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An accurate approximation formula for gamma function

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Abstract

In this paper, we present a very accurate approximation for the gamma function:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\frac{7}{324}\frac{1}{x^3(35x^2+33)}\right) = W_2(x)$$

as $x \to \infty$, and we prove that the function $x \mapsto \ln \Gamma(x+1) - \ln W_2(x)$ is strictly decreasing and convex from $(1,\infty)$ onto $(0,\beta)$, where

$$\beta = \frac{22{,}025}{22{,}032} - \ln \sqrt{2\pi \sinh 1} \approx 0.00002407.$$

MSC: Primary 33B15; 26D15; secondary 26A48; 26A51

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1 Introduction

The Stirling formula states that

$$n! \sim \sqrt{2\pi n} n^n e^{-n} \tag{1.1}$$

for $n \in \mathbb{N}$. The gamma function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ for $x > 0$ is a generalization of the factorial function $n!$ and has important applications in various branches of mathematics; see, for example, [1–6] and the references cited therein.

There are many refinements for the Stirling formula; see, for example, Burnside's [7], Gosper [8], Batir [9], Mortici [10]. Many authors pay attention to find various better approximations for the gamma function, for instance, Ramanujan [11, P. 339], Smith [12, Eq. (42)], [13], Mortici [14], Nemes [15, Corollary 4.1], Yang and Chu [16, Propositions 4 and 5], Chen [17].

More results involving the approximation formulas for the factorial or gamma function can be found in [16, 18–27] and the references cited therein. Several nice inequalities between gamma function and the truncations of its asymptotic series can be found in [28, 29].

Now let us focus on the Windschitl approximation formula (see [12, Eq. (42)], [13]) defined by

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2} := W_0(x) \quad \text{as } x \to \infty. \tag{1.2}$$

As shown in [17], the rate of Windschitl's approximation $W_0(x)$ converging to $\Gamma(x+1)$ is like x^{-5} as $x \to \infty$, and it is faster on replacing $W_0(x)$ by

$$W_1(x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x} + \frac{1}{810x^6}\right)^{x/2} \tag{1.3}$$

(see [13]). These results show that $W_0(x)$ and $W_1(x)$ are excellent approximations for the gamma function.

In 2009, Alzer [30] proved that, for all $x > 0$,

$$\begin{aligned} &\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\alpha}{x^5}\right) \\ &\quad < \Gamma(x+1) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\beta}{x^5}\right) \end{aligned} \tag{1.4}$$

with the best possible constants $\alpha = 0$ and $\beta = 1/1620$. Lu, Song and Ma [31] extended Windschitl's formula to

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left[n\sinh\left(\frac{1}{n} + \frac{a_7}{n^7} + \frac{a_9}{n^9} + \frac{a_{11}}{n^{11}} + \cdots\right)\right]^{n/2}$$

with $a_7 = 1/810, a_9 = -67/42{,}525, a_{11} = 19/8505, \ldots$. An explicit formula for determining the coefficients of n^{-k} ($n \in \mathbb{N}$) was given in [32, Theorem 1] by Chen. Another asymptotic expansion

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2+\sum_{j=0}^{\infty} r_j x^{-j}}, \quad x \to \infty \tag{1.5}$$

was presented in the same reference [32, Theorem 2].

Motivated by the above comments, the aim of this paper is to provide a more accurate Windschitl type approximation:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2} \exp\left(\frac{7}{324}\frac{1}{x^3(35x^2+33)}\right) = W_2(x) \tag{1.6}$$

as $x \to \infty$. Our main result is the following theorem.

Theorem 1 *The function*

$$f_0(x) = \ln\Gamma(x+1) - \ln\sqrt{2\pi} - \left(x+\frac{1}{2}\right)\ln x + x - \frac{x}{2}\ln\left(x\sinh\frac{1}{x}\right) - \frac{7}{324}\frac{1}{x^3(35x^2+33)}$$

is strictly decreasing and convex from $(1,\infty)$ *onto* $(0,f_0(1))$, *where*

$$f_0(1) = \frac{22{,}025}{22{,}032} - \ln\sqrt{2\pi\sinh 1} \approx 0.00002407.$$

2 Lemmas

An important research subject in analyzing inequality is to convert an univariate into the monotonicity of functions [33–35]. Since the function $f_0(x)$ contains gamma and hyperbolic functions, it is very hard to deal with its monotonicity and convexity by usual approaches. For this purpose, we need the following lemmas, which provide a new way to prove our result.

