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New construction and proof techniques of projection algorithm for countable maximal monotone mappings and weakly relatively non-expansive mappings in a Banach space

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Abstract

In a real uniformly convex and uniformly smooth Banach space, some new monotone projection iterative algorithms for countable maximal monotone mappings and countable weakly relatively non-expansive mappings are presented. Under mild assumptions, some strong convergence theorems are obtained. Compared to corresponding previous work, a new projection set involves projection instead of generalized projection, which needs calculating a Lyapunov functional. This may reduce the computational labor theoretically. Meanwhile, a new technique for finding the limit of the iterative sequence is employed by examining the relationship between the monotone projection sets and their projections. To check the effectiveness of the new iterative algorithms, a specific iterative formula for a special example is proved and its computational experiment is conducted by codes of Visual Basic Six. Finally, the application of the new algorithms to a minimization problem is exemplified.

MSC: 47H05; 47H09; 47H10

Keywords: Maximal monotone mapping; Weakly relatively non-expansive mapping; Projection; Limit of a sequence of sets; Uniformly convex and uniformly smooth Banach space

1 Introduction and preliminaries

Let E be a real Banach space with E^* its dual space. Suppose that C is a nonempty closed and convex subset of E. The symbol $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. The symbols "$\to$" and "$\rightharpoonup$" denote strong and weak convergence either in E or in E^*, respectively.

A Banach space E is said to be strictly convex [1] if for $\forall x, y \in E$ which are linearly independent,

$$\|x + y\| < \|x\| + \|y\|.$$

The above inequality is equivalent to the following:

$$\|x\| = \|y\| = 1, \quad x \neq y \quad \Rightarrow \quad \left\|\frac{x+y}{2}\right\| < 1.$$

A Banach space E is said to be uniformly convex [1] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|x_n + y_n\| = 2$, $\lim_{n\to\infty}\|x_n - y_n\| = 0$ holds.

If E is uniformly convex, then it is strictly convex.

The function $\rho_E : [0,+\infty) \to [0,+\infty)$ is called the modulus of smoothness of E [2] if it is defined as follows:

$$\rho_E(t) = \sup\left\{\frac{1}{2}\bigl(\|x+y\| + \|x-y\|\bigr) - 1 : x, y \in E, \|x\| = 1, \|y\| \leq t\right\}.$$

A Banach space E is said to be uniformly smooth [2] if $\frac{\rho_E(t)}{t} \to 0$, as $t \to 0$.

The Banach space E is uniformly smooth if and only if E^* is uniformly convex [2].

We say E has Property (H) if for every sequence $\{x_n\} \subset E$ which converges weakly to $x \in E$ and satisfies $\|x_n\| \to \|x\|$ as $n \to \infty$ necessarily converges to x in the norm.

If E is uniformly convex and uniformly smooth, then E has Property (H).

With each $x \in E$, we associate the set

$$J(x) = \bigl\{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\bigr\}, \quad \forall x \in E.$$

Then the multi-valued mapping $J : E \to 2^{E^*}$ is called the normalized duality mapping [1]. Now, we list some elementary properties of J.

Lemma 1.1 ([1, 2])

(1) *If E is a real reflexive and smooth Banach space, then J is single valued*;

(2) *if E is reflexive, then J is surjective*;

(3) *if E is uniformly smooth and uniformly convex, then J^{-1} is also the normalized duality mapping from E^* into E. Moreover, both J and J^{-1} are uniformly continuous on each bounded subset of E or E^*, respectively*;

(4) *for $x \in E$ and $k \in (-\infty, +\infty)$, $J(kx) = kJ(x)$.*

For a nonlinear mapping U, we use $F(U)$ and $N(U)$ to denote its fixed point set and null point set, respectively; that is, $F(U) = \{x \in D(U) : Ux = x\}$ and $N(U) = \{x \in D(U) : Ux = 0\}$.

Definition 1.2 ([3]) A mapping $T \subset E \times E^*$ is said to be monotone if, for $\forall y_i \in Tx_i$, $i = 1, 2$, we have $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$. The monotone mapping T is called maximal monotone if $R(J + \theta T) = E^*$ for $\theta > 0$.

Definition 1.3 ([4]) The Lyapunov functional $\varphi : E \times E^* \to (0,+\infty)$ is defined as follows:

$$\varphi(x,y) = \|x\|^2 - 2\bigl\langle x, j(y)\bigr\rangle + \|y\|^2, \quad \forall x, y \in E, j(y) \in J(y).$$

Definition 1.4 ([5]) Let $B : C \to C$ be a mapping, then

(1) an element $p \in C$ is said to be an asymptotic fixed point of B if there exists a sequence $\{x_n\}$ in C which converges weakly to p such that $x_n - Bx_n \to 0$, as $n \to \infty$. The set of asymptotic fixed points of B is denoted by $\hat{F}(B)$;

(2) $B : C \to C$ is said to be strongly relatively non-expansive if $\hat{F}(B) = F(B) \neq \emptyset$ and $\varphi(p, Bx) \leq \varphi(p, x)$ for $x \in C$ and $p \in F(B)$;

(3) an element $p \in C$ is said to be a strong asymptotic fixed point of B if there exists a sequence $\{x_n\}$ in C which converges strongly to p such that $x_n - Bx_n \to 0$, as $n \to \infty$. The set of strong asymptotic fixed points of B is denoted by $\tilde{F}(B)$;

(4) $B : C \to C$ is said to be weakly relatively non-expansive if $\tilde{F}(B) = F(B) \neq \emptyset$ and $\varphi(p, Bx) \leq \varphi(p, x)$ for $x \in C$ and $p \in F(B)$.

Remark 1.5 It is easy to see that strongly relatively non-expansive mappings are weakly relatively non-expansive mappings. However, an example in [6] shows that a weakly relatively non-expansive mapping is not a strongly relatively non-expansive mapping.

Lemma 1.6 ([5]) *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed and convex subset of E. If $B : C \to C$ is weakly relatively non-expansive, then $F(B)$ is a closed and convex subset of E.*

Lemma 1.7 ([3]) *Let $T \subset E \times E^*$ be maximal monotone, then*

(1) *$N(T)$ is a closed and convex subset of E;*

(2) *if $x_n \to x$ and $y_n \in Tx_n$ with $y_n \rightharpoonup y$, or $x_n \rightharpoonup x$ and $y_n \in Tx_n$ with $y_n \to y$, then $x \in D(T)$ and $y \in Tx$.*

Definition 1.8 ([4])

(1) If E is a reflexive and strictly convex Banach space and C is a nonempty closed and convex subset of E, then for each $x \in E$ there exists a unique element $v \in C$ such that $\|x - v\| = \inf\{\|x - y\| : y \in C\}$. Such an element v is denoted by $P_C x$ and P_C is called the metric projection of E onto C.

(2) Let E be a real reflexive, strictly convex, and smooth Banach space and C be a nonempty closed and convex subset of E, then for $\forall x \in E$, there exists a unique element $x_0 \in C$ satisfying $\varphi(x_0, x) = \inf\{\varphi(y, x) : y \in C\}$. In this case, $\forall x \in E$, define $\Pi_C : E \to C$ by $\Pi_C x = x_0$, and then Π_C is called the generalized projection from E onto C.

It is easy to see that Π_C is coincident with P_C in a Hilbert space.

Maximal monotone mappings and weakly or strongly relatively non-expansive mappings are different types of important nonlinear mappings due to their practical background. Much work has been done in designing iterative algorithms either to approximate a null point of maximal monotone mappings or a fixed point of weakly or strongly relatively non-expansive mappings, see [5–10] and the references therein. It is a natural idea to construct iterative algorithms to approximate common solutions of a null point of maximal monotone mappings and a fixed point of weakly or strongly relatively non-expansive mappings, which can be seen in [11–15] and the references therein. Now, we list some closely related work.

