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Pointwise approximation of modified conjugate functions by matrix operators of conjugate Fourier series of $2\pi/r$-periodic functions

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Abstract

We extend the results of Xh. Z. Krasniqi (Acta Comment. Univ. Tartu Math. 17:89–101, 2013) and the authors (Acta Comment. Univ. Tartu Math. 13:11–24, 2009; Proc. Est. Acad. Sci. 67:50–60, 2018) to the case when considered function is $2\pi/r$-periodic and the measure of approximation depends on r-differences of the entries of the considered matrices.

MSC: 42A24

Keywords: Rate of approximation; Summability of Fourier series

1 Introduction

Let $L^p_{2\pi/r}$ $(1 \le p < \infty)$ be the class of all $2\pi/r$-periodic real-valued functions, integrable in the Lebesgue sense with the pth power over $Q_r = [-\pi/r, \pi/r]$ with the norm

$$\|f\|_{L^p_{2\pi/r}} = \|f(\cdot)\|_{L^p_{2\pi/r}} := \left(\int_{Q_r} |f(t)|^p \, dt \right)^{1/p},$$

where $r \in \mathbb{N}$. It is clear that $L^p_{2\pi/r} \subset L^p_{2\pi/1} = L^p_{2\pi}$ and for $f \in L^p_{2\pi/r}$

$$\|f\|_{L^p_{2\pi}} = r^{1/p} \|f\|_{L^p_{2\pi/r}}.$$

Taking into account the above relations, we will consider, for $f \in L^1_{2\pi/r}$, the trigonometric Fourier series as such a series of $f \in L^1_{2\pi}$ in the following form:

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} \bigl(a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x\bigr)$$

with the partial sums $S_k f$ and the conjugate one

$$\widetilde{S}f(x) := \sum_{\nu=1}^{\infty} \bigl(a_\nu(f) \sin \nu x - b_\nu(f) \cos \nu x\bigr)$$

with the partial sums $\widetilde{S}_k f$. We also know that if $f \in L^1_{2\pi}$, then

$$\widetilde{f}(x) := -\frac{1}{\pi}\int_0^{\pi} \psi_x(t)\frac{1}{2}\cot\frac{t}{2}\,dt = \lim_{\epsilon\to 0^+}\widetilde{f}(x,\epsilon) = \lim_{\epsilon\to 0^+}\widetilde{f}_r(x,\epsilon),$$

where, for $r \in \mathbb{N}$,

$$\widetilde{f}_r(x,\epsilon) := \begin{cases} -\frac{1}{\pi}(\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\epsilon}^{\frac{2(m+1)\pi}{r}-\epsilon} + \int_{\frac{2[r/2]\pi}{r}+\epsilon}^{\frac{(2[r/2]+1)\pi}{r}})\psi_x(t)\frac{1}{2}\cot\frac{t}{2}\,dt & \text{for an odd } r, \\ -\frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\epsilon}^{\frac{2(m+1)\pi}{r}-\epsilon}\psi_x(t)\frac{1}{2}\cot\frac{t}{2}\,dt & \text{for an even } r, \end{cases}$$

and

$$\widetilde{f}(x,\epsilon) = \widetilde{f}_1(x,\epsilon) := -\frac{1}{\pi}\int_{\epsilon}^{\pi}\psi_x(t)\frac{1}{2}\cot\frac{t}{2}\,dt,$$

with

$$\psi_x(t) := f(x+t) - f(x-t),$$

exist for almost all x (cf. [4, Th. (3.1) IV]).

Let $A := (a_{n,k})$ be an infinite matrix of real numbers such that

$$a_{n,k} \geq 0 \quad \text{when } k,n = 0,1,2,\ldots, \qquad \lim_{n\to\infty} a_{n,k} = 0 \quad \text{and} \quad \sum_{k=0}^{\infty} a_{n,k} = 1,$$

but $A^\circ := (a_{n,k})_{k=0}^{n}$, where

$$a_{n,k} = 0 \quad \text{when } k > n.$$

We will use the notations

$$A_{n,r} = \sum_{k=0}^{\infty}|a_{n,k} - a_{n,k+r}|, \qquad A^\circ_{n,r} = \sum_{k=0}^{n}|a_{n,k} - a_{n,k+r}|$$

for $r \in \mathbb{N}$ and

$$\widetilde{T}_{n,A}f(x) := \sum_{k=0}^{\infty} a_{n,k}\widetilde{S}_k f(x) \quad (n = 0,1,2,\ldots)$$

for the A-transformation of $\widetilde{S}f$.

In this paper, we will study the estimate of $|\widetilde{T}_{n,A}f(x) - \widetilde{f}_r(x,\epsilon)|$ by the function of modulus of continuity type, i.e. a nondecreasing continuous function $\widetilde{\omega}$ having the following properties: $\widetilde{\omega}(0) = 0$, $\widetilde{\omega}(\delta_1 + \delta_2) \leq \widetilde{\omega}(\delta_1) + \widetilde{\omega}(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$. We will also consider functions from the subclass $L^p_{2\pi/r}(\widetilde{\omega})_\beta$ of $L^p_{2\pi/r}$ for $r \in \mathbb{N}$:

$$L^p_{2\pi/r}(\widetilde{\omega})_\beta = \{f \in L^p_{2\pi/r} : \widetilde{\omega}_\beta(f,\delta)_{L^p_{2\pi/r}} = O(\widetilde{\omega}(\delta)) \text{ when } \delta \in [0,2\pi] \text{ and } \beta \geq 0\},$$

where

$$\widetilde{\omega}_\beta f(\delta)_{L^p_{2\pi/r}} = \sup_{0\le|t|\le\delta}\left\{\left|\sin\frac{rt}{2}\right|^\beta \left\|\psi_\cdot(t)\right\|_{L^p_{2\pi/r}}\right\}.$$

