,}896{,}000\pi^{3} \\
& - 5{,}253{,}229{,}707{,}264{,}000\pi^{2} - 15{,}759{,}689{,}121{,}792{,}000\pi \\
& + 33{,}373{,}459{,}316{,}736{,}000\bigr)\bigl(\pi^{4} - 180\pi^{2} + 1680\bigr)^{2}\pi^{6} > 0, \\
\rho_{1,9} = {} & -12\bigl(37\pi^{19} - 11{,}232\pi^{17} + 484{,}323{,}840\pi^{13} - 94{,}650{,}716{,}160\pi^{11} \\
& + 8{,}146{,}603{,}745{,}280\pi^{9} - 3{,}188{,}234{,}649{,}600\pi^{8} - 396{,}337{,}419{,}878{,}400\pi^{7} \\
& + 267{,}811{,}710{,}566{,}400\pi^{6} + 11{,}398{,}735{,}930{,}982{,}400\pi^{5} \\
& - 12{,}854{,}962{,}107{,}187{,}200\pi^{4} - 168{,}721{,}377{,}656{,}832{,}000\pi^{3} \\
& + 289{,}236{,}647{,}411{,}712{,}000\pi^{2} + 867{,}709{,}942{,}235{,}136{,}000\pi \\
& - 1{,}831{,}832{,}100{,}274{,}176{,}000\bigr)\bigl(\pi^{4} - 180\pi^{2} + 1680\bigr)^{2}\pi^{5} > 0, \\
\rho_{1,10} = {} & -624\bigl(\pi^{19} - 95{,}040\pi^{15} + 30{,}412{,}800\pi^{13} - 4{,}523{,}904{,}000\pi^{11} \\
& + 353{,}311{,}580{,}160\pi^{9} - 132{,}843{,}110{,}400\pi^{8} - 16{,}491{,}067{,}084{,}800\pi^{7} \\
& + 11{,}036{,}196{,}864{,}000\pi^{6} + 467{,}704{,}826{,}265{,}600\pi^{5} - 525{,}322{,}970{,}726{,}400\pi^{4} \\
& - 6{,}875{,}550{,}646{,}272{,}000\pi^{3} + 11{,}742{,}513{,}463{,}296{,}000\pi^{2} \\
& + 35{,}227{,}540{,}389{,}888{,}000\pi - 74{,}163{,}242{,}926{,}080{,}000\bigr) \\
& \times \bigl(\pi^{4} - 180\pi^{2} + 1680\bigr)^{2}\pi^{4} < 0,
\end{aligned}
$$

$$\begin{aligned}
\rho_{1,11} = 4\bigl(&\pi^{21} - 224{,}640\pi^{17} + 87{,}429{,}888\pi^{15} - 16{,}109{,}383{,}680\pi^{13} \\
&+ 1{,}712{,}015{,}585{,}280\pi^{11} - 347{,}807{,}416{,}320\pi^{10} - 119{,}419{,}314{,}094{,}080\pi^{9} \\
&+ 44{,}635{,}285{,}094{,}400\pi^{8} + 5{,}512{,}856{,}238{,}489{,}600\pi^{7} \\
&- 3{,}672{,}846{,}316{,}339{,}200\pi^{6} - 155{,}062{,}980{,}417{,}945{,}600\pi^{5} \\
&+ 173{,}541{,}988{,}447{,}027{,}200\pi^{4} + 2{,}265{,}687{,}071{,}391{,}744{,}000\pi^{3} \\
&- 3{,}856{,}488{,}632{,}156{,}160{,}000\pi^{2} - 11{,}569{,}465{,}896{,}468{,}480{,}000\pi \\
&+ 24{,}295{,}878{,}382{,}583{,}808{,}000\bigr)\bigl(\pi^{4} - 180\pi^{2} + 1680\bigr)^{2}\pi^{3} < 0, \\
\rho_{1,12} = 4992\bigl(&\pi^{19} - 597\pi^{17} + 175{,}968\pi^{15} - 29{,}516{,}400\pi^{13} + 3{,}012{,}992{,}640\pi^{11} \\
&- 603{,}832{,}320\pi^{10} - 206{,}228{,}151{,}360\pi^{9} + 76{,}640{,}256{,}000\pi^{8} \\
&+ 9{,}423{,}877{,}478{,}400\pi^{7} - 6{,}253{,}844{,}889{,}600\pi^{6} - 263{,}144{,}318{,}976{,}000\pi^{5} \\
&+ 293{,}562{,}836{,}582{,}400\pi^{4} + 3{,}824{,}042{,}213{,}376{,}000\pi^{3} \\
&- 6{,}489{,}283{,}756{,}032{,}000\pi^{2} - 19{,}467{,}851{,}268{,}096{,}000\pi \\
&+ 40{,}789{,}783{,}609{,}344{,}000\bigr)\bigl(\pi^{4} - 180\pi^{2} + 1680\bigr)^{2}\pi^{2} > 0, \\
\rho_{1,13} = -48\bigl(&\pi^{21} - 740\pi^{19} + 257{,}768\pi^{17} - 52{,}788{,}672\pi^{15} + 7{,}093{,}975{,}680\pi^{13} \\
&- 743{,}178{,}240\pi^{12} - 678{,}927{,}674{,}880\pi^{11} + 135{,}258{,}439{,}680\pi^{10} \\
&+ 45{,}959{,}564{,}851{,}200\pi^{9} - 17{,}003{,}918{,}131{,}200\pi^{8} - 2{,}082{,}714{,}284{,}851{,}200\pi^{7} \\
&+ 1{,}377{,}317{,}368{,}627{,}200\pi^{6} + 57{,}780{,}376{,}554{,}700{,}800\pi^{5} \\
&- 64{,}274{,}810{,}535{,}936{,}000\pi^{4} - 835{,}572{,}536{,}967{,}168{,}000\pi^{3} \\
&+ 1{,}414{,}045{,}831{,}790{,}592{,}000\pi^{2} + 4{,}242{,}137{,}495{,}371{,}776{,}000\pi \\
&- 8{,}869{,}923{,}853{,}959{,}168{,}000\bigr)\bigl(\pi^{4} - 180\pi^{2} + 1680\bigr)^{2}\pi > 0, \\
\rho_{1,14} = 64\bigl(&\pi^{9} - 576\pi^{7} + 145{,}152\pi^{5} - 15{,}482{,}880\pi^{3} + 464{,}486{,}400\pi \\
&- 1{,}021{,}870{,}080\bigr)\bigl(\pi^{12} - 468\pi^{10} + 85{,}800\pi^{8} - 8{,}648{,}640\pi^{6} + 467{,}026{,}560\pi^{4} \\
&- 11{,}416{,}204{,}800\pi^{2} + 74{,}724{,}249{,}600\bigr)\bigl(\pi^{4} - 180\pi^{2} + 1680\bigr)^{2} < 0.
\end{aligned}$$