In [12], Wei et al. presented the following iterative algorithms to approximate a common element of the set of null points of the maximal monotone mapping $T \subset E \times E^*$ and the set of fixed points of the strongly relatively non-expansive mapping $S \subset E \times E$, where E is a real uniformly convex and uniformly smooth Banach space:

$$\begin{cases} x_1 \in E, \quad r_1 > 0, \\ y_n = (J + r_n T)^{-1} J(x_n + e_n), \\ z_n = J^{-1}[\alpha_n Jx_n + (1-\alpha_n) Jy_n], \\ u_n = J^{-1}[\beta_n Jx_n + (1-\beta_n) JSz_n], \\ H_n = \{z \in E : \varphi(z, z_n) \leq \alpha_n \varphi(z, x_n) + (1-\alpha_n)\varphi(z, x_n + e_n)\}, \\ V_n = \{z \in E : \varphi(z, u_n) \leq \beta_n \varphi(z, x_n) + (1-\beta_n)\varphi(z, z_n)\}, \\ W_n = \{z \in E : \langle z - x_n, Jx_1 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap V_n \cap W_n}(x_1), \quad n \in N, \end{cases} \tag{1.1}$$

$$\begin{cases} x_1 \in E, \quad r_1 > 0, \\ y_n = (J + r_n T)^{-1} J(x_n + e_n), \\ z_n = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n) Jy_n], \\ u_n = J^{-1}[\beta_n Jx_1 + (1-\beta_n) JSz_n], \\ H_n = \{z \in E : \varphi(z, z_n) \leq \alpha_n \varphi(z, x_1) + (1-\alpha_n)\varphi(z, x_n + e_n)\}, \\ V_n = \{z \in E : \varphi(z, u_n) \leq \beta_n \varphi(z, x_1) + (1-\beta_n)\varphi(z, z_n)\}, \\ W_n = \{z \in E : \langle z - x_n, Jx_1 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap V_n \cap W_n}(x_1), \quad n \in N, \end{cases} \tag{1.2}$$

and

$$\begin{cases} x_1 \in E, \quad r_1 > 0, \\ y_n = (J + r_n T)^{-1} J(x_n + e_n), \\ z_n = J^{-1}[\alpha_n Jx_n + (1-\alpha_n) Jy_n], \\ u_n = J^{-1}[\beta_n Jx_n + (1-\beta_n) JSz_n], \\ H_1 = \{z \in E : \varphi(z, z_1) \leq \alpha_1 \varphi(z, x_1) + (1-\alpha_1)\varphi(z, x_1 + e_1)\}, \\ V_1 = \{z \in E : \varphi(z, u_1) \leq \beta_1 \varphi(z, x_1) + (1-\beta_1)\varphi(z, z_1)\}, \\ W_1 = E, \\ H_n = \{z \in H_{n-1} \cap V_{n-1} \cap W_{n-1} : \varphi(z, z_n) \leq \alpha_n \varphi(z, x_n) + (1-\alpha_n)\varphi(z, x_n + e_n)\}, \\ V_n = \{z \in H_{n-1} \cap V_{n-1} \cap W_{n-1} : \varphi(z, u_n) \leq \beta_n \varphi(z, x_n) + (1-\beta_n)\varphi(z, z_n)\}, \\ W_n = \{z \in H_{n-1} \cap V_{n-1} \cap W_{n-1} : \langle z - x_n, Jx_1 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap V_n \cap W_n}(x_1), \quad n \in N. \end{cases} \tag{1.3}$$

Under some mild assumptions, $\{x_n\}$ generated by (1.1), (1.2), or (1.3) is proved to be strongly convergent to $\Pi_{N(T)\cap F(S)}(x_1)$. Compared to projective iterative algorithms (1.1) and (1.2), iterative algorithm (1.3) is called monotone projection method since the projection sets H_n, V_n, and W_n are all monotone in the sense that $H_{n+1} \subset H_n$, $V_{n+1} \subset V_n$, and

$W_{n+1} \subset W_n$ for $n \in N$. Theoretically, the monotone projection method will reduce the computation task.

In [13], Klin-eam et al. presented the following iterative algorithm to approximate a common element of the set of null points of the maximal monotone mapping $A \subset E \times E^*$ and the sets of fixed points of two strongly relatively non-expansive mappings $S, T \subset C \times C$, where C is the nonempty closed and convex subset of a real uniformly convex and uniformly smooth Banach space E.

$$\begin{cases} u_n = J^{-1}[\alpha_n Jx_n + (1-\alpha_n)JTz_n], \\ z_n = J^{-1}[\beta_n Jx_n + (1-\beta_n)JS(J + r_n A)^{-1}Jx_n], \\ H_n = \{z \in C : \varphi(z, u_n) \leq \varphi(z, x_n)\}, \\ V_n = \{z \in C : \langle z - x_n, Jx_1 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap V_n}(x_1), \quad n \in N. \end{cases} \tag{1.4}$$

Under some assumptions, $\{x_n\}$ generated by (1.4) is proved to be strongly convergent to $\Pi_{N(A)\cap F(S)\cap F(T)}(x_1)$.

In [14], Wei et al. extended the topic to the case of finite maximal monotone mappings $\{T_i\}_{i=1}^{m_1}$ and finite strongly relatively non-expansive mappings $\{S_j\}_{j=1}^{m_2}$. They constructed the following two iterative algorithms in a real uniformly convex and uniformly smooth Banach space E:

$$\begin{cases} x_1 \in E, \quad r > 0, \\ y_n = J^{-1}[\beta_n Jx_n + \sum_{i=1}^{m_1} \beta_{n,i} J(J + rT_i)^{-1}Jx_n], \\ x_{n+1} = J^{-1}[\alpha_n Jx_n + \sum_{j=1}^{m_2} \alpha_{n,j} JS_j y_n], \quad n \in N, \end{cases} \tag{1.5}$$

and

$$\begin{cases} x_1 \in E, \quad r > 0, \\ y_n = J^{-1}[\beta_n Jx_n + (1-\beta_n)J(J + rT_1)^{-1}J(J + rT_2)^{-1}J\cdots(J + rT_{m_1})^{-1}Jx_n], \\ x_{n+1} = J^{-1}[\alpha_n Jx_n + (1-\alpha_n)JS_1S_2\cdots S_{m_2}y_n], \quad n \in N. \end{cases} \tag{1.6}$$

Under some assumptions, $\{x_n\}$ generated by (1.5) or (1.6) is proved to be weakly convergent to $v = \lim_{n\to\infty} \Pi_{(\bigcap_{i=1}^{m_1} N(T_i))\cap(\bigcap_{j=1}^{m_2} F(S_j))}(x_n)$.

Inspired by the previous work, in Sect. 2.1, we shall construct some new iterative algorithms to approximate the common element of the sets of null points of countable maximal monotone mappings and the sets of fixed points of countable weakly relatively non-expansive mappings. New proof techniques can be found, restrictions are mild, and error is considered. In Sect. 2.2, an example is listed and a specific iterative formula is proved. Computational experiments which show the effectiveness of the new abstract iterative algorithms are conducted. In Sect. 2.3, an application to the minimization problem is demonstrated.

The following preliminaries are also needed in our paper.

Definition 1.9 ([16]) Let $\{C_n\}$ be a sequence of nonempty closed and convex subsets of E, then

(1) s- $\liminf C_n$, which is called strong lower limit, is defined as the set of all $x \in E$ such that there exists $x_n \in C_n$ for almost all n and it tends to x as $n \to \infty$ in the norm.
(2) w- $\limsup C_n$, which is called weak upper limit, is defined as the set of all $x \in E$ such that there exists a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ and $x_{n_k} \in C_{n_k}$ for every n_k and it tends to x as $n_k \to \infty$ in the weak topology;
(3) if s- $\liminf C_n = w$- $\limsup C_n$, then the common value is denoted by $\lim C_n$.

Lemma 1.10 ([16]) *Let $\{C_n\}$ be a decreasing sequence of closed and convex subsets of E, i.e., $C_n \subset C_m$ if $n \geq m$. Then $\{C_n\}$ converges in E and $\lim C_n = \bigcap_{n=1}^{\infty} C_n$.*

Lemma 1.11 ([17]) *Suppose that E is a real reflexive and strictly convex Banach space. If $\lim C_n$ exists and is not empty, then $\{P_{c_n}x\}$ converges weakly to $P_{\lim C_n}x$ for every $x \in E$. Moreover, if E has Property* (H), *the convergence is in norm.*

Lemma 1.12 ([18]) *Let E be a real smooth and uniformly convex Banach space, and let $\{u_n\}$ and $\{v_n\}$ be two sequences of E. If either $\{u_n\}$ or $\{v_n\}$ is bounded and $\varphi(u_n, v_n) \to 0$, as $n \to \infty$, then $u_n - v_n \to 0$, as $n \to \infty$.*

Lemma 1.13 ([19]) *Let E be a real uniformly convex Banach space and $r \in (0, +\infty)$. Then there exists a continuous, strictly increasing, and convex function $\omega : [0, 2r] \to [0, +\infty)$ with $\omega(0) = 0$ such that*

$$\|kx + (1-k)y\|^2 \leq k\|x\|^2 + (1-k)\|y\|^2 - k(1-k)\omega(\|x-y\|)$$

for $k \in [0, 1], x, y \in E$ with $\|x\| \leq r$ and $\|y\| \leq r$.

2 Strong convergence theorems and experiments

2.1 Strong convergence for infinite maximal monotone mappings and infinite weakly relatively non-expansive mappings

In this section, we suppose that the following conditions are satisfied:

(A1) E is a real uniformly convex and uniformly smooth Banach space and $J : E \to E^*$ is the normalized duality mapping;
(A2) $T_i \subset E \times E^*$ is maximal monotone and $S_i : E \to E$ is weakly relatively non-expansive for each $i \in N$;
(A3) $\{s_{n,i}\}$ and $\{\tau_n\}$ are two real number sequences in $(0, +\infty)$ for $i, n \in N$. $\{\alpha_n\}$ is a real number sequence in $(0, 1)$ for $n \in N$;
(A4) $\{\varepsilon_n\}$ is the error sequence in E.

Algorithm 2.1

Step 1. Choose $u_1, \varepsilon_1 \in E$. Let $s_{1,i} \in (0, +\infty)$ for $i \in N$. $\alpha_1 \in (0, 1)$ and $\tau_1 \in (0, +\infty)$. Set $n = 1$, and go to Step 2.