It is easy to see that $\widetilde{\omega}_0 f(\cdot)_{L^p_{2\pi/r}} = \widetilde{\omega} f(\cdot)_{L^p_{2\pi/r}}$ is the classical modulus of continuity. Moreover, it is clear that for $\beta \ge \alpha \ge 0$

$$\widetilde{\omega}_\beta f(\delta)_{L^p_{2\pi/r}} \le \widetilde{\omega}_\alpha f(\delta)_{L^p_{2\pi/r}}$$

and consequently

$$L^p_{2\pi/r}(\widetilde{\omega})_\alpha \subseteq L^p_{2\pi/r}(\widetilde{\omega})_\beta.$$

The deviation $\widetilde{T}_{n,A} f(x) - \widetilde{f}_r(x,\epsilon)$ was estimated with $r = 1$ in [2] and generalized in [1] as follows:

Theorem A ([1, Theorem 8, p. 95]) *If $f \in L^p_{2\pi}(\widetilde{\omega})_\beta$ with $1 < p < \infty$ and $0 \le \beta < 1 - \frac{1}{p}$, where $\widetilde{\omega}$ satisfies the conditions*:

$$\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left(\frac{t^{-\gamma}|\psi_x(t)|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}\,dt\right\}^{1/p} = O_x\bigl((n+1)^\gamma\bigr) \tag{1}$$

with $0 < \gamma < \beta + \frac{1}{p}$ *and*

$$\left\{\int_0^{\frac{\pi}{n+1}}\left(\frac{t|\psi_x(t)|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}\,dt\right\}^{1/p} = O_x\bigl((n+1)^{-1}\bigr), \tag{2}$$

then

$$\left|\widetilde{T}_{n,A^\circ} f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right)\right| = O_x\left((n+1)^{\beta+\frac{1}{p}+1} A^\circ_{n,1}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

The next essential generalizations and improvements in [3, Theorem 1] were given. In these results $\widetilde{f}_r(x,\epsilon)$ and $A_{n,r}$ (with $r \in \mathbb{N}$) instead of $\widetilde{f}_1(x,\epsilon) = \widetilde{f}(x,\epsilon)$ and $A^\circ_{n,1}$, respectively, were taken. We can formulate them as follows.

Theorem B ([3, Theorem 1]) *If $f \in L^p_{2\pi}$, $1 < p < \infty$, $0 \le \beta < 1 - \frac{1}{p}$ and a function $\widetilde{\omega}$ of modulus of continuity type satisfies the conditions*:

$$\left\{\int_0^{\frac{\pi}{r(n+1)}}\left(\frac{t|\psi_x(t)||\sin\frac{rt}{2}|^\beta}{\widetilde{\omega}(t)}\right)^p dt\right\}^{1/p} = O_x\bigl((n+1)^{-1}\bigr) \tag{3}$$

for $r \in \mathbb{N}$,

$$\left\{\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\left(\frac{|\psi_x(t)||\sin\frac{rt}{2}|^\beta}{\widetilde{\omega}(t-\frac{2m\pi}{r})}\right)^p dt\right\}^{1/p} = O_x(1) \tag{4}$$

for a natural $r \geq 3$, *where* $m \in \{1, \ldots [\frac{r}{2}]\}$ *when r is an odd or* $m \in \{1, \ldots [\frac{r}{2}] - 1\}$ *when r is an even natural number, and*

$$\left\{ \int_{\frac{2m\pi}{r} + \frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r} + \frac{\pi}{r}} \left(\frac{|\psi_x(t)| |\sin\frac{rt}{2}|^\beta}{\widetilde{\omega}(t)(t - \frac{2m\pi}{r})^\gamma} \right)^p dt \right\}^{1/p} = O_x\bigl((n+1)^\gamma\bigr), \tag{5}$$

for $r \in \mathbb{N}$ *with* $0 < \gamma < \beta + \frac{1}{p}$, *where* $m \in \{0, \ldots [\frac{r}{2}]\}$ *when r is an odd or* $m \in \{0, \ldots [\frac{r}{2}] - 1\}$ *when r is an even natural number. Moreover, let* $\widetilde{\omega}$ *satisfy, for a natural* $r \geq 2$, *the conditions*:

$$\left\{ \int_{\frac{2(m+1)\pi}{r} - \frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}} \left(\frac{|\psi_x(t)| |\sin\frac{rt}{2}|^\beta}{\widetilde{\omega}(\frac{2(m+1)\pi}{r} - t)} \right)^p dt \right\}^{1/p} = O_x(1), \tag{6}$$

$$\left\{ \int_{\frac{2(m+1)\pi}{r} - \frac{\pi}{r}}^{\frac{2(m+1)\pi}{r} - \frac{\pi}{r(n+1)}} \left(\frac{|\psi_x(t)| |\sin\frac{rt}{2}|^\beta}{\widetilde{\omega}(t)(\frac{2(m+1)\pi}{r} - t)^\gamma} \right)^p dt \right\}^{1/p} = O_x\bigl((n+1)^\gamma\bigr), \tag{7}$$

with $0 < \gamma < \beta + \frac{1}{p}$, *where* $m \in \{0, \ldots [\frac{r}{2}] - 1\}$. *If a matrix A is such that*

$$\sum_{k=0}^{\infty} (k+1)^2 a_{n,k} = O\bigl((n+1)^2\bigr) \tag{8}$$

and

$$\left[\sum_{l=0}^{n} \sum_{k=l}^{r+l-1} a_{n,k} \right]^{-1} = O(1) \tag{9}$$

with $r \in \mathbb{N}$ *are true, then*

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}_r\left(x, \frac{\pi}{r(n+1)}\right) \right| = O_x\left((n+1)^{\beta + \frac{1}{p} + 1} A_{n,r} \widetilde{\omega}\left(\frac{\pi}{n+1} \right) \right).$$