Note that $0 < x^i < (\frac{\pi}{2})^i, i = 2, 3, \forall x \in (0, \pi/2)$, we have $H_1(x) \geq (\rho_{1,0} + \rho_{1,2} \cdot (\frac{\pi}{2})^2 + \rho_{1,3} \cdot (\frac{\pi}{2})^3) + \rho_{1,1}x + (\rho_{1,4} + \rho_{1,6} \cdot (\frac{\pi}{2})^2 + \rho_{1,7} \cdot (\frac{\pi}{2})^3)x^4 + \rho_{1,5}x^5 + (\rho_{1,8} + \rho_{1,10} \cdot (\frac{\pi}{2})^2 + \rho_{1,11} \cdot (\frac{\pi}{2})^3)x^8 + \rho_{1,9}x^9 + (\rho_{1,12} + \rho_{1,14} \cdot (\frac{\pi}{2})^2)x^{12} + \rho_{1,13}x^{13} \approx 9.6 \cdot 10^8 x^{13} + 4.3 \cdot 10^9 * x^12 + 1.5 \cdot 10^{13} x^9 + 5.0 \cdot 10^{13} x^8 + 4.2 \cdot 10^{16} x^5 + 1.2 \cdot 10^{17} x^4 + 1.5 \cdot 10^{19} x + 3.8 \cdot 10^{19} > 0, \forall x \in (0, \pi/2)$. It leads to $\Delta_4(x) \geq 0$ and $F(x) \geq L(x), \forall x \in [0, \pi/2]$.

(3) Finally, we prove that $\Delta_5(x) = (F(x) - R(x)) \cdot x^2 \leq 0, \forall x \in [0, \pi/2]$, which means that $F(x) \leq R(x)$. Combining Eq. (17) with Eq. (16), we have

$$\begin{aligned}
\Delta_5(x) &= \sin(x)^2 \cos(x) - r_1(x)x^2 \cos(x) + \bigl(x - r_2(x)x^2\bigr)\sin(x) \\
&\leq Q_3(x)Q_1(x) - r_1(x)x^2 P_2(x) + \bigl(x - r_2(x)x^2\bigr)Q_1(x) \\
&\triangleq \frac{(\pi - 2x)^3 x^{10}}{-56{,}582{,}064{,}000\bar{\gamma}\pi^{26}} H_2(x),
\end{aligned} \tag{19}$$

where

$$\bar{\gamma} = 5\pi^{10} - 558\pi^8 + 12{,}480\pi^6 - 177{,}120\pi^4 + 1{,}756{,}800\pi^2 - 7{,}257{,}600 \approx -0.12 < 0,$$