Step 2. Compute $v_{n,i} = (J + s_{n,i}T_i)^{-1}J(u_n + \varepsilon_n)$ and $w_{n,i} = J^{-1}[\alpha_n Ju_n + (1-\alpha_n)JS_i v_{n,i}]$ for $i \in N$. If $v_{n,i} = u_n + \varepsilon_n$ and $w_{n,i} = J^{-1}[\alpha_n Ju_n + (1-\alpha_n)J(u_n + \varepsilon_n)]$ for all $i \in N$, then stop; otherwise, go to Step 3.

Step 3. Construct the sets V_n, W_n, and U_n as follows:

$$\begin{cases} V_1 = E, \\ V_{n+1,i} = \{z \in E : \langle v_{n,i} - z, J(u_n + \varepsilon_n) - Jv_{n,i}\rangle \geq 0\}, \\ V_{n+1} = (\bigcap_{i=1}^{\infty} V_{n+1,i}) \cap V_n, \end{cases}$$

$$\begin{cases} W_1 = E, \\ W_{n+1,i} = \{z \in V_{n+1,i} : \varphi(z, w_{n,i}) \leq \alpha_n \varphi(z, u_n) + (1 - \alpha_n)\varphi(z, v_{n,i})\}, \\ W_{n+1} = (\bigcap_{i=1}^{\infty} W_{n+1,i}) \cap W_n, \end{cases}$$

and

$$U_{n+1} = \{z \in W_{n+1} : \|u_1 - z\|^2 \leq \|P_{W_{n+1}}(u_1) - u_1\|^2 + \tau_{n+1}\},$$

go to Step 4.

Step 4. Choose any element $u_{n+1} \in U_{n+1}$ for $n \in N$.

Step 5. Set $n = n + 1$, and return to Step 2.

Theorem 2.1 *If, in Algorithm* 2.1, $v_{n,i} = u_n + \varepsilon_n$ *and* $w_{n,i} = J^{-1}[\alpha_n Ju_n + (1 - \alpha_n)J(u_n + \varepsilon_n)]$ *for all* $i \in N$, *then* $u_n + \varepsilon_n \in (\bigcap_{i=1}^{\infty} N(T_i)) \cap (\bigcap_{i=1}^{\infty} F(S_i))$.

Proof Since $v_{n,i} = u_n + \varepsilon_n$, then from Step 2 in Algorithm 2.1, we know that $Jv_{n,i} + s_{n,i}T_i v_{n,i} = Jv_{n,i}$ for all $i \in N$, which implies that $s_{n,i}T_i v_{n,i} = 0$ for $i \in N$. Therefore, $u_n + \varepsilon_n \in \bigcap_{i=1}^{\infty} N(T_i)$.

Since $w_{n,i} = J^{-1}[\alpha_n Ju_n + (1 - \alpha_n)J(u_n + \varepsilon_n)] = J^{-1}[\alpha_n Ju_n + (1 - \alpha_n)JS_i v_{n,i}]$, then in view of Lemma 1.1 $v_{n,i} = S_i v_{n,i}$ for $i, n \in N$. Thus $v_{n,i} = u_n + \varepsilon_n \in \bigcap_{i=1}^{\infty} F(S_i)$, $n \in N$.

This completes the proof. □

Theorem 2.2 *Suppose* $(\bigcap_{i=1}^{\infty} N(T_i)) \cap (\bigcap_{i=1}^{\infty} F(S_i)) \neq \emptyset$, $\inf_n s_{n,i} > 0$ *for* $i \in N$, $0 < \sup_n \alpha_n < 1$, $\tau_n \to 0$, *and* $\varepsilon_n \to 0$, *as* $n \to \infty$. *Then the iterative sequence* $u_n \to y_0 = P_{\bigcap_{n=1}^{\infty} W_n}(u_1) \in (\bigcap_{i=1}^{\infty} N(T_i)) \cap (\bigcap_{i=1}^{\infty} F(S_i))$, *as* $n \to \infty$.

Proof We split the proof into eight steps.

Step 1. V_n is a nonempty subset of E.

In fact, we shall prove that $(\bigcap_{i=1}^{\infty} N(T_i)) \cap (\bigcap_{i=1}^{\infty} F(S_i)) \subset V_n$, which ensures that $V_n \neq \emptyset$.

For this, we shall use inductive method. Now, $\forall p \in (\bigcap_{i=1}^{\infty} N(T_i)) \cap (\bigcap_{i=1}^{\infty} F(S_i))$.

If $n = 1$, it is obvious that $p \in V_1 = E$. Since T_i is monotone, then

$$\langle v_{1,i} - p, J(u_1 + \varepsilon_1) - Jv_{1,i}\rangle = \langle v_{1,i} - p, s_{1,i}T_i v_{1,i} - s_{1,i}T_i p\rangle \geq 0.$$

Thus $p \in V_{2,i}$, which ensures that $p \in V_2$.

Suppose the result is true for $n = k + 1$. Then, if $n = k + 2$, we have

$$\langle v_{k+1,i} - p, J(u_{k+1} + \varepsilon_{k+1}) - Jv_{k+1,i}\rangle = \langle v_{k+1,i} - p, s_{k+1,i}T_i v_{k+1,i} - s_{k+1,i}T_i p\rangle \geq 0.$$

Then $p \in V_{k+2,i}$, which ensures that $p \in V_{k+2}$.

Therefore, by induction, $(\bigcap_{i=1}^{\infty} N(T_i)) \cap (\bigcap_{i=1}^{\infty} F(S_i)) \subset V_n$ for $n \in N$.

Step 2. W_n is a nonempty closed and convex subset of E for $n \in N$.

Since $\varphi(z, w_{n,i}) \leq \alpha_n \varphi(z, u_n) + (1 - \alpha_n)\varphi(z, v_{n,i})$ is equivalent to $\langle z, 2\alpha_n Ju_n + 2(1 - \alpha_n)Jv_{n,i} - 2Jw_{n,i}\rangle \leq \alpha_n \|u_n\|^2 + (1 - \alpha_n)\|v_{n,i}\|^2 - \|w_{n,i}\|^2$, then it is easy to see that $W_{n,i}$ is closed and convex for $i, n \in N$. Thus W_n is closed and convex for $n \in N$.

Next, we shall use inductive method to show that $(\bigcap_{i=1}^{\infty} N(T_i)) \cap (\bigcap_{i=1}^{\infty} F(S_i)) \subset W_n$ for $n \in N$, which ensures that $W_n \neq \emptyset$ for $n \in N$.

In fact, $\forall p \in (\bigcap_{i=1}^{\infty} N(T_i)) \cap (\bigcap_{i=1}^{\infty} F(S_i))$.

If $n = 1$, it is obvious that $p \in W_1 = E$. Then, from the definition of weakly relatively non-expansive mappings, we have

$$\begin{aligned}\varphi(p, w_{n,i}) &\leq \alpha_1 \varphi(p, u_1) + (1 - \alpha_1)\varphi(p, S_i v_{1,i}) \\ &\leq \alpha_1 \varphi(p, u_1) + (1 - \alpha_1)\varphi(p, v_{1,i}).\end{aligned}$$

Combining this with Step 1, we know that $p \in W_{2,i}$ for $i \in N$. Therefore, $p \in W_2$.

Suppose the result is true for $n = k + 1$. Then, if $n = k + 2$, we know from Step 1 that $p \in V_{k+2,i}$ for $i, k \in N$. Moreover,

$$\begin{aligned}\varphi(p, w_{k+1,i}) &\leq \alpha_{k+1}\varphi(p, u_{k+1}) + (1 - \alpha_{k+1})\varphi(p, S_i v_{k+1,i}) \\ &\leq \alpha_{k+1}\varphi(p, u_{k+1}) + (1 - \alpha_{k+1})\varphi(p, v_{k+1,i}),\end{aligned}$$

which implies that $p \in W_{k+2,i}$, and then $p \in (\bigcap_{i=1}^{\infty} W_{k+2,i}) \cap W_{k+1} = W_{k+2}$. Therefore, by induction,

$$\emptyset \neq \left(\bigcap_{i=1}^{\infty} N(T_i)\right) \cap \left(\bigcap_{i=1}^{\infty} F(S_i)\right) \subset W_n \quad \text{for } n \in N.$$

Step 3. Set $y_n = P_{W_{n+1}}(u_1)$. Then $y_n \to y_0 = P_{\bigcap_{n=1}^{\infty} W_n}(u_1)$, as $n \to \infty$.

From the construction of W_n in Step 3 of Algorithm 2.1, $W_{n+1} \subset W_n$ for $n \in N$. Lemma 1.10 implies that $\lim W_n$ exists and $\lim W_n = \bigcap_{n=1}^{\infty} W_n \neq \emptyset$. Since E has Property (H), then Lemma 1.11 implies that $y_n \to y_0 = P_{\bigcap_{n=1}^{\infty} W_n}(u_1)$, as $n \to \infty$.