Theorem C ([3, Theorem 2]) *Let* $f \in L^p_{2\pi}$, $1 < p < \infty$, $0 \leq \beta < 1 - \frac{1}{p}$ *and a function* $\widetilde{\omega}$ *of modulus of continuity type satisfy, for* $r \in \mathbb{N}$, *the conditions*: (4) *and* (5) *with* $0 < \gamma < \beta + \frac{1}{p}$, *where* $m \in \{0, \ldots [\frac{r}{2}]\}$ *when r is an odd or* $m \in \{0, \ldots [\frac{r}{2}] - 1\}$ *when r is an even natural number. Moreover, let* $\widetilde{\omega}$ *satisfy, for a natural* $r \geq 2$, *the conditions* (6) *and* (7) *with* $0 < \gamma < \beta + \frac{1}{p}$, *where* $m \in \{0, \ldots [\frac{r}{2}] - 1\}$. *If a matrix A is such that*

$$\sum_{k=0}^{\infty} (k+1) a_{n,k} = O(n+1), \tag{10}$$

and (9) *with* $r \in \mathbb{N}$ *are true, then*

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}_r\left(x, \frac{\pi}{r(n+1)}\right) \right| = O_x\left((n+1)^{\beta + \frac{1}{p} + 1} A_{n,r} \widetilde{\omega}\left(\frac{\pi}{n+1} \right) \right).$$

In our theorems we generalize the above results considering $2\pi/r$-periodic functions and using simpler assumptions.

In the paper $\sum_{k=a}^{b} = 0$ when $a > b$.

2 Statement of the results

To begin with, we will present the estimates of the quantities

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{r(n+1)}\right)\right| \quad \text{and} \quad \left\|\widetilde{T}_{n,A}f(\cdot)-\widetilde{f}_r\left(\cdot,\frac{\pi}{r(n+1)}\right)\right\|_{L^p_{2\pi/r}}.$$

Finally, we will formulate some remarks and corollaries.

Theorem 1 *Suppose that* $f\in L^p_{2\pi/r}$, $1<p<\infty$, $r\in\mathbb{N}$, $0\le\beta<1-\frac{1}{p}$ *and a function* $\widetilde{\omega}$ *of the modulus of continuity type satisfies the conditions*:

$$\left\{\int_0^{\frac{\pi}{r(n+1)}}\left(\frac{t|\psi_x(t)||\sin\frac{rt}{2}|^\beta}{\widetilde{\omega}(t)}\right)^p dt\right\}^{1/p}=O_x\left((n+1)^{-1}\right), \tag{11}$$

when $r=1$ *or*

$$\left\{\int_0^{\frac{\pi}{r(n+1)}}\left(\frac{|\psi_x(t)||\sin\frac{rt}{2}|^\beta}{\widetilde{\omega}(t)}\right)^p dt\right\}^{1/p}=O_x(1), \tag{12}$$

when $r\ge 2$, *and*

$$\left\{\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\left(\frac{|\psi_x(t)||\sin\frac{rt}{2}|^\beta}{\widetilde{\omega}(t)t^\gamma}\right)^p dt\right\}^{1/p}=O_x\left((n+1)^\gamma\right), \tag{13}$$

for $r\in\mathbb{N}$ *with* $0<\gamma<\beta+\frac{1}{p}$. *If a matrix* A *is such that* (8) *and* (9) *are true, then*

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{r(n+1)}\right)\right|=O_x\left((n+1)^{\beta+\frac{1}{p}+1}A_{n,r}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

Theorem 2 *Suppose that* $f\in L^p_{2\pi/r}$, $1<p<\infty$, $r\in\mathbb{N}$, $0\le\beta<1-\frac{1}{p}$ *and a function* $\widetilde{\omega}$ *of the modulus of continuity type satisfies the conditions* (12) *and* (13) *for* $r\in\mathbb{N}$ *with* $0<\gamma<\beta+\frac{1}{p}$. *If a matrix* A *is such that* (10) *and* (9) *are true, then*

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{r(n+1)}\right)\right|=O_x\left((n+1)^{\beta+\frac{1}{p}+1}A_{n,r}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

Remark 1 The Hölder inequality gives

$$\begin{aligned}\sum_{k=0}^{\infty}(k+1)a_{n,k}&=\sum_{k=0}^{\infty}(k+1)a_{n,k}^{1/2}a_{n,k}^{1/2}\le\left[\sum_{k=0}^{\infty}(k+1)^2a_{n,k}\right]^{1/2}\left[\sum_{k=0}^{\infty}a_{n,k}\right]^{1/2}\\&=\left[\sum_{k=0}^{\infty}(k+1)^2a_{n,k}\right]^{1/2}\end{aligned}$$

and thus the condition (8) implies (10), but the condition (12) implies (11). Therefore Theorems 1 and 2 are not comparable.

Theorem 3 *Let $f \in L^p_{2\pi/r}(\widetilde{\omega})_\beta$, $1 < p < \infty$, $r \in \mathbb{N}$ and $0 \leq \beta < 1 - \frac{1}{p}$. If a matrix A is such that* (9) *and* (8) *or* (10) *are true, then*

$$\left\| \widetilde{T}_{n,A} f(\cdot) - \widetilde{f}_r\left(\cdot, \frac{\pi}{r(n+1)}\right)\right\|_{L^p_{2\pi/r}} = O_x\left((n+1)^{\beta+\frac{1}{p}+1} A_{n,r} \widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

Corollary 1 *Taking $r = 1$ the conditions* (11) *and* (13) *in Theorem* 1 *reduce to* (1) *and* (2). *Thus we obtain the results from* [2] *and Theorem A* [1, Theorem 8, p. 95], *but in the case of* [3] (*Theorem B and C*) *we reduce the assumptions*.