$$H_2(x) = \sum_{i=0}^{14} \rho_{2,i} x^i,$$

and

$$\begin{aligned}
\rho_{2,0} = {}& -54{,}743{,}040\bigl(10\pi^{14} - 1542\pi^{12} + 72{,}615\pi^{10} - 1{,}559{,}565\pi^8 + 14{,}049{,}000\pi^6 \\
& - 5{,}896{,}800\pi^4 - 635{,}040{,}000\pi^2 + 2{,}667{,}168{,}000\bigr)\pi^{19} < 0, \\
\rho_{2,1} = {}& -17{,}820\bigl(187{,}379\pi^{19} - 27{,}905{,}634\pi^{17} + 1{,}101{,}819{,}840\pi^{15} \\
& + 16{,}808{,}329{,}440\pi^{13} - 4{,}064{,}256{,}000\pi^{12} - 3{,}819{,}046{,}492{,}800\pi^{11} \\
& + 2{,}599{,}498{,}137{,}600\pi^{10} + 167{,}022{,}175{,}219{,}200\pi^9 \\
& - 249{,}629{,}854{,}924{,}800\pi^8 \\
& - 3{,}170{,}509{,}848{,}576{,}000\pi^7 + 5{,}500{,}206{,}415{,}872{,}000\pi^6 \\
& + 40{,}549{,}244{,}682{,}240{,}000\pi^5 - 77{,}445{,}340{,}987{,}392{,}000\pi^4 \\
& - 349{,}559{,}830{,}609{,}920{,}000\pi^3 + 759{,}892{,}318{,}617{,}600{,}000\pi^2 \\
& + 1{,}297{,}856{,}751{,}206{,}400{,}000\pi - 3{,}114{,}856{,}202{,}895{,}360{,}000\bigr)\pi^{13} < 0, \\
\rho_{2,2} = {}& 135\bigl(470{,}935\pi^{21} - 163{,}016{,}586\pi^{19} + 16{,}583{,}895{,}072\pi^{17} \\
& - 399{,}774{,}876{,}480\pi^{15} - 39{,}075{,}261{,}120{,}000\pi^{13} \\
& + 7{,}081{,}559{,}654{,}400\pi^{12} + 2{,}891{,}256{,}628{,}377{,}600\pi^{11} \\
& - 4{,}039{,}064{,}115{,}609{,}600\pi^{10} - 54{,}681{,}296{,}556{,}902{,}400\pi^9 \\
& + 116{,}088{,}651{,}192{,}729{,}600\pi^8 + 29{,}143{,}836{,}868{,}608{,}000\pi^7 \\
& - 519{,}915{,}234{,}263{,}040{,}000\pi^6 + 10{,}877{,}661{,}896{,}048{,}640{,}000\pi^5 \\
& - 19{,}708{,}263{,}780{,}581{,}376{,}000\pi^4 - 126{,}300{,}002{,}703{,}114{,}240{,}000\pi^3 \\
& + 281{,}041{,}609{,}068{,}380{,}160{,}000\pi^2 + 445{,}424{,}437{,}014{,}036{,}480{,}000\pi \\
& - 1{,}083{,}969{,}958{,}607{,}585{,}280{,}000\bigr)\pi^{12} > 0, \\
\rho_{2,3} = {}& 3240\bigl(118{,}457\pi^{21} - 29{,}814{,}542\pi^{19} + 2{,}683{,}328{,}595\pi^{17} - 112{,}738{,}193{,}340\pi^{15} \\
& - 3{,}193{,}344{,}000\pi^{14} + 3{,}713{,}195{,}298{,}960\pi^{13} + 4{,}257{,}366{,}220{,}800\pi^{12} \\
& - 176{,}445{,}931{,}032{,}000\pi^{11} - 538{,}724{,}796{,}825{,}600\pi^{10} \\
& + 5{,}029{,}126{,}333{,}862{,}400\pi^9 + 12{,}839{,}971{,}273{,}113{,}600\pi^8 \\
& - 58{,}269{,}701{,}597{,}184{,}000\pi^7 - 255{,}180{,}783{,}255{,}552{,}000\pi^6 \\
& + 377{,}530{,}820{,}739{,}072{,}000\pi^5 + 3{,}806{,}634{,}452{,}189{,}184{,}000\pi^4 \\
& - 2{,}922{,}752{,}802{,}816{,}000{,}000\pi^3 - 26{,}758{,}510{,}065{,}745{,}920{,}000\pi^2 \\
& + 14{,}276{,}424{,}263{,}270{,}400{,}000\pi + 63{,}335{,}409{,}458{,}872{,}320{,}000\bigr)\pi^{11} > 0,
\end{aligned}$$