Step 4. $\{u_n\}$ is well defined.

It suffices to show that $U_n \neq \emptyset$. From the definitions of $P_{W_{n+1}}(u_1)$ and infimum, we know that for τ_{n+1} there exists $b_n \in W_{n+1}$ such that

$$\|u_1 - b_n\|^2 \leq \left(\inf_{z \in W_{n+1}} \|u_1 - z\|\right)^2 + \tau_{n+1} = \left\|P_{W_{n+1}}(u_1) - u_1\right\|^2 + \tau_{n+1}.$$

This ensures that $U_{n+1} \neq \emptyset$ for $n \to \infty$.

Step 5. $u_{n+1} - y_n \to 0$ as $n \to \infty$.

Since $u_{n+1} \in U_{n+1} \subset W_{n+1}$, then in view of Lemma 1.13 and the fact that W_n is convex, we have, for $\forall k \in (0, 1)$,

$$\begin{aligned}\|y_n - u_1\|^2 &\leq \left\|k y_n + (1 - k)u_{n+1} - u_1\right\|^2 \\ &\leq k\|y_n - u_1\|^2 + (1 - k)\|u_{n+1} - u_1\|^2 - k(1 - k)\omega\left(\|y_n - u_{n+1}\|\right).\end{aligned}$$

Therefore,

$$k\omega\big(\|y_n - u_{n+1}\|\big) \le \|u_{n+1} - u_1\|^2 - \|y_n - u_1\|^2 \le \tau_{n+1}.$$

Letting $k \to 1$, then $y_n - u_{n+1} \to 0$ as $n \to \infty$. Since $y_n \to y_0$, then $u_n \to y_0$, as $n \to \infty$.

Step 6. $u_n - v_{n,i} \to 0$ for $i \in N$, as $n \to \infty$.

Since $y_{n+1} \in W_{n+2} \subset W_{n+1} \subset V_{n+1}$, then

$$\begin{aligned}
0 &\le 2\langle v_{n,i} - y_{n+1}, J(u_n + \varepsilon_n) - Jv_{n,i}\rangle \\
&= 2\langle y_{n+1} - v_{n,i}, Jv_{n,i} - J(u_n + \varepsilon_n)\rangle \\
&= \varphi(y_{n+1}, u_n + \varepsilon_n) - \varphi(y_{n+1}, v_{n,i}) - \varphi(v_{n,i}, u_n + \varepsilon_n) \\
&\le \varphi(y_{n+1}, u_n + \varepsilon_n) - \varphi(v_{n,i}, u_n + \varepsilon_n).
\end{aligned}$$

Thus, by using Step 5 and by letting $\varepsilon_n \to 0$, we have

$$\begin{aligned}
\varphi(v_{n,i}, u_n + \varepsilon_n) \le \varphi(y_{n+1}, u_n + \varepsilon_n)& \\
&= \varphi(y_{n+1}, y_n) + \varphi(y_n, u_n + \varepsilon_n) + 2\langle y_{n+1} - y_n, Jy_n - J(u_n + \varepsilon_n)\rangle \\
&\le \big(\|y_{n+1}\|\|Jy_{n+1} - Jy_n\| + \|y_{n+1} - y_n\|\|y_n\|\big) \\
&\quad + \big(\|y_n\|\big\|Jy_n - J(u_n + \varepsilon_n)\big\| + \|y_{n+1} - u_n - \varepsilon_n\|\|u_n + \varepsilon_n\|\big) \\
&\quad + 2\|y_{n+1} - y_n\|\big\|Jy_n - J(u_n + \varepsilon_n)\big\| \to 0,
\end{aligned}$$

as $n \to \infty$. Using Lemma 1.12, $v_{n,i} - u_n - \varepsilon_n \to 0$ for $i \in N$, as $n \to \infty$. Since $\varepsilon_n \to 0$, then $v_{n,i} - u_n \to 0$ for $i \in N$, as $n \to \infty$. Since $u_n \to y_0$, then $v_{n,i} \to y_0$ for $i \in N$, as $n \to \infty$.

Step 7. $w_{n,i} - u_n \to 0$ for $i \in N$, as $n \to \infty$.

Since $u_{n+1} \in U_{n+1} \subset W_{n+1}$, then noticing Steps 5 and 6,

$$\varphi(u_{n+1}, w_{n,i}) \le \alpha_n \varphi(u_{n+1}, u_n) + (1 - \alpha_n)\varphi(u_{n+1}, v_{n,i}) \to 0,$$

as $n \to \infty$. Lemma 1.12 implies that $u_{n+1} - w_{n,i} \to 0$, as $n \to \infty$. Since $u_n \to y_0$, then $w_{n,i} \to y_0$ for $i \in N$, as $n \to \infty$.

Step 8. $y_0 = P_{\bigcap_{n=1}^{\infty} W_n}(u_1) \in (\bigcap_{i=1}^{\infty} N(T_i)) \cap (\bigcap_{i=1}^{\infty} F(S_i))$.

Since $v_{n,i} = (J + s_{n,i}T_i)^{-1}J(u_n + \varepsilon_n)$, then $Jv_{n,i} + s_{n,i}T_iv_{n,i} = J(u_n + \varepsilon_n)$. Since $v_{n,i} \to y_0$, $u_n \to y_0$, $\varepsilon_n \to 0$ and $\inf_n s_{n,i} > 0$, then $T_iv_{n,i} \to 0$ for $i \in N$, as $n \to \infty$. Using Lemma 1.7, $y_0 \in \bigcap_{i=1}^{\infty} N(T_i)$.

Since $w_{n,i} = J^{-1}[\alpha_n Ju_n + (1 - \alpha_n)JS_iv_{n,i}]$, then in view of Lemma 1.1, $S_iv_{n,i} \to y_0$, as $n \to \infty$. Lemma 1.6 implies that $y_0 \in \bigcap_{i=1}^{\infty} F(S_i)$.

This completes the proof. □

Corollary 2.3 *If $i \equiv 1$, denote by T the maximal monotone mapping and by S the weakly relatively non-expansive mapping, then Algorithm* 2.1 *reduces to the following*:

$$\begin{cases} u_1 \in E, \quad \varepsilon_1 \in E, \\ v_n = (J + s_n T)^{-1} J(u_n + \varepsilon_n), \\ w_n = J^{-1}[\alpha_n Ju_n + (1-\alpha_n)JSv_n], \\ V_1 = W_1 = E, \\ V_{n+1} = \{z \in E : \langle v_n - z, J(u_n + \varepsilon_n) - Jv_n \rangle \geq 0\} \cap V_n, \\ W_{n+1} = \{z \in V_{n+1} : \varphi(z, w_n) \leq \alpha_n \varphi(z, u_n) + (1-\alpha_n)\varphi(z, v_n)\} \cap W_n, \\ U_{n+1} = \{z \in W_{n+1} : \|u_1 - z\|^2 \leq \|P_{W_{n+1}}(u_1) - u_1\|^2 + \tau_{n+1}\}, \\ u_{n+1} \in U_{n+1}, \quad n \in N, \end{cases}$$

where $\{\varepsilon_n\} \subset E$, $\{s_n\} \subset (0,\infty)$, $\{\tau_n\} \subset (0,\infty)$, and $\{\alpha_n\} \subset (0,1)$. Then

(1) *Similar to Theorem* 2.1, *if $v_n = u_n + \varepsilon_n$ and $w_n = J^{-1}[\alpha_n Ju_n + (1-\alpha_n)J(u_n+\varepsilon_n)]$ for all $n \in N$, then $u_n + \varepsilon_n \in N(T) \cap F(S)$.*

(2) *Suppose that E, $\{\varepsilon_n\}$, $\{\tau_n\}$, and $\{\alpha_n\}$ satisfy the same conditions as those in Theorem* 2.2. *If $N(T) \cap F(S) = \emptyset$ and $\inf_n s_n > 0$, then the iterative sequence $u_n \to y_0 = P_{\bigcap_{n=1}^\infty W_n}(u_1) \in N(T) \cap F(S)$, as $n \to \infty$.*

Algorithm 2.2 Only doing the following changes in Algorithm 2.1, we get Algorithm 2.2:

$$w_{n,i} = J^{-1}\bigl[\alpha_n Ju_1 + (1-\alpha_n)JS_i v_{n,i}\bigr] \quad \text{for all } i \in N,$$

and

$$\begin{cases} W_1 = E, \\ W_{n+1,i} = \{z \in V_{n+1,i} : \varphi(z, w_{n,i}) \leq \alpha_n \varphi(z, u_1) + (1-\alpha_n)\varphi(z, v_{n,i})\}, \\ W_{n+1} = (\bigcap_{i=1}^\infty W_{n+1,i}) \cap W_n. \end{cases}$$

Theorem 2.4 *If, in Algorithm* 2.2, *$v_{n,i} = u_n + \varepsilon_n$ and $w_{n,i} = J^{-1}[\alpha_n Ju_1 + (1-\alpha_n)J(u_n+\varepsilon_n)]$ for all $i \in N$, then $u_n + \varepsilon_n \in (\bigcap_{i=1}^\infty N(T_i)) \cap (\bigcap_{i=1}^\infty F(S_i))$.*

Proof Similar to Theorem 2.1, the result follows. □

Theorem 2.5 *We only change the condition that $0 < \sup_n \alpha_n < 1$ in Theorem* 2.2 *by $\alpha_n \to 0$, as $n \to \infty$. Then the iterative sequence $u_n \to y_0 = P_{\bigcap_{n=1}^\infty W_n}(u_1) \in (\bigcap_{i=1}^\infty N(T_i)) \cap (\bigcap_{i=1}^\infty F(S_i))$, as $n \to \infty$.*

Proof Copy Steps 1, 3, 4, 5, and 6 in Theorem 2.2 and make slight changes in the following steps.