Next, using more natural conditions when $\beta > 0$ we can formulate, without proofs, the following theorems.

Theorem 4 *Suppose that $f \in L^p_{2\pi/r}$, $1 < p < \infty$, $r \in \mathbb{N}$, $0 < \beta < 1 - \frac{1}{p}$. Let a function $\widetilde{\omega}$ of the modulus of continuity type satisfy the conditions*:

$$\left\{ \int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}} \left(\frac{t^{-\gamma} |\psi_x(t)| |\sin \frac{rt}{2}|^\beta}{\widetilde{\omega}(t)} \right)^p dt \right\}^{1/p} = O_x\left((n+1)^{\gamma - \frac{1}{p}}\right) \tag{14}$$

for $\gamma \in (\frac{1}{p}, \frac{1}{p} + \beta)$ and $r \in \mathbb{N}$ (*instead of* (13)), *and*

$$\left\{ \int_0^{\frac{\pi}{r(n+1)}} \left(\frac{t |\psi_x(t)| |\sin \frac{rt}{2}|^\beta}{\widetilde{\omega}(t)} \right)^p dt \right\}^{1/p} = O_x\left((n+1)^{-1-\frac{1}{p}}\right)$$

when $r = 1$ or

$$\left\{ \int_0^{\frac{\pi}{r(n+1)}} \left(\frac{|\psi_x(t)| |\sin \frac{rt}{2}|^\beta}{\widetilde{\omega}(t)} \right)^p dt \right\}^{1/p} = O_x\left((n+1)^{-\frac{1}{p}}\right) \tag{15}$$

when $r \geq 2$ (*instead of* (11) *and* (12), *respectively*). *If a matrix A is such that* (9) *and* (8) *are true, then*

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}_r\left(x, \frac{\pi}{r(n+1)}\right)\right| = O_x\left((n+1)^{\beta+1} A_{n,r} \widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right). \tag{16}$$

Moreover, if a function $\widetilde{\omega}$ of the modulus of continuity type and a matrix A satisfy the following conditions: (14) *with $r \in \mathbb{N}$ and $\gamma \in (\frac{1}{p}, \frac{1}{p} + \beta)$*, (15) *with $r \in \mathbb{N}$*, (9) *and* (10), *then the estimate* (16) *is also true.*

Theorem 5 *Let $f \in L^p_{2\pi/r}(\widetilde{\omega})_\beta$ with $1 < p < \infty$, $r \in \mathbb{N}$ and $0 < \beta < 1 - \frac{1}{p}$. If a matrix A is such that* (9) *and* (8) *or* (10) *are true, then*

$$\left\| \widetilde{T}_{n,A} f(\cdot) - \widetilde{f}_r\left(\cdot, \frac{\pi}{r(n+1)}\right)\right\|_{L^p_{2\pi/r}} = O_x\left((n+1)^{\beta+1} A_{n,r} \widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

Remark 2 We note that our extra conditions (9), (8) and (10) for a lower triangular infinite matrix A° always hold.

Corollary 2 *Considering the above remarks and the obvious inequality*

$$A_{n,r} \leq rA_{n,1} \quad \textit{for } r \in \mathbb{N} \tag{17}$$

our results also improve and generalize the mentioned result of Krasniqi [1].

Remark 3 We note that instead of $L^p_{2\pi/r}(\widetilde{\omega})_\beta$ one can consider another subclass of $L^p_{2\pi/r}$ generated by any function of the modulus of continuity type e.g. $\widetilde{\omega}_x$ such that

$$\widetilde{\omega}_x(f,\delta) = \sup_{|t|\leq\delta}\left|\psi_x(t)\right| \leq \widetilde{\omega}_x(\delta)$$

or

$$\widetilde{\omega}_x(f,\delta) = \frac{1}{\delta}\int_0^\delta \left|\psi_x(t)\right|dt \leq \widetilde{\omega}_x(\delta).$$

3 Auxiliary results

We begin this section by some notations from [5] and [4, Sect. 5 of Chapter II]. Let for $r = 1, 2, \ldots$

$$D^\circ_{k,r}(t) = \frac{\sin\frac{(2k+r)t}{2}}{2\sin\frac{rt}{2}}, \qquad \widetilde{D^\circ}_{k,r}(t) = \frac{\cos\frac{(2k+r)t}{2}}{2\sin\frac{rt}{2}}$$

and

$$\widetilde{D}_{k,r}(t) = \frac{\cos\frac{rt}{2} - \cos\frac{(2k+r)t}{2}}{2\sin\frac{rt}{2}} = \frac{\cos\frac{rt}{2}}{2\sin\frac{rt}{2}} - \widetilde{D^\circ}_{k,r}(t).$$

It is clear by [4] that

$$\widetilde{S}_kf(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\widetilde{D}_{k,1}(t)\,dt$$

and

$$\widetilde{T}_{n,A}f(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\sum_{k=0}^{\infty} a_{n,k}\widetilde{D}_{k,1}(t)\,dt.$$

Now, we present a very useful property of the modulus of continuity.

Lemma 1 ([4]) *A function* $\widetilde{\omega}$ *of modulus of continuity type on the interval* $[0, 2\pi]$ *satisfies the following condition*:

$$\delta_2^{-1}\widetilde{\omega}(\delta_2) \leq 2\delta_1^{-1}\widetilde{\omega}(\delta_1) \quad \textit{for } \delta_2 \geq \delta_1 > 0.$$

Next, we present the following well-known estimates.