$$\begin{aligned}
\rho_{2,4} = {} & -1350\bigl(2579\pi^{23} - 1{,}304{,}586\pi^{21} + 190{,}808{,}712\pi^{19} - 4{,}857{,}853{,}680\pi^{17} \\
& - 823{,}686{,}670{,}080\pi^{15} + 272{,}839{,}311{,}360\pi^{14} + 28{,}966{,}274{,}628{,}480\pi^{13} \\
& - 219{,}724{,}548{,}341{,}760\pi^{12} + 166{,}206{,}481{,}943{,}040\pi^{11} \\
& + 5{,}303{,}591{,}552{,}286{,}720\pi^{10} - 1{,}171{,}730{,}648{,}309{,}760\pi^{9} \\
& + 2{,}317{,}601{,}341{,}440{,}000\pi^{8} - 576{,}193{,}032{,}614{,}707{,}200\pi^{7} \\
& - 1{,}263{,}206{,}036{,}039{,}270{,}400\pi^{6} + 15{,}415{,}077{,}252{,}995{,}481{,}600\pi^{5} \\
& + 10{,}344{,}536{,}334{,}139{,}392{,}000\pi^{4} - 149{,}767{,}724{,}873{,}023{,}488{,}000\pi^{3} \\
& + 25{,}334{,}163{,}783{,}548{,}928{,}000\pi^{2} + 507{,}098{,}589{,}831{,}364{,}608{,}000\pi \\
& - 361{,}323{,}319{,}535{,}861{,}760{,}000\bigr)\pi^{10} < 0, \\
\rho_{2,5} = {} & -43{,}200\bigl(498\pi^{23} - 177{,}844\pi^{21} + 26{,}426{,}133\pi^{19} - 1{,}772{,}952{,}897\pi^{17} \\
& + 143{,}700{,}480\pi^{16} + 13{,}733{,}029{,}656\pi^{15} - 119{,}156{,}438{,}016\pi^{14} \\
& + 4{,}355{,}924{,}207{,}040\pi^{13} + 7{,}627{,}468{,}197{,}888\pi^{12} - 240{,}282{,}095{,}518{,}080\pi^{11} \\
& - 77{,}912{,}484{,}249{,}600\pi^{10} + 5{,}350{,}872{,}626{,}668{,}800\pi^{9} \\
& - 3{,}604{,}409{,}633{,}341{,}440\pi^{8} - 61{,}395{,}244{,}517{,}376{,}000\pi^{7} \\
& + 111{,}158{,}598{,}116{,}966{,}400\pi^{6} + 395{,}514{,}763{,}370{,}496{,}000\pi^{5} \\
& - 1{,}303{,}336{,}590{,}822{,}604{,}800\pi^{4} - 1{,}470{,}286{,}291{,}009{,}536{,}000\pi^{3} \\
& + 7{,}275{,}723{,}144{,}560{,}640{,}000\pi^{2} + 2{,}725{,}499{,}177{,}533{,}440{,}000\pi \\
& - 16{,}197{,}252{,}255{,}055{,}872{,}000\bigr)\pi^{9} < 0, \\
\rho_{2,6} = {} & 36\bigl(3055\pi^{25} - 1{,}943{,}682\pi^{23} + 419{,}154{,}420\pi^{21} - 44{,}118{,}680{,}220\pi^{19} \\
& + 2{,}140{,}947{,}712{,}800\pi^{17} - 130{,}288{,}435{,}200\pi^{16} + 86{,}530{,}606{,}128{,}000\pi^{15} \\
& + 3{,}142{,}250{,}496{,}000\pi^{14} - 20{,}960{,}647{,}460{,}121{,}600\pi^{13} \\
& - 17{,}423{,}671{,}703{,}961{,}600\pi^{12} + 1{,}023{,}195{,}300{,}994{,}944{,}000\pi^{11} \\
& + 344{,}467{,}294{,}617{,}600{,}000\pi^{10} - 23{,}018{,}674{,}839{,}164{,}928{,}000\pi^{9} \\
& + 5{,}473{,}622{,}558{,}638{,}080{,}000\pi^{8} + 281{,}073{,}089{,}819{,}934{,}720{,}000\pi^{7} \\
& - 255{,}696{,}320{,}798{,}392{,}320{,}000\pi^{6} - 1{,}893{,}832{,}571{,}360{,}378{,}880{,}000\pi^{5} \\
& + 3{,}403{,}944{,}523{,}821{,}219{,}840{,}000\pi^{4} + 6{,}320{,}191{,}562{,}160{,}537{,}600{,}000\pi^{3} \\
& - 20{,}096{,}013{,}935{,}679{,}897{,}600{,}000\pi^{2} - 7{,}101{,}872{,}142{,}601{,}420{,}800{,}000\pi \\
& + 44{,}480{,}146{,}577{,}345{,}740{,}800{,}000\bigr)\pi^{8} > 0,
\end{aligned}$$