Step 2. W_n is a nonempty closed and convex subset of E for $n \in N$.

Since $\varphi(z, w_{n,i}) \leq \alpha_n \varphi(z, u_1) + (1-\alpha_n)\varphi(z, v_{n,i})$ is equivalent to $\langle z, 2\alpha_n Ju_1 + 2(1-\alpha_n)Jv_{n,i} - 2Jw_{n,i}\rangle \leq \alpha_n \|u_1\|^2 + (1-\alpha_n)\|v_{n,i}\|^2 - \|w_{n,i}\|^2$, then it is easy to see that $W_{n,i}$ is closed and convex for $i, n \in N$. Thus W_n is closed and convex for $n \in N$.

Next, we shall use inductive method to show that $(\bigcap_{i=1}^{\infty} N(T_i)) \cap (\bigcap_{i=1}^{\infty} F(S_i)) \subset W_n$ for $n \in N$, which ensures that $W_n \neq \emptyset$ for $n \in N$.

In fact, $\forall p \in (\bigcap_{i=1}^{\infty} N(T_i)) \cap (\bigcap_{i=1}^{\infty} F(S_i))$.

If $n = 1$, it is obvious that $p \in W_1 = E$. Then, from the definition of weakly relatively non-expansive mappings, we have

$$\begin{aligned}\varphi(p, w_{1,i}) &\leq \alpha_1 \varphi(p, u_1) + (1 - \alpha_1)\varphi(p, S_i v_{1,i}) \\ &\leq \alpha_1 \varphi(p, u_1) + (1 - \alpha_1)\varphi(p, v_{1,i}).\end{aligned}$$

Combining this with Step 1, we know that $p \in W_{2,i}$ for $i \in N$. Therefore, $p \in W_2$.

Suppose the result is true for $n = k + 1$. Then, if $n = k + 2$, we know from Step 1 that $p \in V_{k+2,i}$ for $i, k \in N$. Moreover,

$$\begin{aligned}\varphi(p, w_{k+1,i}) &\leq \alpha_{k+1} \varphi(p, u_1) + (1 - \alpha_{k+1})\varphi(p, S_i v_{k+1,i}) \\ &\leq \alpha_{k+1} \varphi(p, u_1) + (1 - \alpha_{k+1})\varphi(p, v_{k+1,i}),\end{aligned}$$

which implies that $p \in W_{k+2,i}$ and then $p \in (\bigcap_{i=1}^{\infty} W_{k+2,i}) \cap W_{k+1} = W_{k+2}$. Therefore, by induction, $\emptyset \neq (\bigcap_{i=1}^{\infty} N(T_i)) \cap (\bigcap_{i=1}^{\infty} F(S_i)) \subset W_n$ for $n \in N$.

Step 7. $w_{n,i} - u_n \to 0$ for $i \in N$, as $n \to \infty$.

Since $u_{n+1} \in U_{n+1} \subset W_{n+1}$, then in view of the facts that $\alpha_n \to 0$ and Step 6,

$$\varphi(u_{n+1}, w_{n,i}) \leq \alpha_n \varphi(u_{n+1}, u_1) + (1 - \alpha_n)\varphi(u_{n+1}, v_{n,i}) \to 0,$$

as $n \to \infty$, for $i \in N$. Lemma 1.12 implies that $w_{n,i} - u_n \to 0$ for $i \in N$, as $n \to \infty$.

Step 8. $y_0 = P_{\bigcap_{n=1}^{\infty} W_n}(u_1) \in (\bigcap_{i=1}^{\infty} N(T_i)) \cap (\bigcap_{i=1}^{\infty} F(S_i))$.

In the same way as Step 8 in Theorem 2.2, we have $y_0 \in \bigcap_{i=1}^{\infty} N(T_i)$. Since $w_{n,i} = J^{-1}[\alpha_n Ju_1 + (1 - \alpha_n)JS_i v_{n,i}]$, then $S_i v_{n,i} \to y_0$, as $n \to \infty$. Thus in view of Lemma 1.6, $y_0 \in \bigcap_{i=1}^{\infty} F(S_i)$.

This completes the proof. □

Corollary 2.6 *If* $i \equiv 1$, *denote by* T *the maximal monotone mapping and by* S *the weakly relatively non-expansive mapping, then Algorithm* 2.2 *reduces to the following*:

$$\begin{cases} u_1 \in E, \quad \varepsilon_1 \in E, \\ v_n = (J + s_n T)^{-1} J(u_n + \varepsilon_n), \\ w_n = J^{-1}[\alpha_n Ju_1 + (1 - \alpha_n)JSv_n], \\ V_1 = W_1 = E, \\ V_{n+1} = \{z \in E : \langle v_n - z, J(u_n + \varepsilon_n) - Jv_n \rangle \geq 0\} \cap V_n, \\ W_{n+1} = \{z \in V_{n+1} : \varphi(z, w_n) \leq \alpha_n \varphi(z, u_1) + (1 - \alpha_n)\varphi(z, v_n)\} \cap W_n, \\ U_{n+1} = \{z \in W_{n+1} : \|u_1 - z\|^2 \leq \|P_{W_{n+1}}(u_1) - u_1\|^2 + \tau_{n+1}\}, \\ u_{n+1} \in U_{n+1}, \quad n \in N, \end{cases}$$

where $\{\varepsilon_n\} \subset E$, $\{s_n\} \subset (0, \infty)$, $\{\tau_n\} \subset (0, \infty)$ *and* $\{\alpha_n\} \subset (0, 1)$. *Then*

(1) *Similar to Theorem* 2.4, *if* $v_n = u_n + \varepsilon_n$ *and* $w_n = J^{-1}[\alpha_n Ju_1 + (1-\alpha_n)J(u_n+\varepsilon_n)]$, *then* $u_n + \varepsilon_n \in N(T) \cap F(S)$ *for all* $n \in N$.
(2) *Suppose that* E, $\{\varepsilon_n\}$, $\{\tau_n\}$, *and* $\{\alpha_n\}$ *satisfy the same conditions as those in Theorem* 2.5. *If* $N(T) \cap F(S) = \emptyset$ *and* $\inf_n s_n > 0$, *then the iterative sequence* $u_n \to y_0 = P_{\bigcap_{n=1}^{\infty} W_n}(u_1) \in N(T) \cap F(S)$ *as* $n \to \infty$.

Remark 2.7 Compared to the existing related work, e.g., [12–14], strongly relatively non-expansive mappings are extended to weakly relatively non-expansive mappings. Moreover, in our paper, the discussion on this topic is extended to the case of infinite maximal monotone mappings and infinite weakly relatively non-expansive mappings.

Remark 2.8 Calculating the generalized projection $\Pi_{H_n \cap V_n \cap W_n}(x_1)$ in [12] or $\Pi_{H_n \cap V_n}(x_1)$ in [13] is replaced by calculating the projection $P_{W_{n+1}}(u_1)$ in Step 3 in our Algorithms 2.1 and 2.2, which makes the computation easier.

Remark 2.9 A new proof technique for finding the limit $y_0 = P_{\bigcap_{n=1}^{\infty} W_n}(u_1)$ is employed in our paper by examining the properties of the projective sets W_n sufficiently, which is quite different from that for finding the limit $\Pi_{N(T)\cap F(S)}(x_1)$ in [12] or $\Pi_{N(A)\cap F(S)\cap F(T)}(x_1)$ in [13].

Remark 2.10 Theoretically, the projection is easier for calculating than the generalized projection in a general Banach space since the generalized projection involves a Lyapunov functional. In this sense, iterative algorithms constructed in our paper are new and more efficient.

2.2 Special cases in Hilbert spaces and computational experiments

Corollary 2.11 *If* E *reduces to a Hilbert space* H, *then iterative Algorithm* 2.1 *becomes the following one*:

$$\begin{cases} u_1 \in H, \quad \varepsilon_1 \in H, \\ v_{n,i} = (I + s_{n,i}T_i)^{-1}(u_n + \varepsilon_n), \\ w_{n,i} = \alpha_n u_n + (1-\alpha_n)S_i v_{n,i}, \\ V_1 = W_1 = H, \\ V_{n+1,i} = \{z \in H : \langle v_{n,i} - z, u_n + \varepsilon_n - v_{n,i}\rangle \geq 0\}, \\ V_{n+1} = (\bigcap_{i=1}^{\infty} V_{n+1,i}) \cap V_n, \\ W_{n+1,i} = \{z \in V_{n+1,i} : \|z - w_{n,i}\|^2 \leq \alpha_n \|z - u_n\|^2 + (1-\alpha_n)\|z - v_{n,i}\|^2\}, \\ W_{n+1} = (\bigcap_{i=1}^{\infty} W_{n+1,i}) \cap W_n, \\ U_{n+1} = \{z \in W_{n+1} : \|u_1 - z\|^2 \leq \|P_{W_{n+1}}(u_1) - u_1\|^2 + \tau_{n+1}\}, \\ u_{n+1} \in U_{n+1}, \quad n \in N. \end{cases} \tag{2.1}$$

The results of Theorems 2.1 and 2.2 are true for this special case.

