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Optimal bounds for the generalized Euler–Mascheroni constant

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Abstract

We provide several sharp upper and lower bounds for the generalized Euler–Mascheroni constant. As consequences, some previous bounds for the Euler–Mascheroni constant are improved.

MSC: 11Y60; 40A05; 33B15

Keywords: Euler–Mascheroni constant; gamma function; psi function; Asymptotic formula

1 Introduction

Let $a > 0$. Then the generalized Euler–Mascheroni constant $\gamma(a)$ [1] is given by

$$\gamma(a) = \lim_{n\to\infty}\left[\frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \log\left(\frac{a+n-1}{a}\right)\right].$$

We clearly see that the generalized Euler–Mascheroni constant $\gamma(a)$ is the natural generalization of the classical Euler–Mascheroni constant [2–5]

$$\gamma = \gamma(1) = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.577215664901\ldots.$$

Recently, the two bounds for γ and $\gamma(a)$ have attracted the attention of many mathematicians. In particular, many remarkable inequalities and asymptotic formulas for γ and $\gamma(a)$ can be found in the literature [6–10].

Let

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n,$$

$$R_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log\left(n + \frac{1}{2}\right),$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{2n} - \log n,$$

$$T_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log\left(n + \frac{1}{2} + \frac{1}{24n}\right),$$

$$y_n(a) = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \log\left(\frac{a+n-1}{a}\right),$$

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$$\alpha_n(a) = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-2} + \frac{1}{2(a+n-1)} - \log\left(\frac{a+n-1}{a}\right), \tag{1.1}$$

$$\beta_n(a) = \frac{1}{a} + \frac{1}{a+1} + \frac{1}{a+n-1} - \log\left(\frac{a+n-1/2}{a}\right), \tag{1.2}$$

$$\lambda_n(a) = \frac{1}{a} + \frac{1}{a+1} + \frac{1}{a+n-1} - \log\left(\frac{a+n-1/2}{a} + \frac{1}{24a(a+n-1)}\right), \tag{1.3}$$

$$\mu_n(a) = y_n(a) - \frac{1}{2(a+n-1)} + \frac{1}{12(a+n-1)^2} - \frac{1}{120(a+n-1)^4}. \tag{1.4}$$

Negoi [11] proved that the two-sided inequality

$$\frac{1}{48(n+1)^3} \le \gamma - T_n \le \frac{1}{48n^3} \tag{1.5}$$

is valid for $n \ge 1$.

Qiu and Vuorinen [12] proved that the two-sided inequality

$$\frac{1}{2n} - \frac{\lambda}{n^2} < \gamma_n - \gamma \le \frac{1}{2n} - \frac{\mu}{n^2} \tag{1.6}$$

is valid for $n \ge 1$ if and only if $\lambda \ge 1/12$ and $\mu \le \gamma - 1/2$.

In [13], DeTemple proved that the double inequality

$$\frac{1}{24(n+1)^2} \le R_n - \gamma \le \frac{1}{24n^2} \tag{1.7}$$

holds for all $n \ge 1$.

Chen [14] proved that $\alpha = 1/\sqrt{12\gamma - 6} - 1$ and $\beta = 0$ are the best possible constants such that the double inequality

$$\frac{1}{12(n+\alpha)^2} \le \gamma - S_n \le \frac{1}{12(n+\beta)^2} \tag{1.8}$$

holds for $n \ge 1$.

Sîntămărian [15], and Berinde and Mortici [16] proved that the double inequalities

$$\frac{1}{2(n+a)} \le y_n(a) - \gamma(a) \le \frac{1}{2(n+a-1)}, \tag{1.9}$$

$$\frac{1}{24(n+a)^2} \le \beta_n(a) - \gamma(a) \le \frac{1}{24(n+a-1)^2} \tag{1.10}$$

are valid for all $a > 0$ and $n \ge 1$.

The main purpose of this article is to find the best possible constants α_1, α_2, α_3, α_4, β_1, β_2, β_3 and β_4 such that the double inequalities

$$\frac{1}{12(a+n-\alpha_1)^2} \le \gamma(a) - \alpha_n(a) < \frac{1}{12(a+n-\beta_1)^2},$$

$$\frac{1}{24(a+n-\alpha_2)^2} \le \beta_n(a) - \gamma(a) < \frac{1}{24(a+n-\beta_2)^2},$$

$$\frac{1}{48(a+n-\alpha_3)^3} \le \gamma(a) - \lambda_n(a) < \frac{1}{48(a+n-\beta_3)^3},$$

$$\frac{\alpha_4}{(a+n-1)^6} \le \gamma(a) - \mu_n(a) < \frac{\beta_4}{(a+n-1)^6}$$

hold for all $a > 0$ and $n \ge n_0$ and improve the bounds for the Euler–Mascheroni constant.