Lemma 2 ([4]) *If* $0 < |t| \leq \pi$ *then*

$$\left|\widetilde{D^\circ}_{k,1}(t)\right| \leq \frac{\pi}{2|t|}, \qquad \left|\widetilde{D}_{k,1}(t)\right| \leq \frac{\pi}{|t|}$$

and, for any real t, we have

$$\left|D^{\circ}_{k,1}(t)\right| \le k+\frac{1}{2}, \qquad \left|\widetilde{D}_{k,1}(t)\right| \le \frac{1}{2}k(k+1)|t|, \qquad \left|\widetilde{D}_{k,1}(t)\right| \le k+1.$$

Lemma 3 ([5, 6]) *Let $r \in N$, $l \in Z$ and $(a_n) \subset \mathbb{C}$. If $t \neq \frac{2l\pi}{r}$, then for every $m \ge n$*

$$\sum_{k=n}^{m} a_k \sin kt = -\sum_{k=n}^{m}(a_k - a_{k+r})\widetilde{D^{\circ}}_{k,r}(t) + \sum_{k=m+1}^{m+r} a_k\widetilde{D^{\circ}}_{k,-r}(t) - \sum_{k=n}^{n+r-1} a_k\widetilde{D^{\circ}}_{k,-r}(t),$$

$$\sum_{k=n}^{m} a_k \cos kt = \sum_{k=n}^{m}(a_k - a_{k+r})D^{\circ}_{k,r}(t) - \sum_{k=m+1}^{m+r} a_k D^{\circ}_{k,-r}(t) + \sum_{k=n}^{n+r-1} a_k D^{\circ}_{k,-r}(t).$$

We additionally need the following estimate as a consequence of Lemma 3.

Lemma 4 *Let $r \in \mathbb{N}$, $l \in \mathbb{Z}$ and $(a_{n,k}) \subset \mathbb{R}^+_0$ for $n.k \in \mathbb{N}_0$. If $t \neq \frac{2l\pi}{r}$, then*

$$\left|\frac{1}{2}\sum_{k=0}^{\infty} a_{n,k}\cos\frac{(2k+1)t}{2}\right| \le \frac{1}{2|\sin\frac{rt}{2}|}\left(A_{n,r} + \sum_{k=0}^{r-1} a_{n,k}\right) \le \frac{1}{|\sin\frac{rt}{2}|}A_{n,r}.$$

Proof By Lemma 3,

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{\infty} a_{n,k}\cos\frac{(2k+1)t}{2}\\
&\quad= \frac{1}{2}\left(\sum_{k=0}^{\infty} a_{n,k}\cos kt\cos\frac{t}{2} - \sum_{k=0}^{\infty} a_{n,k}\sin kt\sin\frac{t}{2}\right)\\
&\quad= \frac{\cos\frac{t}{2}}{2}\left(\sum_{k=0}^{\infty}(a_{n,k}-a_{n,k+r})D^{\circ}_{k,r}(t) + \sum_{k=0}^{r-1} a_{n,k}D^{\circ}_{k,-r}(t)\right)\\
&\qquad - \frac{\sin\frac{t}{2}}{2}\left(-\sum_{k=0}^{\infty}(a_{n,k}-a_{n,k+r})\widetilde{D^{\circ}}_{k,r}(t) - \sum_{k=0}^{r-1} a_{n,k}\widetilde{D^{\circ}}_{k,-r}(t)\right)
\end{aligned}$$

and our inequalities follow. □

We also need some special conditions which follow from the ones mentioned above.

Lemma 5 *Suppose that $f \in L^p_{2\pi/r}$, where $1 \le p < \infty$ and $r \in \mathbb{N}$. If the condition* (12) *holds with any function $\widetilde{\omega}$ of the modulus of continuity type and $\beta \ge 0$, then*

$$\left\{\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{|\psi_x(t)|}{\widetilde{\omega}(\frac{2(m+1)\pi}{r}-t)}\right)^p\left|\sin\frac{rt}{2}\right|^{\beta p}dt\right\}^{\frac{1}{p}} = O_x(1),$$

where $m \in \{0,\dots[\frac{r}{2}]-1\}$.

Proof By the substitution $t = \frac{2(m+1)\pi}{r} - u$, we obtain

$$\left\{\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{|\psi_x(t)|}{\widetilde{\omega}(\frac{2(m+1)\pi}{r}-t)}\right)^p\left|\sin\frac{rt}{2}\right|^{\beta p}dt\right\}^{1/p}$$

$$= \left\{ \int_0^{\frac{\pi}{r(n+1)}} \left(\frac{|\psi_x(\frac{2(m+1)\pi}{r} - u)|}{\widetilde{\omega}(u)} \left| \sin \frac{r}{2}\left(\frac{2(m+1)\pi}{r} - u \right) \right|^{\beta} \right)^p du \right\}^{1/p}$$
$$= \left\{ \int_0^{\frac{\pi}{r(n+1)}} \left(\frac{|\psi_x(u)|}{\widetilde{\omega}(u)} \left| \sin \frac{ru}{2} \right|^{\beta} \right)^p du \right\}^{1/p}.$$

Hence, by (12) our estimate follows. □

Lemma 6 *Suppose that* $f \in L^p_{2\pi/r}$, *where* $1 \le p < \infty$ *and* $r \in \mathbb{N}$. *If the condition* (12) *holds with any function* $\widetilde{\omega}$ *of the modulus of continuity type and* $\beta \ge 0$, *then*

$$\left\{ \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} \left(\frac{|\psi_x(t)|}{\widetilde{\omega}(t - \frac{2m\pi}{r})} \right)^p \left| \sin \frac{rt}{2} \right|^{\beta p} dt \right\}^{\frac{1}{p}} = O_x(1),$$

where $m \in \{0, \dots [\frac{r}{2}]\}$.