Lemma 1 *The inequality*

$$\psi'\left(x+\frac{1}{2}\right) > x\frac{x^4+\frac{227}{66}x^2+\frac{4237}{2640}}{x^6+\frac{155}{44}x^4+\frac{329}{176}x^2+\frac{375}{4928}}$$

holds for $x > 0$.

Proof Let

$$g_1(x) = \psi'\left(x+\frac{1}{2}\right) - x\frac{x^4+\frac{227}{66}x^2+\frac{4237}{2640}}{x^6+\frac{155}{44}x^4+\frac{329}{176}x^2+\frac{375}{4928}}.$$

Then by the recurrence formula [36, p. 260, (6.4.6)]

$$\psi'(x+1) - \psi'(x) = -\frac{1}{x^2}$$

we have

$$\begin{aligned}
&g_1(x+1) - g_1(x) \\
&\quad = \psi'\left(x+\frac{3}{2}\right) - \frac{(x+1)((x+1)^4+\frac{227}{66}(x+1)^2+\frac{4237}{2640})}{(x+1)^6+\frac{155}{44}(x+1)^4+\frac{329}{176}(x+1)^2+\frac{375}{4928}} \\
&\qquad - \psi'\left(x+\frac{1}{2}\right) + \frac{x(x^4+\frac{227}{66}x^2+\frac{4237}{2640})}{x^6+\frac{155}{44}x^4+\frac{329}{176}x^2+\frac{375}{4928}} \\
&\quad = -58{,}982{,}400(2x+1)^{-2}\left(4928x^6+17{,}360x^4+9212x^2+375\right)^{-1} \\
&\qquad \times \left(4928x^6+29{,}568x^5+91{,}280x^4+168{,}000x^3+187{,}292x^2\right. \\
&\qquad \left. +117{,}432x+31{,}875\right)^{-1} \\
&\quad < 0.
\end{aligned}$$

It then follows that

$$g_1(x) > g_1(x+1) > \cdots > \lim_{n\to\infty} g_1(x+n) = 0,$$

which proves the desired inequality, and the proof is done. □

Lemma 2 *The inequalities*

$$\frac{t}{\sinh t} > 1 - \frac{1}{6}t^2 + \frac{7}{360}t^4 - \frac{31}{15{,}120}t^6 + \frac{127}{604{,}800}t^8 - \frac{73}{3{,}421{,}440}t^{10} > 0 \tag{2.1}$$

hold for $t \in (0,1]$.

Proof It was proved in [29, Theorem 1] that, for integer $n \geq 0$, the double inequality

$$-\sum_{i=0}^{2n+1} \frac{2(2^{2i-1}-1)B_{2i}}{(2i)!}t^{2i-1} < \frac{1}{\sinh t} < -\sum_{i=0}^{2n} \frac{2(2^{2i-1}-1)B_{2i}}{(2i)!}t^{2i-1} \tag{2.2}$$

holds for $x > 0$. Taking $n = 2$ yields

$$\frac{1}{\sinh t} > \frac{1}{t} - \frac{1}{6}t + \frac{7}{360}t^3 - \frac{31}{15{,}120}t^5 + \frac{127}{604{,}800}t^7 - \frac{73}{3{,}421{,}440}t^9 := \frac{h(t)}{t},$$

which is equivalent to the first inequality of (2.1) for all $t > 0$.

Since $x \in (0,1]$, making a change of variable $t^2 = 1 - x \in (0,1]$ we obtain

$$\begin{aligned} h(t) = {} & \frac{73}{3{,}421{,}440}x^5 + \frac{12{,}371}{119{,}750{,}400}x^4 + \frac{85{,}243}{59{,}875{,}200}x^3 \\ & + \frac{858{,}623}{59{,}875{,}200}x^2 + \frac{15{,}950{,}191}{119{,}750{,}400}x + \frac{14{,}556{,}793}{17{,}107{,}200} > 0, \end{aligned}$$

which proves the second one, and the proof is complete. □

The following lemma offers a simple criterion to determine the sign of a class of special polynomial on given interval contained in $(0,\infty)$ without using Descartes' rule of signs, which play an important role in studying certain special functions; see for example [37, 38]. A series version can be found in [39].