$$\begin{aligned}
\rho_{2,7} = {}& 864\big(5\pi^{10} - 558\pi^{8} + 12{,}480\pi^{6} - 177{,}120\pi^{4} + 1{,}756{,}800\pi^{2} \\
& - 7{,}257{,}600\big)\big(139\pi^{15} - 30{,}800\pi^{13} + 638{,}668{,}800\pi^{9} - 106{,}444{,}800\pi^{8} \\
& - 64{,}399{,}104{,}000\pi^{7} + 60{,}673{,}536{,}000\pi^{6} + 2{,}557{,}229{,}875{,}200\pi^{5} \\
& - 3{,}344{,}069{,}836{,}800\pi^{4} - 45{,}600{,}952{,}320{,}000\pi^{3} + 86{,}373{,}568{,}512{,}000\pi^{2} \\
& + 255{,}365{,}332{,}992{,}000\pi - 579{,}400{,}335{,}360{,}000\big)\pi^{7} > 0, \\
\rho_{2,8} = {}& -2\big(199\pi^{17} - 95{,}040\pi^{15} + 5{,}364{,}817{,}920\pi^{11} - 1{,}051{,}887{,}513{,}600\pi^{9} \\
& + 91{,}968{,}307{,}200\pi^{8} + 77{,}391{,}330{,}508{,}800\pi^{7} - 61{,}250{,}892{,}595{,}200\pi^{6} \\
& - 2{,}652{,}044{,}090{,}572{,}800\pi^{5} + 3{,}321{,}895{,}256{,}064{,}000\pi^{4} \\
& + 45{,}115{,}972{,}780{,}032{,}000\pi^{3} - 84{,}515{,}195{,}584{,}512{,}000\pi^{2} \\
& - 250{,}300{,}944{,}875{,}520{,}000\pi + 563{,}640{,}646{,}238{,}208{,}000\big)\big(5\pi^{10} - 558\pi^{8} \\
& + 12{,}480\pi^{6} - 177{,}120\pi^{4} + 1{,}756{,}800\pi^{2} - 7{,}257{,}600\big)\pi^{6} < 0, \\
\rho_{2,9} = {}& -1728\big(5\pi^{10} - 558\pi^{8} + 12{,}480\pi^{6} - 177{,}120\pi^{4} + 1{,}756{,}800\pi^{2} - 7{,}257{,}600\big) \\
& \times \big(\pi^{17} - 129{,}360\pi^{13} + 33{,}707{,}520\pi^{11} - 2{,}956{,}800\pi^{10} - 3{,}626{,}515{,}200\pi^{9} \\
& + 1{,}774{,}080{,}000\pi^{8} + 211{,}718{,}707{,}200\pi^{7} - 163{,}924{,}992{,}000\pi^{6} \\
& - 7{,}086{,}030{,}336{,}000\pi^{5} + 8{,}762{,}535{,}936{,}000\pi^{4} + 118{,}562{,}476{,}032{,}000\pi^{3} \\
& - 219{,}957{,}534{,}720{,}000\pi^{2} - 652{,}361{,}859{,}072{,}000\pi \\
& + 1{,}459{,}230{,}474{,}240{,}000\big)\pi^{5} < 0, \\
\rho_{2,10} = {}& 4\big(5\pi^{10} - 558\pi^{8} + 12{,}480\pi^{6} - 177{,}120\pi^{4} + 1{,}756{,}800\pi^{2} - 7{,}257{,}600\big) \\
& \times \big(\pi^{19} - 475{,}200\pi^{15} + 212{,}889{,}600\pi^{13} - 40{,}715{,}136{,}000\pi^{11} \\
& + 2{,}554{,}675{,}200\pi^{10} + 3{,}886{,}427{,}381{,}760\pi^{9} - 1{,}778{,}053{,}939{,}200\pi^{8} \\
& - 214{,}297{,}651{,}814{,}400\pi^{7} + 162{,}232{,}093{,}900{,}800\pi^{6} \\
& + 6{,}999{,}156{,}051{,}148{,}800\pi^{5} - 8{,}559{,}674{,}287{,}718{,}400\pi^{4} \\
& - 115{,}416{,}546{,}803{,}712{,}000\pi^{3} + 212{,}292{,}282{,}875{,}904{,}000\pi^{2} \\
& + 630{,}387{,}564{,}871{,}680{,}000\pi - 1{,}401{,}685{,}291{,}302{,}912{,}000\big)\pi^{4} > 0, \\
\rho_{2,11} = {}& 4608\big(5\pi^{10} - 558\pi^{8} + 12{,}480\pi^{6} - 177{,}120\pi^{4} + 1{,}756{,}800\pi^{2} - 7{,}257{,}600\big) \\
& \times \big(5\pi^{17} - 2919\pi^{15} + 717{,}120\pi^{13} - 40{,}320\pi^{12} - 95{,}264{,}400\pi^{11} \\
& + 25{,}724{,}160\pi^{10} + 7{,}974{,}046{,}080\pi^{9} - 3{,}548{,}160{,}000\pi^{8} - 429{,}305{,}184{,}000\pi^{7} \\
& + 319{,}973{,}068{,}800\pi^{6} + 13{,}775{,}287{,}680{,}000\pi^{5} - 16{,}684{,}583{,}731{,}200\pi^{4} \\
& - 224{,}249{,}389{,}056{,}000\pi^{3} + 409{,}335{,}607{,}296{,}000\pi^{2} \\
& + 1{,}216{,}740{,}704{,}256{,}000\pi - 2{,}690{,}992{,}668{,}672{,}000\big)\pi^{3} > 0,
\end{aligned}$$