Proof By the substitution $t = \frac{2m\pi}{r} + u$, analogously to the above proof, we obtain

$$\left\{ \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} \left(\frac{|\psi_x(t)|}{\widetilde{\omega}(t - \frac{2m\pi}{r})} \right)^p \left| \sin \frac{rt}{2} \right|^{\beta p} dt \right\}^{1/p}$$
$$= \left\{ \int_0^{\frac{\pi}{r(n+1)}} \left(\frac{|\psi_x(\frac{2m\pi}{r} + u)|}{\widetilde{\omega}(u)} \left| \sin \frac{r}{2}\left(\frac{2m\pi}{r} + u \right) \right|^{\beta} \right)^p du \right\}^{1/p}$$
$$\le \left\{ \int_0^{\frac{\pi}{r(n+1)}} \left(\frac{|\psi_x(u)|}{\widetilde{\omega}(u)} \left| \sin \frac{ru}{2} \right|^{\beta} \right)^p dt \right\}^{1/p} = O_x(1)$$

and we have the desired estimate. □

Now, we formulate another two lemmas without proofs. We can prove them in the same way as Lemmas 5 and 6, respectively.

Lemma 7 *Suppose that* $f \in L^p_{2\pi/r}$, *where* $1 \le p < \infty$ *and* $r \in \mathbb{N}$. *If the condition* (13) *holds with any function* $\widetilde{\omega}$ *of the modulus of continuity type and* $\gamma, \beta \ge 0$, *then*

$$\left\{ \int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}} \left(\frac{|\psi_x(t)| |\sin \frac{rt}{2}|^{\beta}}{\widetilde{\omega}(t)(\frac{2(m+1)\pi}{r} - t)^{\gamma}} \right)^p dt \right\}^{1/p} = O_x\big((n+1)^{\gamma}\big),$$

where $m \in \{0, \dots [\frac{r}{2}] - 1\}$.

Lemma 8 *Suppose that* $f \in L^p_{2\pi/r}$, *where* $1 \le p < \infty$ *and* $r \in \mathbb{N}$. *If the condition* (13) *holds with any function* $\widetilde{\omega}$ *of the modulus of continuity type and* $\gamma, \beta \ge 0$, *then*

$$\left\{ \int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} \left(\frac{|\psi_x(t)| |\sin \frac{rt}{2}|^{\beta}}{\widetilde{\omega}(t)(t - \frac{2m\pi}{r})^{\gamma}} \right)^p dt \right\}^{1/p} = O_x\big((n+1)^{\gamma}\big),$$

where $m \in \{0, \dots [\frac{r}{2}]\}$.

4 Proofs of theorems

4.1 Proof of Theorem 1

It is clear that for odd r

$$\begin{aligned}
&\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{r(n+1)}\right)\\
&=-\frac{1}{\pi}\int_0^{\pi}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D}_{k,1}(t)\,dt\\
&\quad+\frac{1}{\pi}\left(\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}+\int_{\frac{2[r/2]\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{(2[r/2]+1)\pi}{r}}\right)\psi_x(t)\frac{1}{2}\cot\frac{t}{2}\,dt\\
&=-\frac{1}{\pi}\left(\int_0^{\frac{\pi}{r(n+1)}}+\sum_{m=1}^{[r/2]}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}+\sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\right)\\
&\quad\times\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D}_{k,1}(t)\,dt\\
&\quad+\frac{1}{\pi}\left(\sum_{m=0}^{[r/2]}\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}+\sum_{m=0}^{[r/2]-1}\int_{\frac{(2m+1)\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\right)\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)\,dt\\
&=I_0(x)+I_1(x)+I_2(x)+I_3(x)+I_4(x)
\end{aligned}$$

and for even r

$$\begin{aligned}
&\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{r(n+1)}\right)\\
&=-\frac{1}{\pi}\int_0^{\pi}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D}_{k,1}(t)\,dt+\frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\psi_x(t)\frac{1}{2}\cot\frac{t}{2}\,dt\\
&=-\frac{1}{\pi}\left(\int_0^{\frac{\pi}{r(n+1)}}+\sum_{m=1}^{[r/2]-1}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}+\sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\right)\\
&\quad\times\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D}_{k,1}(t)\,dt\\
&\quad+\frac{1}{\pi}\left(\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{(2m+1)\pi}{r}}+\sum_{m=0}^{[r/2]-1}\int_{\frac{(2m+1)\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\right)\\
&\quad\times\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)\,dt\\
&=I_0(x)+I_1'(x)+I_2(x)+I_3'(x)+I_4(x),
\end{aligned}$$

whence

$$\begin{aligned}
&\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{r(n+1)}\right)\right|\\
&\quad\le\left|I_0(x)\right|+\left|I_1(x)\right|+\left|I_1'(x)\right|+\left|I_2(x)\right|+\left|I_3(x)\right|+\left|I_3'(x)\right|+\left|I_4(x)\right|.
\end{aligned}$$

Next, using Lemma 2, (8), the Hölder inequality with $p > 1$ and $q = \frac{p}{p-1}$ and (11) when $r = 1$ or (12) when $r \geq 2$ we get

$$\begin{aligned}
&\left|I_0(x)\right| \\
&= O\bigl((n+1)^2\bigr)\int_0^{\frac{\pi}{r(n+1)}} t\left|\psi_x(t)\right| dt \\
&\leq O\bigl((n+1)^2\bigr)\left\{\int_0^{\frac{\pi}{r(n+1)}}\left(\frac{t|\psi_x(t)|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{rt}{2}\,dt\right\}^{1/p}\left\{\int_0^{\frac{\pi}{r(n+1)}}\left(\frac{\widetilde{\omega}(t)}{\sin^\beta\frac{rt}{2}}\right)^q dt\right\}^{\frac{1}{q}} \\
&\leq O\bigl((n+1)^2\bigr)O_x\bigl((n+1)^{-1}\bigr)\widetilde{\omega}\left(\frac{\pi}{r(n+1)}\right)\left\{\int_0^{\frac{\pi}{r(n+1)}}\left(\frac{\pi}{rt}\right)^{\beta q} dt\right\}^{\frac{1}{q}} \\
&= O_x\bigl((n+1)\bigr)\widetilde{\omega}\left(\frac{\pi}{r(n+1)}\right)\left(\frac{\pi}{r(n+1)}\right)^{\frac{1}{q}-\beta} = O_x\bigl((n+1)^{\beta+\frac{1}{p}}\bigr)\widetilde{\omega}\left(\frac{\pi}{n+1}\right),
\end{aligned}$$