Lemma 3 ([37, Lemma 7]) *Let* $n \in \mathbb{N}$ *and* $m \in \mathbb{N} \cup \{0\}$ *with* $n > m$ *and let* $P_n(t)$ *be a polynomial of degree* n *defined by*

$$P_n(t) = \sum_{i=m+1}^{n} a_i t^i - \sum_{i=0}^{m} a_i t^i, \tag{2.3}$$

where $a_n, a_m > 0$, $a_i \geq 0$ *for* $0 \leq i \leq n-1$ *with* $i \neq m$. *Then there is a unique number* $t_{m+1} \in (0,\infty)$ *satisfying* $P_n(t) = 0$ *such that* $P_n(t) < 0$ *for* $t \in (0, t_{m+1})$ *and* $P_n(t) > 0$ *for* $t \in (t_{m+1}, \infty)$.

Consequently, for given $t_0 > 0$, *if* $P_n(t_0) > 0$ *then* $P_n(t) > 0$ *for* $t \in (t_0, \infty)$ *and if* $P_n(t_0) < 0$ *then* $P_n(t) < 0$ *for* $t \in (0, t_0)$.

3 Proof of Theorem 1

With the aid of the lemmas in Sect. 2, we can prove Theorem 1.

Proof of Theorem 1 Differentiation yields

$$f_0'(x) = \psi(x+1) - \frac{1}{2}\ln\left(x \sinh\frac{1}{x}\right) + \frac{1}{2x}\coth\frac{1}{x}$$

$$
\begin{aligned}
&\quad - \ln x - \frac{1}{2x} - \frac{1}{2} + \frac{7}{324}\frac{175x^2+99}{x^4(35x^2+33)^2}, \\
f_0''(x) &= \psi'(x+1) + \frac{1}{2x^3}\frac{1}{\sinh^2(1/x)} \\
&\quad - \frac{3}{2x} + \frac{1}{2x^2} - \frac{7}{54}\frac{6125x^4+6545x^2+2178}{x^5(35x^2+33)^3}.
\end{aligned}
$$

Since $\lim_{x\to\infty} f_0(x) = \lim_{x\to\infty} f_0'(x) = 0$, it suffices to prove $f_0''(x) > 0$ for $x \geq 1$. Replacing x by $(x + 1/2)$ in Lemma 1 leads to

$$
\psi'(x+1) > \frac{7}{30}\frac{(2x+1)(165x^4+330x^3+815x^2+650x+417)}{77x^6+231x^5+560x^4+735x^3+623x^2+294x+60},
$$

which indicates that

$$
\begin{aligned}
f_0''(x) &> \frac{7}{30}\frac{(2x+1)(165x^4+330x^3+815x^2+650x+417)}{77x^6+231x^5+560x^4+735x^3+623x^2+294x+60} + \frac{1}{2x^3}\frac{1}{\sinh^2(1/x)} \\
&\quad - \frac{3}{2x} + \frac{1}{2x^2} - \frac{7}{54}\frac{6125x^4+6545x^2+2178}{x^5(35x^2+33)^3} := f_{01}\left(\frac{1}{x}\right).
\end{aligned}
$$

Arranging gives

$$
\begin{aligned}
f_{01}(t) &= \frac{t}{2}\left(\frac{t}{\sinh t}\right)^2 + \frac{7}{30}\frac{t(t+2)(417t^4+650t^3+815t^2+330t+165)}{60t^6+294t^5+623t^4+735t^3+560t^2+231t+77} \\
&\quad - \frac{3}{2}t + \frac{1}{2}t^2 - \frac{7}{54}t^7\frac{2178t^4+6545t^2+6125}{(33t^2+35)^3},
\end{aligned}
$$