$$
\begin{aligned}
\rho_{2,12} = {} & -96\bigl(5\pi^{10} - 558\pi^{8} + 12{,}480\pi^{6} - 177{,}120\pi^{4} + 1{,}756{,}800\pi^{2} - 7{,}257{,}600\bigr) \\
& \times \bigl(\pi^{19} - 995\pi^{17} + 410{,}592\pi^{15} - 88{,}549{,}200\pi^{13} + 3{,}870{,}720\pi^{12} \\
& + 11{,}047{,}639{,}680\pi^{11} - 2{,}841{,}108{,}480\pi^{10} - 893{,}605{,}426{,}560\pi^{9} \\
& + 388{,}310{,}630{,}400\pi^{8} + 47{,}103{,}101{,}337{,}600\pi^{7} - 34{,}641{,}395{,}712{,}000\pi^{6} \\
& - 1{,}488{,}093{,}194{,}649{,}600\pi^{5} + 1{,}787{,}128{,}145{,}510{,}400\pi^{4} \\
& + 23{,}948{,}547{,}194{,}880{,}000\pi^{3} - 43{,}416{,}398{,}462{,}976{,}000\pi^{2} \\
& - 129{,}167{,}648{,}096{,}256{,}000\pi + 284{,}292{,}431{,}216{,}640{,}000\bigr)\pi^{2} < 0, \\
\rho_{2,13} = {} & 256\bigl(5\pi^{10} - 558\pi^{8} + 12{,}480\pi^{6} - 177{,}120\pi^{4} + 1{,}756{,}800\pi^{2} - 7{,}257{,}600\bigr) \\
& \times \bigl(\pi^{19} - 1734\pi^{17} + 1{,}043{,}280\pi^{15} - 318{,}349{,}440\pi^{13} + 14{,}515{,}200\pi^{12} \\
& + 56{,}498{,}601{,}600\pi^{11} - 11{,}554{,}099{,}200\pi^{10} - 6{,}203{,}534{,}752{,}800\pi^{9} \\
& + 2{,}843{,}353{,}497{,}600\pi^{8} + 419{,}195{,}855{,}232{,}000\pi^{7} - 354{,}261{,}919{,}334{,}400\pi^{6} \\
& - 16{,}350{,}918{,}880{,}665{,}600\pi^{5} + 22{,}731{,}806{,}490{,}624{,}000\pi^{4} \\
& + 314{,}092{,}921{,}798{,}656{,}000\pi^{3} - 646{,}147{,}253{,}993{,}472{,}000\pi^{2} \\
& - 1{,}856{,}398{,}674{,}493{,}440{,}000\pi + 4{,}477{,}605{,}791{,}662{,}080{,}000\bigr)\pi < 0, \\
\rho_{2,14} = {} & -64\bigl(\pi^{9} - 1728\pi^{7} + 725{,}760\pi^{5} - 108{,}380{,}160\pi^{3} + 23{,}224{,}320\pi^{2} \\
& + 4{,}180{,}377{,}600\pi - 10{,}218{,}700{,}800\bigr)\bigl(5\pi^{10} - 558\pi^{8} + 12{,}480\pi^{6} - 177{,}120\pi^{4} \\
& + 1{,}756{,}800\pi^{2} - 7{,}257{,}600\bigr)\bigl(3\pi^{10} - 1100\pi^{8} + 166{,}320\pi^{6} - 11{,}975{,}040\pi^{4} \\
& + 365{,}904{,}000\pi^{2} - 2{,}594{,}592{,}000\bigr) > 0.
\end{aligned}
$$

Note that $0 < x^i < (\frac{\pi}{2})^i, i = 2, 3, \forall x \in (0, \pi/2)$, we have $H_2(x) \leq (\rho_{2,0} + \rho_{2,2} \cdot (\frac{\pi}{2})^2 + \rho_{2,3} \cdot (\frac{\pi}{2})^3) + \rho_{2,1}x + (\rho_{2,4} + \rho_{2,6} \cdot (\frac{\pi}{2})^2 + \rho_{2,7} \cdot (\frac{\pi}{2})^3)x^4 + \rho_{2,5}x^5 + (\rho_{2,8} + \rho_{2,10} \cdot (\frac{\pi}{2})^2 + \rho_{2,11} \cdot (\frac{\pi}{2})^3)x^8 + \rho_{2,9}x^9 + (\rho_{2,12} + \rho_{2,14} \cdot (\frac{\pi}{2})^2)x^{12} + \rho_{2,13}x^{13} \approx -1.6 \cdot 10^6x^{13} - 1.2 \cdot 10^7x^{12} - 2.7 \cdot 10^{10}x^9 - 1.4 \cdot 10^{11}x^8 - 7.2 \cdot 10^{-13}x^5 - 3.7 \cdot 10^{14}x^4 - 2.5 \cdot 10^{16}x - 1.1 \cdot 10^{17} < 0, \forall x \in (0, \pi/2)$. So we have $\Delta_5(x) \leq 0$ and $F(x) \leq R(x)$, $\forall x \in [0, \pi/2]$.

From the above discussions, we have completed the proof. □

4 Discussions and conclusions

In principle, one can prove that $L_i(x) \leq L(x) \leq F(x) \leq R(x) \leq R_i(x)$, $\forall x \in [0, \pi/2]$ in a similar way, where $L_i(x)$ and $R_i(x)$, $i = 2, 3$, are two bounding functions in Eq. (6) and Eq. (7), respectively. The maximum errors between $F(x)$ and its different bounds are listed in Table 1. It shows that the bounds in this paper achieve a much better approximation than those of the bounds in Eq. (6) and Eq. (7).