2 Main results

In order to prove our main results, we need several formulas and lemmas which we present in this section.

For $x > 0$, the classical gamma function Γ and its logarithmic derivative, the so-called psi function ψ are defined [17–24] as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively.

The psi function ψ has the recurrence and asymptotic formulas [25] as follows:

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{2.1}$$

$$\psi(x) \sim \log x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots \quad (x \to \infty). \tag{2.2}$$

Lemma 2.1 (See [14, Proof of Theorem 1]) *The function*

$$f_1(x) = \frac{1}{\sqrt{12(\log x - \psi(x+1) + \frac{1}{2x})}} - x \tag{2.3}$$

is strictly decreasing on $[2, \infty)$ *with* $f_1(\infty) = 0$.

Lemma 2.2 (See [26, Proof of Theorem 1], [27, Remark 4]) *The function*

$$f_2(x) = \frac{1}{\sqrt{24(\psi(x+1) - \log(x+1/2))}} - x \tag{2.4}$$

is strictly decreasing on $[2, \infty)$ *with* $f_2(\infty) = 1/2$.

Lemma 2.3 (See [28, Proof of Theorem 2]) *The function*

$$f_3(x) = \frac{1}{\sqrt[3]{48[\log(x + \frac{1}{2} + \frac{1}{24x}) - \psi(x+1)]}} - x \tag{2.5}$$

is strictly decreasing on $[5, \infty)$ *with* $f_3(\infty) = 83/360$.

Lemma 2.4 (See [29, Theorem 1.2(2)]) *The function*

$$f_4(x) = \frac{x^2}{120} - \left(\psi(x) - \log x + \frac{1}{2x} + \frac{1}{12x^2}\right)x^6 \tag{2.6}$$

is strictly increasing from $(0, \infty)$ *onto* $(0, 1/252)$.

Theorem 2.5 *Let $\alpha_n(a)$ and $f_1(x)$ be, respectively, defined by* (1.1) *and* (2.3). *Then* $\alpha_1 = 1 - f_1(a+2)$ *and* $\beta_1 = 1$ *are the best possible constants such that the double inequality*

$$\frac{1}{12(a+n-\alpha_1)^2} \leq \gamma(a) - \alpha_n(a) < \frac{1}{12(a+n-\beta_1)^2} \tag{2.7}$$

holds for all $a > 0$ *and* $n \geq 3$.

Proof It follows from (1.1), (2.1) and (2.2) that

$$\begin{aligned} \gamma(a) - \alpha_n(a) &= \lim_{n\to\infty}\left[\psi(n+a) - \psi(a) - \log\left(\frac{a+n-1}{a}\right)\right] \\ &\quad - \left[\psi(n+a) - \psi(a) - \frac{1}{2(a+n-1)} - \log\left(\frac{a+n-1}{a}\right)\right] \\ &= \lim_{n\to\infty}\left[\psi(n+a) - \log(a+n-1)\right] \\ &\quad - \psi(n+a) + \frac{1}{2(a+n-1)} + \log(a+n-1) \\ &= \log(a+n-1) - \psi(n+a) + \frac{1}{2(a+n-1)}. \end{aligned} \tag{2.8}$$

From (2.3) and (2.8) we clearly see that inequality (2.7) is equivalent to

$$\alpha_1 \leq 1 - f_1(n+a-1) < \beta_1. \tag{2.9}$$

Therefore, Theorem 2.5 follows easily from Lemma 2.1 and (2.19). □

Theorem 2.6 *Let $\beta_n(a)$ and $f_2(x)$ be, respectively, defined by* (1.2) *and* (2.4). *Then* $\alpha_2 = 1 - f_2(a+2)$ *and* $\beta_2 = 1/2$ *are the best possible constants such that the double inequality*

$$\frac{1}{24(a+n-\alpha_2)^2} \leq \beta_n(a) - \gamma(a) < \frac{1}{24(a+n-\beta_2)^2} \tag{2.10}$$

holds for all $a > 0$ *and* $n \geq 3$.