where $t = 1/x \in (0, 1)$. Applying the first inequality of (2.1) we have

$$
\begin{aligned}
f_{01}(t) &> \frac{t}{2}\left(1 - \frac{1}{6}t^2 + \frac{7}{360}t^4 - \frac{31}{15{,}120}t^6 + \frac{127}{604{,}800}t^8 - \frac{73}{3{,}421{,}440}t^{10}\right)^2 \\
&\quad + \frac{7}{30}\frac{t(t+2)(417t^4+650t^3+815t^2+330t+165)}{60t^6+294t^5+623t^4+735t^3+560t^2+231t+77} \\
&\quad - \frac{3}{2}t + \frac{1}{2}t^2 - \frac{7}{54}t^7\frac{2178t^4+6545t^2+6125}{(33t^2+35)^3} \\
&= \frac{t^{11}\times p_{22}(t)}{(33t^2+35)^3(60t^6+294t^5+623t^4+735t^3+560t^2+231t+77)},
\end{aligned}
$$

where $p_{22}(t) = \sum_{k=0}^{22} a_k t^k$ with $a_0 = \frac{2{,}341{,}955}{27}$, $a_1 = \frac{2{,}341{,}955}{9}$, $a_2 = \frac{4{,}592{,}761{,}525{,}177}{41{,}057{,}280}$, $a_3 = \frac{3{,}740{,}791{,}861{,}177}{13{,}685{,}760}$, $a_4 = -\frac{21{,}774{,}907{,}040{,}747}{615{,}859{,}200}$, $a_5 = \frac{1{,}776{,}198{,}096{,}757}{51{,}321{,}600}$, $a_6 = -\frac{2{,}348{,}474{,}362{,}865{,}491}{59{,}122{,}483{,}200}$, $a_7 = -\frac{444{,}392{,}576{,}792{,}851}{19{,}707{,}494{,}400}$, $a_8 = \frac{722{,}576{,}509{,}559{,}549}{344{,}881{,}152{,}000}$, $a_9 = \frac{734{,}284{,}235{,}570{,}623}{229{,}920{,}768{,}000}$, $a_{10} = -\frac{27{,}685{,}269{,}148{,}007{,}477}{74{,}494{,}328{,}832{,}000}$, $a_{11} = -\frac{13{,}202{,}571{,}814{,}150{,}457}{24{,}831{,}442{,}944{,}000}$, $a_{12} = \frac{1{,}859{,}898{,}503{,}651{,}431}{585{,}312{,}583{,}680{,}000}$, $a_{13} = \frac{40{,}990{,}762{,}057{,}313{,}921}{682{,}864{,}680{,}960{,}000}$, $a_{14} = \frac{1{,}227{,}464{,}630{,}525{,}327}{573{,}606{,}332{,}006{,}400}$, $a_{15} = -\frac{107{,}829{,}513{,}340{,}517}{19{,}510{,}419{,}456{,}000}$, $a_{16} = -\frac{1{,}469{,}516{,}232{,}022{,}339}{4{,}780{,}052{,}766{,}720{,}000}$, $a_{17} = \frac{224{,}320{,}158{,}179}{492{,}687{,}360{,}000}$, $a_{18} = \frac{214{,}165{,}238{,}137}{6{,}437{,}781{,}504{,}000}$, $a_{19} = -\frac{402{,}182{,}039}{11{,}943{,}936{,}000}$, $a_{20} = -\frac{150{,}639{,}953}{50{,}164{,}531{,}200}$, $a_{21} = \frac{2{,}872{,}331}{1{,}194{,}393{,}600}$, $a_{22} = \frac{58{,}619}{119{,}439{,}360}$.

It remains to prove $p_{22}(t) = \sum_{k=0}^{22} a_k t^k > 0$ for $t \in (0, 1]$. Since $a_k > 0$ for $k = 0, 1, 2, 3, 8, 9, 12, 13, 14, 17, 18, 21, 22$ and $a_k < 0$ for $k = 4, 6, 7, 10, 11, 15, 16, 19, 20$, we have

$$
p_{22}(t) = \sum_{k=0}^{22} a_k t^k = \sum_{a_k>0} a_k t^k + \sum_{a_k<0} a_k t^k > \sum_{k=4,6,7,10,11,15,16,19,20} a_k t^k + \sum_{k=0}^{3} a_k t^k := p_{20}(t).
$$