for $0 \leq \beta < 1 - \frac{1}{p}$. We note that applying the condition (9) we have

$$\begin{aligned}
\left[(n+1)A_{n,r}\right]^{-1} &= \left[\sum_{l=0}^{n} A_{n,r}\right]^{-1} \leq \left[\sum_{l=0}^{n}\sum_{k=l}^{\infty}|a_{n,k}-a_{n,k+r}|\right]^{-1} \\
&\leq \left[\sum_{l=0}^{n}\left|\sum_{k=l}^{\infty}(a_{n,k}-a_{n,k+r})\right|\right]^{-1} = \left[\sum_{l=0}^{n}\sum_{k=l}^{r+l-1}a_{n,k}\right]^{-1} = O(1),
\end{aligned}$$

whence

$$\left|I_0(x)\right| = O_x\left((n+1)^{1+\beta+\frac{1}{p}}A_{n,r}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

By Lemma 2

$$\begin{aligned}
&\left|I_1(x)\right| + \left|I_1'(x)\right| + \left|I_2(x)\right| \\
&\leq \frac{1}{\pi}\left(\sum_{m=1}^{[r/2]}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} + \sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\right)\frac{|\psi_x(t)|}{t}\,dt \\
&\leq \frac{1}{\pi}\left(\sum_{m=1}^{[r/2]}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} + \sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\right)\frac{|\psi_x(t)|}{\pi/r}\,dt
\end{aligned}$$

and using the Hölder inequality with $p > 1$ and $q = \frac{p}{p-1}$

$$\begin{aligned}
&\left|I_1(x)\right| + \left|I_1'(x)\right| + \left|I_2(x)\right| \\
&\leq O_x(1)\sum_{m=1}^{[r/2]}\left[\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\left(\frac{|\psi_x(t)|\sin^\beta\frac{rt}{2}}{\widetilde{\omega}(t-\frac{2m\pi}{r})}\right)^p dt\right]^{\frac{1}{p}} \\
&\quad\times\left[\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\left(\frac{\widetilde{\omega}(t-\frac{2m\pi}{r})}{\sin^\beta\frac{rt}{2}}\right)^q dt\right]^{\frac{1}{q}} \\
&\quad+ O_x(1)\sum_{m=1}^{[r/2]-1}\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{|\psi_x(t)|\sin^\beta\frac{rt}{2}}{\widetilde{\omega}(\frac{2(m+1)\pi}{r}-t)}\right)^p dt\right]^{\frac{1}{p}}
\end{aligned}$$

$$\times \left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}} \left(\frac{\widetilde{\omega}(\frac{2(m+1)\pi}{r}-t)}{\sin^{\beta}\frac{rt}{2}} \right)^{q} dt \right]^{\frac{1}{q}}.$$

Hence, by Lemmas 5 and 6 with (12) and (9),

$$\begin{aligned}
&\left|I_1(x)\right| + \left|I_1'(x)\right| + \left|I_2(x)\right| \\
&\quad = O_x(1)\widetilde{\omega}\left(\frac{\pi}{r(n+1)}\right)\left[\int_0^{\frac{\pi}{r(n+1)}} \left(\frac{1}{\sin^{\beta}\frac{rt}{2}}\right)^{q} dt\right]^{\frac{1}{q}} \\
&\quad = O_x\left((n+1)^{\beta-\frac{1}{q}}\right)\widetilde{\omega}\left(\frac{\pi}{n+1}\right) = O_x\left((n+1)^{\beta+\frac{1}{p}} A_{n,r}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right),
\end{aligned}$$

for $0 \le \beta < 1 - \frac{1}{p}$.

In the case of the last integrals, applying Lemma 4 we obtain

$$\begin{aligned}
&\left|I_3(x)\right| + \left|I_3'(x)\right| + \left|I_4(x)\right| \\
&\quad \le \frac{1}{\pi}\left(\sum_{m=0}^{[r/2]} \int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{(2m+1)\pi}{r}} + \sum_{m=0}^{[r/2]-1} \int_{\frac{(2m+1)\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\right) \frac{|\psi_x(t)|}{|\sin\frac{t}{2}\sin\frac{rt}{2}|} A_{n,r}\,dt.
\end{aligned}$$

Using the estimates $|\sin\frac{t}{2}| \ge \frac{|t|}{\pi}$ for $t \in [0,\pi]$, $|\sin\frac{rt}{2}| \ge \frac{rt}{\pi} - 2m$ for $t \in [\frac{2m\pi}{r} + \frac{\pi}{r(n+1)}, \frac{(2m+1)\pi}{r}]$, where $m \in \{0,\ldots,[r/2]\}$ and $|\sin\frac{rt}{2}| \ge 2(m+1) - \frac{rt}{\pi}$ for $t \in [\frac{(2m+1)\pi}{r}, \frac{2(m+1)\pi}{r} - \frac{\pi}{r(n+1)}]$, where $m \in \{0,\ldots,[r/2]-1\}$, we obtain

$$\begin{aligned}
&\left|I_3(x)\right| + \left|I_3'(x)\right| + \left|I_4(x)\right| \\
&\quad \le A_{n,r}\sum_{m=0}^{[r/2]} \int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{(2m+1)\pi}{r}} \frac{|\psi_x(t)|}{\frac{rt}{\pi}(t-\frac{2m\pi}{r})}\,dt \\
&\qquad + A_{n,r}\sum_{m=0}^{[r/2]-1} \int_{\frac{(2m+1)\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}} \frac{|\psi_x(t)|}{\frac{rt}{\pi}[\frac{2(m+1)\pi}{r}-t]}\,dt.
\end{aligned}$$