Proof It follows from (1.2), (2.1) and (2.2) that

$$\beta_n(a) - \gamma(a) = \psi(n+a) - \log\left(a+n-\frac{1}{2}\right). \tag{2.11}$$

From (2.4) and (2.11) we clearly see that inequality (2.10) can be rewritten as

$$\alpha_2 \leq 1 - f_2(n+a-1) < \beta_2. \tag{2.12}$$

Therefore, Theorem 2.6 follows easily from Lemma 2.2 and (2.12). □

Remark 2.1 We clearly see that both the upper and the lower bounds given in (2.10) for $\beta_n(a) - \gamma(a)$ are better than that given in (1.10) for $n \geq 3$ due to $1 - f_2(2) = 3 - 1/\sqrt{36 - 24(\gamma + \log 5 - \log 2)} = 0.466904841516\ldots$.

Theorem 2.7 *Let $\lambda_n(a)$ and $f_3(x)$ be, respectively, defined by* (1.3) *and* (2.5). *Then $\alpha_3 = 1 - f_3(a+5)$ and $\beta_3 = 277/360$ are the best possible constants such that the double inequality*

$$\frac{1}{48(a+n-\alpha_3)^3} \le \gamma(a) - \lambda_n(a) < \frac{1}{48(a+n-\beta_3)^3} \tag{2.13}$$

holds for all $a > 0$ and $n \ge 6$.

Proof From (1.3), (2.1) and (2.2) we have

$$\gamma(a) - \lambda_n(a) = \log\left(a + n - \frac{1}{2} + \frac{1}{24(a+n-1)}\right) - \psi(a+n). \tag{2.14}$$

It follows from (2.5) and (2.14) that inequality (2.13) can be rewritten as

$$\alpha_3 \le 1 - f_3(a+n-1) < \beta_3. \tag{2.15}$$

Therefore, Theorem 2.7 follows easily from Lemma 2.3 and (2.15). □

Theorem 2.8 *Let $\mu_n(a)$ and $f_4(x)$ be, respectively, defined by* (1.4) *and* (2.6). *Then $\alpha_4 = f_4(a)$ and $\beta_4 = 1/252$ are the best possible constants such that the double inequality*

$$\frac{\alpha_4}{(a+n-1)^6} \le \gamma(a) - \mu_n(a) < \frac{\beta_4}{(a+n-1)^6} \tag{2.16}$$

holds for all $a > 0$ and $n \ge 1$.

Proof It follows from (1.4), (2.1) and (2.2) that

$$\begin{aligned}
&\gamma(a) - \mu_n(a) \\
&\quad = \frac{1}{120(n+a-1)^4} \\
&\qquad - \left[\psi(n+a-1) - \log(n+a-1) + \frac{1}{2(n+a-1)} + \frac{1}{12(n+a-1)^2}\right].
\end{aligned} \tag{2.17}$$

From (2.6) and (2.17) we clearly see that inequality (2.16) is equivalent to

$$\alpha_4 \le f_4(n+a-1) < \beta_4. \tag{2.18}$$

Therefore, Theorem 2.8 follows easily from Lemma 2.4 and (2.18). □

Remark 2.2 Note that

$$\alpha_n(a) = y_n(a) - \frac{1}{2(a+n-1)}. \tag{2.19}$$

It follows from (1.4), Theorem 2.5, Theorem 2.8 and (2.19) that $\alpha_1 = 1 - f_1(a+2)$, $\beta_1 = 1$,

$\alpha_4 = f_4(a)$ and $\beta_4 = 1/252$ are the best possible constants such that the double inequalities

$$\begin{aligned} \frac{1}{2(a+n-1)} - \frac{1}{12(a+n-\beta_1)^2} &< y_n(a) - \gamma(a) \\ &\leq \frac{1}{2(a+n-1)} - \frac{1}{12(a+n-\alpha_1)^2}, \end{aligned} \tag{2.20}$$

$$\begin{aligned} &\frac{1}{2(a+n-1)} - \frac{1}{12(a+n-1)^2} + \frac{1}{120(a+n-1)^4} - \frac{\beta_4}{(a+n-1)^6} \\ &\quad < y_n(a) - \gamma(a) \\ &\quad \leq \frac{1}{2(a+n-1)} - \frac{1}{12(a+n-1)^2} + \frac{1}{120(a+n-1)^4} - \frac{\alpha_4}{(a+n-1)^6}, \end{aligned} \tag{2.21}$$

hold for all $a > 0$ and $n \geq 3$.

We clearly see that the two inequalities (2.20) and (2.21) are the improvements of the inequality (1.9) for $n \geq 3$.