Clearly, the coefficients of the polynomial $-p_{20}(t)$ satisfy the conditions in Lemma 3, and

$$-p_{20}(1) = \sum_{k=4,6,7,10,11,15,16,19,20} (-a_k) - \sum_{k=0}^{3} a_k = -\frac{1{,}135{,}768{,}202{,}621{,}781{,}774{,}901}{1{,}792{,}519{,}787{,}520{,}000} < 0.$$

It then follows that $p_{20}(t) > 0$ for $t \in (0,1]$, and so is $p_{22}(t)$, which implies $f_{01}(t) > 0$ for $t \in (0,1]$. Consequently, $f_0''(x) > 0$ for all $x \geq 1$. This completes the proof. □

As a direct consequence of Theorem 1, we immediately get the following.

Corollary 1 *For $n \in \mathbb{N}$, the double inequality*

$$\exp\frac{7}{324n^3(35n^2+33)} < \frac{n!}{\sqrt{2\pi n}(n/e)^n(n\sinh n^{-1})^{n/2}} < \lambda\exp\frac{7}{324n^3(35n^2+33)}$$

holds with the best constant

$$\lambda = \exp f_0(1) = \frac{1}{\sqrt{2\pi\sinh 1}}\exp\frac{22{,}025}{22{,}032} \approx 1.000024067.$$

Set

$$D_0(y) = y - \ln(1+y), \quad y = \frac{7}{324x^3(35x^2+33)}.$$

Then it is easy to check that, for $x > 1$,

$$\begin{aligned}
\frac{dD_0(y)}{dx} &= -\frac{49}{324}\frac{175x^2+99}{x^4(35x^2+33)^2(11{,}340x^5+10{,}692x^3+7)} < 0,\\
\frac{d^2D_0(y)}{dx^2} &= \frac{343}{54}\frac{(18{,}191{,}250x^9+37{,}110{,}150x^7+24{,}992{,}550x^5+6125x^4+5{,}821{,}794x^3+6545x^2+2178)}{x^5(35x^2+33)^3(11{,}340x^5+10{,}692x^3+7)^2}\\
&> 0.
\end{aligned}$$

That is to say, $x \mapsto D_0(y)$ is decreasing and convex on $(1,\infty)$, and so is the function $f_0^*(x) := f_0(x) + D_0(y)$ by Theorem 1.

Corollary 2 *The function*

$$\begin{aligned}
f_0^*(x) = {} & \ln\Gamma(x+1) - \ln\sqrt{2\pi} - \left(x+\frac{1}{2}\right)\ln x + x - \frac{x}{2}\ln\left(x\sinh\frac{1}{x}\right)\\
& - \ln\left(1+\frac{7}{324x^3(35x^2+33)}\right)
\end{aligned}$$

is strictly decreasing and convex from $(1,\infty)$ onto $(0, f_0^*(1))$, where

$$f_0^*(1) = 1 - \ln\frac{22{,}039}{22{,}032} - \ln\sqrt{2\pi\sinh 1} \approx 0.00002412.$$

Remark 1 Corollary 2 offers another approximation formula

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{7}{324}\frac{1}{x^3(35x^2+33)}\right) = W_2^*(x). \tag{3.1}$$

Also, for $n \in \mathbb{N}$,

$$1 + \frac{7}{324n^3(35n^2+33)} < \frac{n!}{\sqrt{2\pi n}(n/e)^n(n\sinh n^{-1})^{n/2}} < \lambda^*\left(1 + \frac{7}{324n^3(35n^2+33)}\right)$$

with the best constant

$$\lambda^* = \exp f_0^*(1) = \frac{22{,}032}{22{,}039}\frac{e}{\sqrt{2\pi\sinh 1}} \approx 1.000024117.$$

4 Numerical comparisons

It is well known that an excellent approximation for the gamma function is fairly accurate but relatively simple. In this section, we list some known approximation formulas for the gamma function and compare them with $W_1(x)$ given by (1.3) and our new one $W_2(x)$ defined by (1.6).

