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A method for estimating the power of moments

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Abstract

Let X be an observable random variable with unknown distribution function $F(x) = \mathbb{P}(X \le x)$, $-\infty < x < \infty$, and let

$$\theta = \sup\{r \ge 0 : \mathbb{E}|X|^r < \infty\}.$$

We call θ the power of moments of the random variable X. Let $X_1, X_2, \dots, X_n$ be a random sample of size n drawn from $F(\cdot)$. In this paper we propose the following simple point estimator of θ and investigate its asymptotic properties:

$$\hat{\theta}_n = \frac{\log n}{\log \max_{1 \le k \le n} |X_k|},$$

where $\log x = \ln(e \vee x)$, $-\infty < x < \infty$. In particular, we show that

$$\hat{\theta}_n \to_{\mathbb{P}} \theta \quad \text{if and only if} \quad \lim_{x\to\infty} x^r \mathbb{P}(|X| > x) = \infty \quad \forall r > \theta.$$

This means that, under very reasonable conditions on $F(\cdot)$, $\hat{\theta}_n$ is actually a consistent estimator of θ.

MSC: 62F10; 60F15; 62F12

Keywords: Asymptotic theorems; Consistent estimator; Point estimator; Power of moments

1 Motivation

The motivation of the current work arises from the following problem concerning parameter estimation. Let X be an observable random variable with unknown distribution function $F(x) = \mathbb{P}(X \le x)$, $-\infty < x < \infty$, and let

$$\theta = \sup\{r \ge 0 : \mathbb{E}|X|^r < \infty\}.$$

We call θ the *power of moments* of the random variable X. Clearly θ is a parameter of the distribution of the random variable X. Now let $X_1, X_2, \dots, X_n$ be a random sample of size n drawn from the random variable X; i.e., $X_1, X_2, \dots, X_n$ are independent and identically distributed (i.i.d.) random variables whose common distribution function is $F(\cdot)$. It is natural to pose the following question: Can we estimate the parameter θ based on the random sample $X_1, \dots, X_n$?

This is a serious and important problem. For example, if $\theta > 2$ and if the distribution of X is nondegenerate, then it is clear that $0 < \operatorname{Var} X < \infty$ and so by the classical Lévy central limit theorem, the distribution of

$$\frac{S_n - n\mu}{\sqrt{n}}$$

is approximately normal (for all sufficiently large n) with mean 0 and variance $\sigma^2 = \operatorname{Var} X = \mathbb{E}(X - \mu)^2$ where $\mu = \mathbb{E}X$. Thus the problem that we are facing is how can we conclude with a high degree of confidence that $\theta > 2$.

In this paper we propose the following point estimator of θ and will investigate its asymptotic properties:

$$\hat{\theta}_n = \frac{\log n}{\log \max_{1 \le k \le n} |X_k|}.$$

Here and below $\log x = \ln(e \vee x)$, $-\infty < x < \infty$.

Our main results will be stated in Sect. 2 and they all pertain to a sequence of i.i.d. random variables $\{X_n; n \ge 1\}$ drawn from the distribution function $F(\cdot)$ of the random variable X. The proofs of our main results will be provided in Sect. 3.

2 Statement of the main results

Throughout, X is a random variable with unknown distribution $F(x) = \mathbb{P}(X \le x)$, $-\infty < x < \infty$ and write

$$\rho_1 = \sup\Bigl\{r \ge 0 : \lim_{x \to \infty} x^r \mathbb{P}(X > x) = 0\Bigr\} \quad \text{and} \quad \rho_2 = \sup\Bigl\{r \ge 0 : \liminf_{x \to \infty} x^r \mathbb{P}(X > x) = 0\Bigr\}.$$

Clearly, just as θ as defined in Sect. 1 is a parameter of the distribution $F(\cdot)$ of the random variable X, so are ρ_1 and ρ_2. These parameters satisfy

$$0 \le \rho_1 \le \rho_2 \le \infty.$$

The main results of this paper are Theorems 2.1–2.5.

Theorem 2.1 *Let $\{X_n; n \ge 1\}$ be a sequence of i.i.d. random variables drawn from the distribution function $F(\cdot)$ of the random variable X. Then*

$$\limsup_{n \to \infty} \frac{\log \max_{1 \le k \le n} X_k}{\log n} = \frac{1}{\rho_1} \quad \textit{a.s.} \tag{2.1}$$

and there exists an increasing positive integer sequence $\{l_n; n \ge 1\}$ (which depends on the probability distribution of X when $\rho_1 < \infty$) such that

$$\lim_{n \to \infty} \frac{\log \max_{1 \le k \le l_n} X_k}{\log l_n} = \frac{1}{\rho_1} \quad \textit{a.s.} \tag{2.2}$$

Theorem 2.2 *Let $\{X_n; n \geq 1\}$ be a sequence of i.i.d. random variables drawn from the distribution function $F(\cdot)$ of the random variable X. Then*

$$\liminf_{n\to\infty} \frac{\log \max_{1\leq k\leq n} X_k}{\log n} = \frac{1}{\rho_2} \quad a.s. \tag{2.3}$$

and there exists an increasing positive integer sequence $\{m_n; n \geq 1\}$ (which depends on the probability distribution of X when $\rho_2 > 0$) such that

$$\lim_{n\to\infty} \frac{\log \max_{1\leq k\leq m_n} X_k}{\log m_n} = \frac{1}{\rho_2} \quad a.s. \tag{2.4}$$