Let $a = 1$ and

$$\begin{aligned} c_1 &= f_1(3) = 1/\sqrt{12(\gamma + \log 3) - 20} - 3 = 0.015998\ldots, \\ c_2 &= f_2(3) = 1/\sqrt{44 - 24(\gamma + \log 7 - \log 2)} - 3 = 0.5242567\ldots, \\ c_3 &= f_3(6) = -6 + 1/\sqrt[3]{48(\gamma - 49/20 + \log 937 - \log 144)} = 0.242347\ldots \end{aligned}$$

and

$$c_4 = f_4(1) = \gamma - 23/40 = 0.00221566\ldots.$$

Then

$$\begin{aligned} &\gamma(1) = \gamma, \qquad \alpha_n(1) = \gamma_n - \frac{1}{2n} = S_n, \qquad \beta_n(1) = R_n, \\ &\lambda_n(1) = T_n, \qquad \mu_n(1) = \gamma_n - \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4}. \end{aligned}$$

Therefore, Theorems 2.5–2.8 lead to Corollaries 2.1–2.5 immediately.

Corollary 2.1 *The double inequality*

$$\frac{1}{2n} - \frac{1}{12n^2} < \gamma_n - \gamma \leq \frac{1}{2n} - \frac{1}{12(n+c_1)^2} \tag{2.22}$$

holds for all $n \geq 3$.

Corollary 2.2 *The double inequality*

$$\frac{1}{12(n+c_1)^2} \leq \gamma - S_n < \frac{1}{12n^2} \tag{2.23}$$

holds for all $n \geq 3$.

Corollary 2.3 *The double inequality*

$$\frac{1}{24(n+c_2)^2} \le R_n - \gamma < \frac{1}{24(n+1/2)^2} \tag{2.24}$$

holds for all $n \ge 3$.

Corollary 2.4 *The double inequality*

$$\frac{1}{48(n+c_3)^2} \le \gamma - T_n < \frac{1}{48(n+83/360)^2} \tag{2.25}$$

holds for all $n \ge 6$.

Corollary 2.5 *The double inequality*

$$\frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} < \gamma_n - \gamma \le \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{c_4}{n^6} \tag{2.26}$$

holds for all $n \ge 1$.

Remark 2.3 We clearly see that the upper bound given in (2.22) is better than that given in (1.6) for $n \ge 3$ due to $n > \sqrt{12(\gamma-1/2)}c_1/(1-\sqrt{12(\gamma-1/2)}) = 0.4117\ldots$ is the solution of the inequality $1/[12(n+c_1)^2] > (\gamma-1/2)/n^2$, the lower bound given in (2.23) is better than that given in (1.8) for $n \ge 3$ due to $c_1 < 1\sqrt{12\gamma-6}-1 = 0.03885914\ldots$, both the upper and the lower bounds given in (2.24) are improvements of that given in (1.7) for $n \ge 3$, inequality (2.25) is stronger than inequality (1.5) for $n \ge 6$, the lower bound given in (2.26) is better than that given in (1.6) for $n \ge 1$, and the upper bound given in (2.26) is stronger than that given in (1.6) for $n \ge 2$ due to

$$n > \left(\frac{1+\sqrt{1-4800[1-12(\gamma-1/2)]c_4}}{20[1-12(\gamma-1/2)]}\right)^{1/2} = 1.00000000006823\ldots$$

being the solution of the inequality

$$\frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{c_4}{n^6} < \frac{1}{2n} - \frac{\gamma-1/2}{n^2}.$$

3 Results and discussion

As the natural generalization of the Euler–Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right) = 0.5772156649\ldots,$$

the generalized Euler–Mascheroni constant is defined by

$$\gamma(a) = \lim_{n\to\infty}\left[\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\log\left(\frac{a+n-1}{a}\right)\right]$$

for $a > 0$.

Recently, the evaluations for γ and $\gamma(a)$ have been the subject of intensive research. In the article, we provide several sharp upper and lower bounds for the generalized Euler–Mascheroni constant $\gamma(a)$. As applications, we improve some previously results on the Euler–Mascheroni constant γ. The idea presented may stimulate further research in the theory of special function.

4 Conclusion

In this paper, we present several best possible approximations for the generalized Euler–Mascheroni constant

$$\gamma(a) = \lim_{n\to\infty}\left[\frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \log\left(\frac{a+n-1}{a}\right)\right]$$