Corollary 2.12 *If E reduces to a Hilbert space H, then iterative Algorithm* 2.2 *becomes the following one*:

$$\begin{cases} u_1 \in H, \quad \varepsilon_1 \in H, \\ v_{n,i} = (I + s_{n,i} T_i)^{-1}(u_n + \varepsilon_n), \\ w_{n,i} = \alpha_n u_1 + (1 - \alpha_n) S_i v_{n,i}, \\ V_1 = W_1 = H, \\ V_{n+1,i} = \{z \in H : \langle v_{n,i} - z, u_n + \varepsilon_n - v_{n,i}\rangle \geq 0\}, \\ V_{n+1} = (\bigcap_{i=1}^{\infty} V_{n+1,i}) \cap V_n, \\ W_{n+1,i} = \{z \in V_{n+1,i} : \|z - w_{n,i}\|^2 \leq \alpha_n \|z - u_1\|^2 + (1 - \alpha_n)\|z - v_{n,i}\|^2\}, \\ W_{n+1} = (\bigcap_{i=1}^{\infty} W_{n+1,i}) \cap W_n, \\ U_{n+1} = \{z \in W_{n+1} : \|u_1 - z\|^2 \leq \|P_{W_{n+1}}(u_1) - u_1\|^2 + \tau_{n+1}\}, \\ u_{n+1} \in U_{n+1}, \quad n \in N. \end{cases} \tag{2.2}$$

The results of Theorems 2.4 and 2.5 are true for this special case.

Corollary 2.13 *If, further $i \equiv 1$, then* (2.1) *and* (2.2) *reduce to the following two cases*:

$$\begin{cases} u_1 \in H, \quad \varepsilon_1 \in H, \\ v_n = (I + s_n T)^{-1}(u_n + \varepsilon_n), \\ w_n = \alpha_n u_n + (1 - \alpha_n) S v_n, \\ V_1 = W_1 = H, \\ V_{n+1} = \{z \in H : \langle v_n - z, u_n + \varepsilon_n - v_n\rangle \geq 0\} \cap V_n, \\ W_{n+1} = \{z \in V_{n+1} : \|z - w_n\|^2 \leq \alpha_n \|z - u_n\|^2 + (1 - \alpha_n)\|z - u_n\|^2\} \cap W_n, \\ U_{n+1} = \{z \in W_{n+1} : \|u_1 - z\|^2 \leq \|P_{W_{n+1}}(u_1) - u_1\|^2 + \tau_{n+1}\}, \\ u_{n+1} \in U_{n+1}, \quad n \in N, \end{cases} \tag{2.3}$$

and

$$\begin{cases} u_1 \in H, \quad \varepsilon_1 \in H, \\ v_n = (I + s_n T)^{-1}(u_n + \varepsilon_n), \\ w_n = \alpha_n u_1 + (1 - \alpha_n) S v_n, \\ V_1 = W_1 = H, \\ V_{n+1} = \{z \in H : \langle v_n - z, u_n + \varepsilon_n - v_n\rangle \geq 0\} \cap V_n, \\ W_{n+1} = \{z \in V_{n+1} : \|z - w_n\|^2 \leq \alpha_n \|z - u_1\|^2 + (1 - \alpha_n)\|z - u_n\|^2\} \cap W_n, \\ U_{n+1} = \{z \in W_{n+1} : \|u_1 - z\|^2 \leq \|P_{W_{n+1}}(u_1) - u_1\|^2 + \tau_{n+1}\}, \\ u_{n+1} \in U_{n+1}, \quad n \in N. \end{cases} \tag{2.4}$$

The results of Corollaries 2.3 and 2.6 are true for the special cases, respectively.

Remark 2.14 Take $H = (-\infty, +\infty)$, $Tx = 2x$, and $Sx = x$ for $x \in (-\infty, +\infty)$. Let $\varepsilon_n = \alpha_n = \tau_n = \frac{1}{n}$ and $s_n = 2^{n-1}$ for $n \in N$. Then T is maximal monotone and S is weakly relatively non-expansive. Moreover, $N(T) \cap F(S) = \{0\}$.

Remark 2.15 Taking the example in Remark 2.14 and choosing the initial value $u_1 = 1 \in (-\infty, +\infty)$, we can get an iterative sequence $\{u_n\}$ by algorithm (2.3) in the following way:

$$\begin{cases} u_1 = 1 \in (-\infty, +\infty), \\ u_{n+1} = \frac{u_1 + v_n - \sqrt{(u_1 - v_n)^2 + \tau_{n+1}}}{2}, \quad n \in N, \end{cases} \tag{2.5}$$

where $v_n = \frac{u_n + \varepsilon_n}{1 + 2s_n}$, $n \in N$. Moreover, $u_n \to 0 \in N(T) \cap F(S)$, as $n \to \infty$.

Proof We can easily see from iterative algorithm (2.3) that

$$v_n = \frac{u_n + \varepsilon_n}{1 + 2s_n} \quad \text{for } n \in N \tag{2.6}$$

and

$$w_n = \alpha_n u_n + (1 - \alpha_n) v_n \quad \text{for } n \in N. \tag{2.7}$$

To analyze the construction of set W_n, we notice that $|z - w_n|^2 \le \alpha_n |z - u_n|^2 + (1 - \alpha_n)|z - v_n|^2$ is equivalent to

$$\left[2\alpha_n u_n + 2(1 - \alpha_n) v_n - 2w_n\right] z \le \alpha_n u_n^2 + (1 - \alpha_n) v_n^2 - w_n^2. \tag{2.8}$$

In view of (2.7), compute the left-hand side of (2.8):

$$\begin{aligned} &\left[2\alpha_n u_n + 2(1 - \alpha_n) v_n - 2w_n\right] z \\ &\quad = \left[2\alpha_n u_n + 2(1 - \alpha_n) v_n - 2\alpha_n u_n - 2(1 - \alpha_n) v_n\right] z \\ &\quad \equiv 0 \quad \text{for } n \in N. \end{aligned} \tag{2.9}$$

Meanwhile, compute the right-hand side of (2.8):

$$\begin{aligned} &\alpha_n u_n^2 + (1 - \alpha_n) v_n^2 - w_n^2 \\ &\quad = \alpha_n u_n^2 + (1 - \alpha_n) v_n^2 - \alpha_n^2 u_n^2 - 2\alpha_n (1 - \alpha_n) u_n v_n - (1 - \alpha_n)^2 v_n^2 \\ &\quad = \alpha_n (1 - \alpha_n) u_n^2 + \alpha_n (1 - \alpha_n) v_n^2 - 2\alpha_n (1 - \alpha_n) u_n v_n \\ &\quad = \alpha_n (1 - \alpha_n)(u_n - v_n)^2 \quad \text{for } n \in N. \end{aligned} \tag{2.10}$$

Using (2.8)–(2.10), we get

$$W_{n+1} = V_{n+1} \cap W_n \quad \text{for } n \in N. \tag{2.11}$$

Next, we shall use inductive method to show that

$$\begin{cases} 0 < v_{n+1} < v_n < 1, \\ v_n > \frac{1}{2^{n+1}(n+1)}, \\ V_{n+1} = (-\infty, v_n], \\ W_{n+1} = V_{n+1}, \\ U_{n+1} = [u_1 - \sqrt{(u_1 - v_n)^2 + \tau_{n+1}}, v_n], \\ \text{we may choose } u_{n+1} = \frac{u_1 + v_n - \sqrt{(u_1 - v_n)^2 + \tau_{n+1}}}{2} \quad \text{for } n \in N. \end{cases} \tag{2.12}$$

In fact, if $n = 1$, then $v_1 = \frac{u_1+\varepsilon_1}{1+2s_1} = \frac{2}{3}$, thus $V_2 = (-\infty, v_1] \cap V_1 = (-\infty, v_1]$. From (2.11), $W_2 = V_2 \cap W_1 = V_2$. And then $P_{W_2}(u_1) = v_1 = \frac{2}{3}$. So we have

$$\begin{aligned} U_2 &= \left\{ z \in W_2 : |u_1 - z| \le \sqrt{\left|P_{W_2}(u_1) - u_1\right|^2 + \tau_2} \right\} \\ &= \left[1 - \sqrt{\frac{1}{9} + \frac{1}{2}}, 1 + \sqrt{\frac{1}{9} + \frac{1}{2}} \right] \cap \left(-\infty, \frac{2}{3} \right] \\ &= \left[1 - \sqrt{\frac{1}{9} + \frac{1}{2}}, \frac{2}{3} \right] \\ &= \left[u_1 - \sqrt{(u_1 - v_1)^2 + \tau_2}, v_1 \right]. \end{aligned}$$

Therefore, we may choose $u_2 \in U_2$ as follows:

$$u_2 = \frac{u_1 + v_1 - \sqrt{(u_1 - v_1)^2 + \tau_2}}{2}.$$

From (2.6), $v_2 = \frac{u_2+\varepsilon_2}{1+2s_2} = \frac{4}{15} - \frac{\sqrt{22}}{60}$. Then $0 < v_2 < v_1 < 1$. And it is easy to see $v_1 > \frac{1}{2^{1+1}(1+1)}$. Thus (2.12) is true for $n + 1$.