It has been shown in [17] that, as $x \to \infty$, Ramanujan's [11, P. 339] approximation formula holds,

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6}\left(1+O\left(\frac{1}{x^4}\right)\right) := R(x),$$

and Smith's one [12, Eq. (42)],

$$\Gamma\left(x+\frac{1}{2}\right) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x\left(2x\tanh\frac{1}{2x}\right)^{x/2}\left(1+O\left(\frac{1}{x^5}\right)\right) := S(x),$$

Nemes' one [15, Corollary 4.1],

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{12x^2-1/10}\right)^x\left(1+O\left(\frac{1}{x^5}\right)\right) =: N_1(x),$$

and Chen's one [17],

$$\begin{aligned}\Gamma(x+1) &\sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{12x^3+24x/7-1/2}\right)^{x^2+53/210}\left(1+O\left(\frac{1}{x^7}\right)\right)\\ &:= C(x).\end{aligned} \tag{4.1}$$

Moreover, it is easy to check that Nemes' result [13] is another one,

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\exp\left(\frac{210x^2+53}{360x(7x^2+2)}\right)\left(1+O\left(\frac{1}{x^7}\right)\right) := N_2(x), \tag{4.2}$$

and so are Yang and Chu's [16, Propositions 4 and 5] ones,

$$\Gamma\left(x+\frac{1}{2}\right) = \sqrt{2\pi}\left(\frac{x}{e}\right)^x\exp\left(-\frac{1}{24}\frac{x}{x^2+7/120}\right)\left(1+O\left(\frac{1}{x^5}\right)\right) := Y_1(x),$$

$$\Gamma\left(x+\frac{1}{2}\right) = \sqrt{2\pi}\left(\frac{x}{e}\right)^x\exp\left(-\frac{1}{24x}+\frac{7}{2880x}\frac{1}{x^2+31/98}\right)\left(1+O\left(\frac{1}{x^7}\right)\right) := Y_2(x),$$

Table 1 Comparison among N_2 (4.2), C (4.1), W_1 (1.3) and W_2 (1.6)

x	$\lvert\frac{N_2(x)-\Gamma(x+1)}{\Gamma(x+1)}\rvert$	$\lvert\frac{C(x)-\Gamma(x+1)}{\Gamma(x+1)}\rvert$	$\lvert\frac{W_1(x)-\Gamma(x+1)}{\Gamma(x+1)}\rvert$	$\lvert\frac{W_2(x)-\Gamma(x+1)}{\Gamma(x+1)}\rvert$
1	1.114×10^{-4}	1.398×10^{-4}	1.832×10^{-4}	2.407×10^{-5}
2	1.900×10^{-6}	2.222×10^{-6}	2.668×10^{-6}	2.308×10^{-7}
5	4.353×10^{-9}	4.956×10^{-9}	5.743×10^{-9}	1.249×10^{-10}
10	3.609×10^{-11}	4.088×10^{-11}	4.710×10^{-11}	2.785×10^{-13}
20	2.864×10^{-13}	3.240×10^{-13}	3.727×10^{-13}	5.634×10^{-16}
50	4.713×10^{-16}	5.330×10^{-16}	6.129×10^{-16}	1.492×10^{-19}
100	3.684×10^{-18}	4.166×10^{-18}	4.791×10^{-18}	2.918×10^{-22}

and we have Windschitl one [13],

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x} + \frac{1}{810x^6}\right)^{x/2}\left(1 + O\left(\frac{1}{x^7}\right)\right) = W_1(x).$$

For our new ones $W_2(x)$ given in (1.6) and its counterpart $W_2^*(x)$ given in (3.1), we easily check that

$$\lim_{x\to\infty} \frac{\ln\Gamma(x+1) - \ln W_2(x)}{x^{-9}} = \lim_{x\to\infty} \frac{\ln\Gamma(x+1) - \ln W_2^*(x)}{x^{-9}} = \frac{869}{2{,}976{,}750},$$

which show that the rates of $W_2(x)$ and $W_2^*(x)$ converging to $\Gamma(x+1)$ are both as x^{-9}.

From these, we see that our new Windschitl type approximation formulas $W_2(x)$ and $W_2^*(x)$ are best among those listed above, which can also be seen from Table 1.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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