Table 1 Maximum errors between $F(x)$ and its different bounds

Bounds	$L_1(x)$	$R_1(x)$	$L_2(x)$	$R_2(x)$	$L(x)$	$R(x)$
Error	2.8e–2	2.5e–4	2.09e–3	1.7e–3	1.37e–5	4.89e–6

The new method can be applied to refine the Becker–Stark inequality, which is studied in [5, 16, 24] and is known as

$$\frac{8}{\pi^2-4x^2} < \frac{\tan(x)}{x} < \frac{\pi^2}{\pi^2-4x^2}, \quad \forall x \in (0,\pi/2). \tag{20}$$

Zhu [24] refined it as

$$\begin{aligned}\alpha_l(x) &= \frac{8}{\pi^2-4x^2} + \frac{2}{\pi^2} - \frac{(\pi^2-9)}{6\pi^4}\cdot\left(\pi^2-4x^2\right) < \frac{\tan(x)}{x} \\ &< \frac{8}{\pi^2-4x^2} + \frac{2}{\pi^2} - \frac{(10-\pi^2)}{\pi^4}\cdot\left(\pi^2-4x^2\right) = \alpha_r(x), \quad \forall x \in \left(0,\frac{\pi}{2}\right),\end{aligned} \tag{21}$$

while it is refined in [16] as follows:

$$\begin{aligned}\alpha_{2l}(x) &= \frac{8+\mu(x)}{\pi^2-4x^2} < \frac{\tan(x)}{x} \\ &< \frac{8+\mu(x)+(\frac{32}{\pi^3}-\frac{8}{3\pi})(\frac{\pi}{2}-x)^3}{\pi^2-4x^2} = \alpha_{2r}(x), \quad \forall x \in \left(0,\frac{\pi}{2}\right),\end{aligned} \tag{22}$$

where $\mu(x) = \frac{8}{\pi}(\frac{\pi}{2}-x) + (\frac{16}{\pi^2}-\frac{8}{3})(\frac{\pi}{2}-x)^2$.

By applying the method in Sect. 2 and using the form $\frac{\sum_{i=0}^{6} v_i x^i}{\pi^2-4x^2}$, one obtains the resulting bounds, $\beta_l(x) = \frac{\kappa_1(x)}{45\pi^6(\pi^2-4x^2)}$ and $\beta_r(x) = \frac{\kappa_2(x)}{3\pi^6(\pi^2-4x^2)}$, where $\kappa_1(x) = 45\pi^8 + (-2\pi^8 - 3660\pi^6 + 36{,}000\pi^4)x^2 + (16\pi^7 + 21{,}000\pi^5 - 208{,}800\pi^3)x^3 + (-48\pi^6 - 49{,}440\pi^4 + 492{,}480\pi^2)x^4 + (64\pi^5 + 54{,}240\pi^3 - 541{,}440\pi)x^5 + (-32\pi^4 - 23{,}040\pi^2 + 230{,}400)x^6$ and $\kappa_2(x) = 3\pi^8 + (-12\pi^6+\pi^8)x^2 + (5280\pi^3 - 456\pi^5 - 8\pi^7x^3) + (-24{,}768\pi^2 + 2272\pi^4 + 24\pi^6)x^4 + (40{,}704\pi - 3808\pi^3 - 32\pi^5)x^5 + (-23{,}040 + 2176\pi^2 + 16\pi^4)x^6$, such that

$$\beta_l(x) < \frac{\tan(x)}{x} < \beta_r(x), \quad \forall x \in \left(0,\frac{\pi}{2}\right).$$

By using the Maple software, $\forall x \in (0,\frac{\pi}{2})$, it can be verified that $\beta_l(x) - \alpha_l(x) = -\frac{(\pi-2x)^3}{90\pi^6} \times (57{,}600x^3 - 8\pi^4x^3 - 5760\pi^2x^3 + 4920\pi^3x^2 - 48{,}960\pi x^2 + 4\pi^5x^2 + 6210\pi^2x - 630\pi^4x - 105\pi^5 + 1035\pi^3) \approx -\frac{(\pi-2x)^3}{90\pi^6}(-28.1940986x^3 - 37.4163x^2 - 77.48403x - 40.57055) > 0$, $\beta_r(x) - \alpha_r(x) = \frac{1}{3\pi^6}(\pi-2x)^2x^2(-5760x^2 + 544\pi^2x^2 + 4\pi^4x^2 + 4416\pi x - 408\pi^3x - 4\pi^5x - 216\pi^2 + 12\pi^4 + \pi^6) \approx \frac{1}{3\pi^6}(\pi-2x)^2x^2(-1.298840x^2 - 1.36647x - 1.5362637) < 0$, $\beta_l(x) - \alpha_{2l}(x) = -\frac{(\pi-2x)^3}{45\pi^6}(28{,}800x^3 - 4\pi^4x^3 - 2880\pi^2x^3 - 24{,}480\pi x^2 + 2\pi^5x^2 + 2460\pi^3x^2 + 3240\pi^2x - 330\pi^4x - 75\pi^5 + 720\pi^3) \approx -\frac{(\pi-2x)^3}{45\pi^6}(-14.0970443x^3 - 18.7081400x^2 - 167.48179x - 626.95716) > 0$ and $\beta_l(x) - \alpha_{2r}(x) = \frac{(\pi-2x)^4}{3\pi^6}(\pi^4x^2 - 1440x^2 + 136\pi^2x^2 - 336\pi x + 34\pi^3x - 60\pi^2 + 6\pi^4) \approx \frac{(\pi-2x)^4}{3\pi^6}(-0.324710x^2 - 1.361725x - 7.7217177) < 0$. So the bounds $\beta_l(x)$ and $\beta_r(x)$ achieve a better approximation than those results in both [24] and [16].