Suppose (2.12) is true for $n = k$, that is,

$$\begin{cases} 0 < v_{k+1} < v_k < 1, \\ v_k > \frac{1}{2^{k+1}(k+1)}, \\ V_{k+1} = (-\infty, v_k], \\ W_{k+1} = V_{k+1}, \\ U_{k+1} = [u_1 - \sqrt{(u_1 - v_k)^2 + \tau_{k+1}}, v_k], \\ \text{we may choose } u_{k+1} = \frac{u_1 + v_k - \sqrt{(u_1 - v_k)^2 + \tau_{k+1}}}{2}. \end{cases}$$

Then, for $n = k + 1$, we first analyze the set V_{k+2}.

Note that $u_{k+1} + \varepsilon_{k+1} - v_{k+1} = (1 + 2s_{k+1})v_{k+1} - v_{k+1} = 2s_{k+1}v_{k+1} > 0$, then $\langle v_{k+1} - z, u_{k+1} + \varepsilon_{k+1} - v_{k+1} \rangle \ge 0$ is equivalent to $z \le v_{k+1}$. Then

$$V_{k+2} = (-\infty, v_{k+1}] \cap V_{k+1} = (-\infty, v_{k+1}] \cap (-\infty, v_k] = (-\infty, v_{k+1}].$$

From (2.11),

$$W_{k+2} = V_{k+2} \cap W_{k+1} = (-\infty, v_{k+1}] \cap V_{k+1} = V_{k+2}.$$

Now, we analyze set U_{k+2}.

Since $0 < v_{k+1} < 1 = u_1$, then $P_{W_{k+2}}(u_1) = v_{k+1}$. Thus $|u_1 - z| \leq \sqrt{|P_{W_{k+2}}(u_1) - u_1|^2 + \tau_{k+2}}$ is equivalent to $u_1 - \sqrt{(u_1 - v_{k+1})^2 + \tau_{k+2}} \leq z \leq u_1 + \sqrt{(u_1 - v_{k+1})^2 + \tau_{k+2}}$.

It is easy to check that $u_1 + \sqrt{(u_1 - v_{k+1})^2 + \tau_{k+2}} > 1 > v_{k+1}$, and $u_1 - \sqrt{(u_1 - v_{k+1})^2 + \tau_{k+2}} < u_1 - (u_1 - v_{k+1}) = v_{k+1}$.

Thus $U_{k+2} = [u_1 - \sqrt{(u_1 - v_{k+1})^2 + \tau_{k+2}}, v_{k+1}]$. Then we may choose $u_{k+2} \in U_{k+2}$ such that

$$u_{k+2} = \frac{u_1 + v_{k+1} - \sqrt{(u_1 - v_{k+1})^2 + \tau_{k+2}}}{2}.$$

Now, we show that $v_{k+2} > 0$.

Since

$$\begin{aligned} v_{k+2} &= \frac{u_{k+2} + \varepsilon_{k+2}}{1 + 2s_{k+2}} \\ &= \frac{\frac{u_1 + v_{k+1} - \sqrt{(u_1 - v_{k+1})^2 + \tau_{k+2}}}{2} + \frac{1}{k+2}}{1 + 2^{k+2}} \\ &= \frac{1}{(k+2)(1+2^{k+2})} + \frac{1 + v_{k+1} - \sqrt{(u_1 - v_{k+1})^2 + \frac{1}{k+2}}}{2(1+2^{k+2})}, \end{aligned}$$

then

$$\begin{aligned} v_{k+2} > 0 \quad &\Leftrightarrow \quad \frac{1}{k+2} + \frac{1 + v_{k+1}}{2} > \frac{\sqrt{(1 - v_{k+1})^2 + \frac{1}{k+2}}}{2} \\ &\Leftrightarrow \quad \frac{1}{(k+2)^2} + \frac{1}{k+2} + \frac{v_{k+1}}{k+2} + v_{k+1} > \frac{1}{4(k+2)}, \end{aligned}$$

which is obviously true. Thus $v_{k+2} > 0$.

Next, we show that $v_{k+1} > \frac{1}{2^{k+2}(k+2)}$.

Since $v_{k+1} = \frac{u_{k+1} + \varepsilon_{k+1}}{1 + 2s_{k+1}} = \frac{(k+1)u_{k+1}}{(k+1)(1+2^{k+1})} + \frac{1}{(k+1)(1+2^{k+1})}$, then

$$\begin{aligned} & v_{k+1} > \frac{1}{2^{k+2}(k+2)} \\ \Leftrightarrow \quad & (k+1)u_{k+1} + 1 > \frac{(k+1)(1+2^{k+1})}{2^{k+2}(k+2)} \\ \Leftrightarrow \quad & (k+1)\frac{1 + v_k - \sqrt{(1 - v_k)^2 + \frac{1}{k+1}}}{2} > \frac{k + 1 - k2^{k+1} - 3 \cdot 2^{k+1}}{2^{k+2}(k+2)} \\ \Leftrightarrow \quad & (1 + v_k) + \frac{3+k}{(k+1)(k+2)} - \frac{1}{2^{k+1}(k+2)} > \sqrt{(1 - v_k)^2 + \frac{1}{k+1}} \\ \Leftrightarrow \quad & \left[\frac{3+k}{(k+1)(k+2)} - \frac{1}{2^{k+1}(k+2)}\right]^2 + 4v_k + 2v_k\left[\frac{3+k}{(k+1)(k+2)} - \frac{1}{2^{k+1}(k+2)}\right] \\ & \quad + 2\left[\frac{3+k}{(k+1)(k+2)} - \frac{1}{2^{k+1}(k+2)}\right] > \frac{1}{k}. \end{aligned} \tag{2.13}$$

Note that

$$2\left[\frac{3+k}{(k+1)(k+2)} - \frac{1}{2^{k+1}(k+2)}\right] - \frac{1}{k+1} = \frac{k2^{k+1} + 2^{k+3} - 2k - 2}{2^{k+1}(k+1)(k+2)} > 0,$$

then (2.13) is true, which implies that $v_{k+1} > \frac{1}{2^{k+2}(k+2)}$.

Finally, we show that $v_{k+2} < v_{k+1}$.

From the definition of u_{k+2}, we have $u_{k+2} < \frac{1+v_{k+1}-(1-v_{k+1})}{2} = v_{k+1}$. Then $v_{k+2} < \frac{v_{k+1}+\frac{1}{k+2}}{1+2^{k+2}}$. Since $v_{k+1} > \frac{1}{2^{k+2}(k+2)}$, then $\frac{v_{k+1}+\frac{1}{k+2}}{1+2^{k+2}} - v_{k+1} = \frac{\frac{1}{k+2}-2^{k+2}v_{k+1}}{1+2^{k+2}} < 0$, which implies that $v_{k+2} < v_{k+1}$.

Therefore, by induction, (2.12) is true for $n \in N$. Since $0 < v_{n+1} < v_n < 1$, then $\lim_{n\to\infty} v_n$ exists. Set $a = \lim_{n\to\infty} v_n$. From (2.12), $\lim_{n\to\infty} u_n = a$ and from (2.6), $a = 0$. Then in view of (2.7), $\lim_{n\to\infty} w_n = 0$. That is, $\lim_{n\to\infty} w_n = \lim_{n\to\infty} v_n = \lim_{n\to\infty} u_n = 0$.

This completes the proof. □

Remark 2.16 We next do a computational experiment on (2.5) in Remark 2.15 to check the effectiveness of iterative algorithm (2.3). By using the codes of Visual Basic Six, we get Table 1 and Fig. 1, from which we can see the convergence of $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$.