By the Hölder inequality with $p > 1$ and $q = \frac{p}{p-1}$ we have

$$\begin{aligned}
&\left|I_3(x)\right| + \left|I_3'(x)\right| + \left|I_4(x)\right| \\
&\quad \le \frac{\pi}{r}A_{n,r}\sum_{m=0}^{[r/2]}\left[\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} \left(\frac{|\psi_x(t)|}{\widetilde{\omega}(t)(t-\frac{2m\pi}{r})^{\gamma}}\left|\sin\frac{rt}{2}\right|^{\beta}\right)^{p} dt\right]^{\frac{1}{p}} \\
&\qquad \times \left[\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} \left(\frac{\widetilde{\omega}(t)(t-\frac{2m\pi}{r})^{\gamma}}{t(t-\frac{2m\pi}{r})|\sin\frac{rt}{2}|^{\beta}}\right)^{q} dt\right]^{\frac{1}{q}} \\
&\qquad + \frac{\pi}{r}A_{n,r}\sum_{m=0}^{[r/2]-1}\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}} \left(\frac{|\psi_x(t)|}{\widetilde{\omega}(t)(\frac{2(m+1)\pi}{r}-t)^{\gamma}}\left|\sin\frac{rt}{2}\right|^{\beta}\right)^{p} dt\right]^{\frac{1}{p}} \\
&\qquad \times \left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}} \left(\frac{\widetilde{\omega}(t)(\frac{2(m+1)\pi}{r}-t)^{\gamma}}{t(\frac{2(m+1)\pi}{r}-t)|\sin\frac{rt}{2}|^{\beta}}\right)^{q} dt\right]^{\frac{1}{q}}.
\end{aligned}$$

Further, using Lemmas 7 and 8 with (13) and Lemma 1 we get

$$
\begin{aligned}
&\left|I_3(x)\right| + \left|I_3'(x)\right| + \left|I_4(x)\right| \\
&\le O_x(1)A_{n,r}\sum_{m=0}^{[r/2]}(n+1)^{\gamma}\left[\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\left(\frac{\widetilde{\omega}(t)(t-\frac{2m\pi}{r})^{\gamma}}{t(t-\frac{2m\pi}{r})|\sin\frac{rt}{2}|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}} \\
&\quad + O_x(1)A_{n,r}\sum_{m=0}^{[r/2]-1}(n+1)^{\gamma}\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\left(\frac{\widetilde{\omega}(t)(\frac{2(m+1)\pi}{r}-t)^{\gamma}}{t(\frac{2(m+1)\pi}{r}-t)|\sin\frac{rt}{2}|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}} \\
&= O_x(1)A_{n,r}\left[\sum_{m=0}^{[r/2]}(n+1)^{\gamma}\left\{\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\left(\frac{\widetilde{\omega}(t+\frac{2m\pi}{r})t^{\gamma-1}}{(t+\frac{2m\pi}{r})|\sin\frac{rt}{2}|^{\beta}}\right)^{q}dt\right\}^{\frac{1}{q}}\right. \\
&\quad \left. + \sum_{m=0}^{[r/2]-1}(n+1)^{\gamma}\left\{\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\left(\frac{\widetilde{\omega}(\frac{2(m+1)\pi}{r}-t)t^{\gamma-1}}{(\frac{2(m+1)\pi}{r}-t)|\sin\frac{rt}{2}|^{\beta}}\right)^{q}dt\right\}^{\frac{1}{q}}\right] \\
&= O_x(1)A_{n,r}(n+1)^{\gamma}\left\{\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\left(\frac{\widetilde{\omega}(t)t^{\gamma-1}}{t|\sin\frac{rt}{2}|^{\beta}}\right)^{q}dt\right\}^{\frac{1}{q}} \\
&= O_x(1)A_{n,r}(n+1)^{1+\gamma}\widetilde{\omega}\left(\frac{\pi}{r(n+1)}\right)\left(\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}t^{(\gamma-1-\beta)q}\,dt\right)^{\frac{1}{q}} \\
&= O_x(1)A_{n,r}(n+1)^{1+\gamma}\widetilde{\omega}\left(\frac{\pi}{r(n+1)}\right)(n+1)^{1+\beta-\gamma-\frac{1}{q}} \\
&= O_x\left((n+1)^{1+\beta+\frac{1}{p}}A_{n,r}\widetilde{\omega}\left(\frac{\pi}{(n+1)}\right)\right)
\end{aligned}
$$

for $0<\gamma<\beta+\frac{1}{p}$.

Collecting the partial estimates our statement follows.

4.2 Proof of Theorem 2

The proof is the same as above, but for estimate of $|I_0(x)|$ we only used the inequality $|\widetilde{D}_{k,1}(t)|\le k+1$ from Lemma 2, and the condition (10) instead of (8).

4.3 Proof of Theorem 3

We note that for the estimate of $\|\widetilde{T}_{n,A}f(\cdot)-\widetilde{f}_r(\cdot,\frac{\pi}{(n+1)})\|_{L^p_{2\pi}}$ we need the conditions on $\widetilde{\omega}$ from the assumptions of Theorems 1 or 2. These conditions always hold with $\|\psi_{\cdot}(t)\|_{L^p_{2\pi/r}}$ instead of $|\psi_x(t)|$ and thus the desired result follows.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

MK, WŁ and BS contributed equally in all stages to the writing of the paper. All authors read and approved the final manuscript.

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