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References
1. Baker, G.A. Jr., Graves-Morris, P.: Padé Approximants. Cambridge University Press, New York (1996)
2. Banjac, B., Makragić, M., Malešević, B.: Some notes on a method for proving inequalities by computer. Results Math. **69**(1), 161–176 (2016)
3. Chen, C.P., Paris, R.B.: Series representations of the remainders in the expansions for certain trigonometric functions and some related inequalities. Math. Inequal. Appl. **20**(4), 1003–1016 (2017)
4. Davis, P.J.: Interpolation and Approximation. Dover Publications, New York (1975)
5. Debnath, L., Mortici, C., Zhu, L.: Refinements of Jordan–Steckin and Becker–Stark inequalities. Results Math. **67**(1–2), 207–215 (2015)
6. Jiang, W.D., Luo, Q.M., Qi, F.: Refinements and sharpening of some Huygens and Wilker type inequalities. Math. Inequal. Appl. **6**(1), 19–22 (2014)
7. Lutovac, T., Malešsević, B., Mortici, C.: The natural algorithmic approach of mixed trigonometric-polynomial problems. J. Inequal. Appl. **2017**, 116 (2017)
8. Malešević, B., Banjac, B., Jović, I.: A proof of two conjectures of Chao–Ping Chen for inverse trigonometricfunctions. J. Math. Inequal. **11**(1), 151–162 (2017)
9. Malešević, B., Lutovac, T., Banjac, B.: A proof of an open problem of Yusuke Nishizawa for a power-exponential function. J. Math. Inequal. **12**(2), 473–485 (2018)
10. Malešević, B., Lutovac, T., Rašajski, M., et al.: Extensions of the natural approach to refinements and generalizations of some trigonometric inequalities. Adv. Differ. Equ. **2018**(1), 90 (2018)
11. Malešević, B., Lutovac, T., Rašajski, M., Mortici, C.: Extensions of the natural approach to refinements and generalizations of some trigonometric inequalities. Adv. Differ. Equ. **2018**, 90 (2018)
12. Malešević, B., Makragic, M.: A method for proving some inequalities on mixed trigonometric polynomial functions. J. Math. Inequal. **10**, 849–876 (2015)
13. Mortici, C.: The natural approach of Wilker–Cusa–Huygens inequalities. Math. Inequal. Appl. **14**, 535–541 (2011)
14. Mortici, C.: A subtly analysis of Wilker inequation. Appl. Math. Comput. **231**, 516–520 (2014)
15. Nenezić, M., Malesević, B., Mortici, C.: New approximations of some expressions involving trigonometric functions. Appl. Math. Comput. **283**, 299–315 (2016)
16. Nenezić, M., Zhu, L.: Some improvements of Jordan–Steckin and Becker–Stark inequalities. Appl. Anal. Discrete Math. **12**, 244–256 (2018)
17. Neuman, E.: Wilker and Huygens-type inequalities for the generalized trigonometric and for the generalized hyperbolic functions. Appl. Math. Comput. **230**(3), 211–217 (2014)
18. Sumner, J.S., Jagers, A.A., Vowe, M., Anglesio, J.: Inequalities involving trigonometric functions. Am. Math. Mon. **98**(3), 264–267 (1991)
19. Wilker, J.B.: Problem E-3306. Am. Math. Mon. **96**, 55 (1989)
20. Wu, S.H., Li, S.G., Bencze, M.: Sharpened versions of Mitrinovic–Adamovic, Lazarevic and Wilker's inequalities for trigonometric and hyperbolic functions. J. Nonlinear Sci. Appl. **9**(5), 2688–2696 (2016)
21. Wu, S.H., Srivastava, H.M.: A further refinement of Wilker's inequality. Integral Transforms Spec. Funct. **19**(10), 757–765 (2008)
22. Wu, S.H., Yu, H.P., Deng, Y.P., et al.: Several improvements of Mitrinovic–Adamovic and Lazarevic's inequalities with applications to the sharpening of Wilker-type inequalities. J. Nonlinear Sci. Appl. **9**(4), 1755–1765 (2016)
23. Yang, Z.H., Chu, Y.M., Zhang, X.H.: Sharp Cusa type inequalities with two parameters and their applications. Appl. Math. Comput. **268**, 1177–1198 (2015)
24. Zhu, L.: A refinement of the Becker–Stark inequalities. Math. Notes **93**(3–4), 421–425 (2013)