Table 1 Numerical results of $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ with initial $u_1 = 1.0$

n	v_n	w_n	u_n
1	0.666666666666667	1.00000000000000	1.00000000000000
2	0.188493070669609	0.315479212008828	0.442465353348047
3	0.047734978022387	0.063917141637640	0.096281468868147
4	0.013887781581545	0.006938907907725	−0.01390771311373
5	0.005016751133393	−0.00287604161289	−0.03444721259803
6	0.002022073632571	−0.00418691873111	−0.03523188054954
7	0.000854971429905	−0.00391942854572	−0.03256582839944
8	0.000371596957448	−0.00362300404227	−0.02949958193595
9	0.000164574841194	−0.00281862431655	−0.02668421757849
10	0.000073908605586	−0.002357850182411	−0.02424367927438

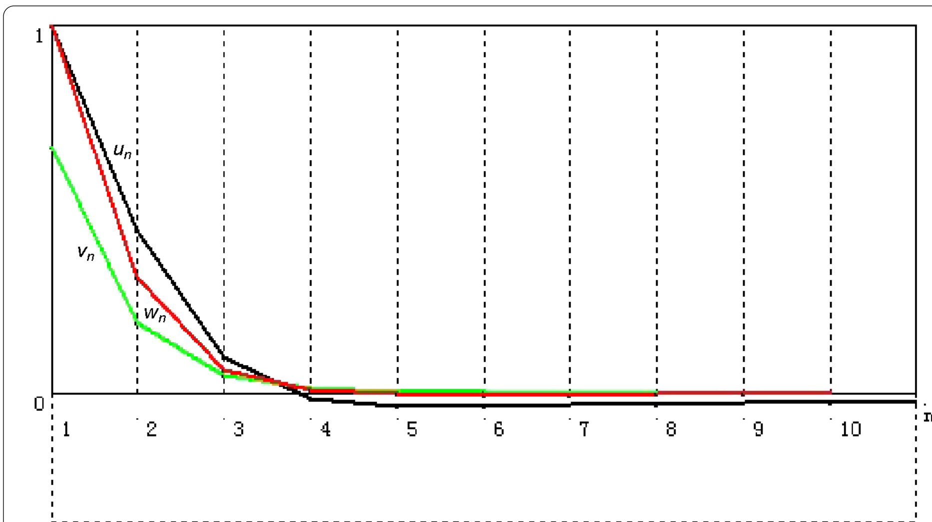

Figure 1 Convergence of $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$

Remark 2.17 Similar to Remark 2.15, considering the same example in Remark 2.14 and choosing the initial value $u_1 = 1 \in (-\infty, +\infty)$, we can get an iterative sequence $\{u_n\}$ by algorithm (2.4) in the following way:

$$\begin{cases} u_1 = 1 \in (-\infty, +\infty), \\ u_{n+1} = \frac{u_1 + v_n - \sqrt{(u_1 - v_n)^2 + \tau_{n+1}}}{2}, \quad n \in N, \end{cases} \tag{2.14}$$

where $v_n = \frac{u_n + \varepsilon_n}{1 + 2s_n}$ and $w_n = \alpha_n u_1 + (1 - \alpha_n) v_n$ for $n \in N$. Then $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ converge strongly to $0 \in N(T) \cap F(S)$, as $n \to \infty$.

Remark 2.18 We do a computational experiment on (2.14) in Remark 2.17 to check the effectiveness of iterative algorithm (2.4). By using the codes of Visual Basic Six, we get Table 2 and Fig. 2, from which we can see the convergence of $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$.

2.3 Applications to minimization problems

Let $h : E \to (-\infty, +\infty]$ be a proper convex, lower-semicontinuous function. The subdifferential ∂h of h is defined as follows: $\forall x \in E$,

$$\partial h(x) = \{ z \in E^* : h(x) + \langle y - x, z \rangle \leq h(y), \forall y \in E \}.$$

Theorem 2.19 *Let E, S, $\{\varepsilon_n\}$, $\{s_n\}$, $\{\tau_n\}$, and $\{\alpha_n\}$ be the same as those in Corollary* 2.3. *Let $h : E \to (-\infty, +\infty]$ be a proper convex, lower-semicontinuous function. Let $\{u_n\}$ be gener-*

Table 2 Numerical esults of $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ with initial $u_1 = 1.0$

n	v_n	w_n	u_n
1	0.666666666666667	1.00000000000000	1.00000000000000
2	0.188493070669609	0.594246535334805	0.442465353348047
3	0.047734978022387	0.365156652014924	0.096281468868147
4	0.013887781581545	0.260415836186159	−0.01390771311373
5	0.005016751133393	0.204013400906715	−0.03444721259803
6	0.002022073632571	0.168351728027143	−0.03523188054954
7	0.000854971429905	0.143589975511347	−0.03256582839944
8	0.000371596957448	0.125325147337767	−0.02949958193595
9	0.000164574841194	0.111257399858839	−0.02668421757849
10	0.000073908605586	0.100066517745027	−0.02424367927438
11	0.000033552200238	0.090939592909307	−0.02216063262202
12	0.000015364834636	0.083347417765083	−0.02038360583157
13	0.000007086981657	0.076929618752290	−0.01885943628695
14	0.000003288762206	0.071431625279192	−0.01754220267938
15	0.000001534136645	0.066668098527535	−0.01639454294823
16	0.000000718881060	0.062500673950994	−0.01538669196834
17	0.000000338196904	0.058823847714733	−0.01449504667360
18	0.000000159662486	0.055555706347903	−0.01370083322728
19	0.000000075612039	0.052631650579827	−0.01298901840146
20	0.000000035908223	0.050000034112812	−0.01234746359706

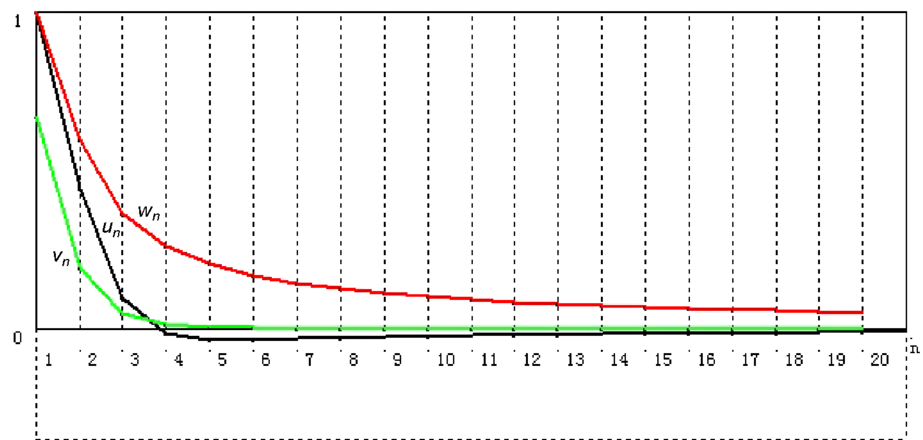

Figure 2 Convergence of $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$

ated by

$$\begin{cases} u_1 \in E, \quad \varepsilon_1 \in E, \\ v_n = \arg\min_{z\in E}\{h(z) + \frac{1}{2s_n}\|z\|^2 - \frac{1}{s_n}\langle z, J(u_n+\varepsilon_n)\rangle\}, \\ w_n = J^{-1}[\alpha_n Ju_n + (1-\alpha_n)JSv_n], \\ V_1 = W_1 = E, \\ V_{n+1} = \{z \in E : \langle v_n - z, J(u_n+\varepsilon_n) - Jv_n\rangle \geq 0\} \cap V_n, \\ W_{n+1} = \{z \in V_{n+1} : \varphi(z, w_n) \leq \alpha_n\varphi(z,u_n) + (1-\alpha_n)\varphi(z,v_n)\} \cap W_n, \\ U_{n+1} = \{z \in W_{n+1} : \|u_1 - z\|^2 \leq \|P_{W_{n+1}}(u_1) - u_1\|^2 + \tau_{n+1}\}, \\ u_{n+1} \in U_{n+1}, \quad n \in N. \end{cases}$$

Then

(1) *if* $v_n = u_n + \varepsilon_n$ *and* $w_n = J^{-1}[\alpha_n Ju_n + (1-\alpha_n)J(u_n+\varepsilon_n)]$ *for all* $n \in N$, *then* $u_n + \varepsilon_n \in N(\partial h) \cap F(S)$.

(2) *If* $N(\partial h) \cap F(S) \neq \emptyset$ *and* $\inf_n s_n > 0$, *then the iterative sequence* $u_n \to y_o = P_{\bigcap_{n=1}^{\infty} W_n}(u_1) \in N(\partial h) \cap F(S)$, *as* $n \to \infty$.

Proof Similar to [11], $v_n = \arg\min_{z\in E}\{h(z) + \frac{1}{2s_n}\|z\|^2 - \frac{1}{s_n}\langle z, J(u_n+\varepsilon_n)\rangle\}$ is equivalent to $0 \in \partial h(v_n) + \frac{1}{s_n}Ju_n - \frac{1}{s_n}J(u_n+\varepsilon_n)$. Then $v_n = (J + s_n\partial h)^{-1}J(u_n+\varepsilon_n)$. So, Corollary 2.3 ensures the desired results.

This completes the proof. □

Theorem 2.20 *We only do the following changes in Theorem* 2.19: $w_n = J^{-1}[\alpha_n Ju_1 + (1-\alpha_n)JSv_n]$ *and* $W_{n+1} = \{z \in V_{n+1} : \varphi(z,w_n) \leq \alpha_n\varphi(z,u_1) + (1-\alpha_n)\varphi(z,v_n)\} \cap W_n$. *Then, under the assumptions of Corollary* 2.6, *we still have the result of Theorem* 2.19.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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