and improve some well-known bounds for the Euler–Mascheroni constant,

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.5772156649\ldots.$$

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Knopp, K.: Theory and Applications of Infinite Series. Dover Publications, New York (1990)
2. Chen, C.-P., Qi, F.: The best bounds of the *n*-th harmonic number. Glob. J. Appl. Math. Math. Sci. **1**(1), 41–49 (2008)
3. Niu, D.-W., Zhang, Y.-J., Qi, F.: A double inequality for the harmonic number in terms of the hyperbolic cosine. Turk. J. Anal. Number Theory **2**(6), 223–225 (2014)
4. Wang, M.-K., Chu, Y.-M.: Landen inequalities for a class of hypergeometric functions with applications. Math. Inequal. Appl. **21**(2), 521–537 (2018)
5. Wang, M.-K., Qiu, S.-L., Chu, Y.-M.: Infinite series formula for Hübner upper bound function with applications to Hersch–Pfluger distortion function. Math. Inequal. Appl. **21**(3), 629–648 (2018)
6. Alzer, H.: Inequalities for the gamma and polygamma functions. Abh. Math. Semin. Univ. Hamb. **68**, 363–372 (1998)
7. Mortici, C.: New approximations of the gamma function in terms of the digamma function. Appl. Math. Lett. **23**(1), 97–100 (2010)
8. Mortici, C.: Optimizing the rate of convergence in some new classes of sequences convergent to Euler's constant. Anal. Appl. **8**(1), 99–107 (2010)
9. Mortici, C.: Improved convergence towards generalized Euler–Mascheroni constant. Appl. Math. Comput. **215**(9), 3443–3448 (2010)
10. Chen, C.-P., Srivastava, H.M.: New representations for the Lugo and Euler–Mascheroni constants. Appl. Math. Lett. **24**(7), 1239–1244 (2011)
11. Negoi, T.: A faster convergence to the constant of Euler. Gaz. Mat., Ser. A **15**, 111–113 (1997)
12. Qiu, S.-L., Vuorinen, M.: Some properties of the gamma and psi functions, with applications. Math. Comput. **74**(250), 723–742 (2005)
13. DeTemple, D.W.: A geometric look at sequences that converge to Euler's constant. Coll. Math. J. **37**(2), 128–131 (2006)

14. Chen, C.-P.: The best bounds in Vernescu's inequalities for the Euler's constant. RGMIA Res. Rep. Collect. **12**(3), Article ID 11 (2009)
15. Sîntămărian, A.: A generalization of Euler's constant. Numer. Algorithms **46**(2), 141–151 (2007)
16. Berinde, V., Mortici, C.: New sharp estimates of the generalized Euler–Mascheroni constant. Math. Inequal. Appl. **16**(1), 279–288 (2013)
17. Chu, Y.-M., Wang, M.-K.: Optimal Lehmer mean bounds for the Toader mean. Results Math. **61**(3–4), 223–229 (2012). https://doi.org/10.1007/s00025-010-0090-9
18. Guo, B.-N., Qi, F.: Sharp inequalities for the psi function and harmonic numbers. Analysis **34**(2), 201–208 (2014)
19. Yang, Z.-H., Chu, Y.-M.: A monotonicity property involving generalized elliptic integral of the first kind. Math. Inequal. Appl. **20**(3), 729–735 (2017)
20. Yang, Z.-H., Chu, Y.-M., Zhang, X.-H.: Sharp Stolarsky mean bounds for the complete elliptic integral of the second kind. J. Nonlinear Sci. Appl. **10**(3), 929–936 (2017)
21. Yang, Z.-H., Zhang, W., Chu, Y.-M.: Sharp Gautsch inequality for parameter $0 < p < 1$ with applications. Math. Inequal. Appl. **20**(4), 1107–1120 (2017)
22. Guo, B.-N., Qi, F.: Sharp bounds for harmonic numbers. Appl. Math. Comput. **218**(3), 991–995 (2011)
23. Yang, Z.-H., Qian, W.-M., Chu, Y.-M., Zhang, W.: On approximating the error function. Math. Inequal. Appl. **21**(2), 469–479 (2018)
24. Yang, Z.-H., Qian, W.-M., Chu, Y.-M., Zhang, W.: On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind. J. Math. Anal. Appl. **462**(2), 1714–1726 (2018)
25. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. U. S. Government Printing Office, Washington (1964)
26. Chen, C.-P.: Inequalities for the Euler–Mascheroni constant. Appl. Math. Lett. **23**(2), 161–164 (2010)
27. Chen, C.-P.: Inequalities and monotonicity properties for some special functions. J. Math. Inequal. **3**(1), 79–91 (2009)
28. Chen, C.-P., Mortici, C.: Limits and inequalities associated with the Euler–Mascheroni constant. Appl. Math. Comput. **219**(18), 9755–9761 (2013). https://doi.org/10.1016/j.amc.2013.03.089
29. Qiu, S.-L., Zhao, X.: Some properties of the psi function and evaluations of γ. Appl. Math. J. Chin. Univ. Ser. A **31**(1), 103–111 (2016). https://doi.org/10.1007/s11766-016-3272-8

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