Remark 2.1 We must point out that (2.2) and (2.4) are two interesting conclusions. To see this, let $\{U_n; n \geq 1\}$ be a sequence of independent random variables with

$$\mathbb{P}(U_n = 1) = \mathbb{P}(U_n = 3) = \frac{1}{2n} \quad \text{and} \quad \mathbb{P}(U_n = 2) = 1 - \frac{1}{n}, \quad n \geq 1.$$

Since

$$\sum_{n=1}^{\infty} \mathbb{P}(U_n = 3) = \sum_{n=1}^{\infty} \mathbb{P}(U_n = 1) = \sum_{n=1}^{\infty} \frac{1}{2n} = \infty,$$

it follows from the Borel–Cantelli lemma that

$$\limsup_{n\to\infty} U_n = 3 \quad \text{a.s.} \quad \text{and} \quad \liminf_{n\to\infty} U_n = 1 \quad \text{a.s.}$$

However, for any sequences $\{l_n; n \geq 1\}$ and $\{m_n; n \geq 1\}$ of increasing positive integers,

$$\text{neither} \quad \lim_{n\to\infty} U_{l_n} = 3 \quad \text{a.s.} \quad \text{nor} \quad \lim_{n\to\infty} U_{m_n} = 1 \quad \text{a.s.} \quad \text{holds.}$$

Remark 2.2 For an observable random variable X, it is often the case that $\rho_1 = \rho_2$. However, for any given constants ρ_1 and ρ_2 with $0 \leq \rho_1 < \rho_2 \leq \infty$, one can construct a random variable X such that

$$\sup\Big\{r \geq 0 : \lim_{x\to\infty} x^r \mathbb{P}(X > x) = 0\Big\} = \rho_1 \quad \text{and} \quad \sup\Big\{r \geq 0 : \liminf_{x\to\infty} x^r \mathbb{P}(X > x) = 0\Big\} = \rho_2.$$

For example, if $0 < \rho_1 < \rho_2 < \infty$, a random variable X can be constructed having probability distribution given by

$$\mathbb{P}(X = d_n) = \frac{c}{d_n^{\rho_1}}, \quad n \geq 1,$$

where $d_n = 2^{(\rho_2/\rho_1)^n}$, $n \geq 1$ and

$$c = \left(\sum_{n=1}^{\infty} \frac{1}{d_n^{\rho_1}}\right)^{-1} > 0.$$

Combining Theorems 2.1 and 2.2, we establish a law of large numbers for $\log \max_{1\leq k\leq n} X_k$, $n \geq 1$ as follows.

Theorem 2.3 *Let $\{X_n; n \geq 1\}$ be a sequence of i.i.d. random variables drawn from the distribution function $F(\cdot)$ of the random variable X and let $\rho \in [0,\infty]$. Then the following four statements are equivalent*:

$$\frac{\log \max_{1\leq k\leq n} X_k}{\log n} \xrightarrow{a.s.} \frac{1}{\rho}, \tag{2.5}$$

$$\frac{\log \max_{1\leq k\leq n} X_k}{\log n} \xrightarrow{\mathbb{P}} \frac{1}{\rho}, \tag{2.6}$$

$$\rho_1 = \rho_2 = \rho, \tag{2.7}$$

$$\lim_{x\to\infty} x^r \mathbb{P}(X > x) = \begin{cases} 0 & \forall r < \rho \textit{ if } \rho > 0, \\ \infty & \forall r > \rho \textit{ if } \rho < \infty. \end{cases} \tag{2.8}$$

If $0 \leq \rho < \infty$, then anyone of (2.5)–(2.8) *holds if and only if there exists a function $L(\cdot) : (0,\infty) \to (0,\infty)$ such that*

$$\mathbb{P}(X > x) \sim \frac{L(x)}{x^\rho} \quad \textit{as } x \to \infty \quad \textit{and} \quad \lim_{x\to\infty} \frac{\ln L(x)}{\ln x} = 0. \tag{2.9}$$

The following result concerns convergence in distribution for $\log \max_{1\leq k\leq n} X_k$, $n \geq 1$.

Theorem 2.4 *Let $\{X_n; n \geq 1\}$ be a sequence of i.i.d. random variables drawn from the distribution function $F(\cdot)$ of the random variable X. Suppose that there exist constants $0 < \rho < \infty$ and $-\infty < \tau < \infty$ and a monotone function $h(\cdot) : [0,\infty) \to (0,\infty)$ with $\lim_{x\to\infty} h(x^2)/h(x) = 1$ such that*

$$\mathbb{P}(X > x) \sim \frac{(\log x)^\tau h(x)}{x^\rho} \quad \textit{as } x \to \infty. \tag{2.10}$$

Then

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{P}\left(\log \max_{1\leq k\leq n} X_k \leq \frac{\ln n + \tau \ln\ln n + \ln h(n) - \tau \ln \rho + x}{\rho}\right) \\ &\quad = \exp\left(-e^{-x}\right) \quad \forall -\infty < x < \infty. \end{aligned} \tag{2.11}$$

Also, by Theorems 2.1–2.3, we have the following result for the point estimator $\hat{\theta}_n$.

Theorem 2.5 *Let $\{X_n; n \geq 1\}$ be a sequence of i.i.d. random variables drawn from the distribution function $F(\cdot)$ of the random variable X. Let*

$$\hat{\theta}_n = \frac{\log n}{\log \max_{1\leq k\leq n} |X_k|}, \quad n \geq 1.$$

Then we have

$$\liminf_{n\to\infty} \hat{\theta}_n = \theta = \sup\left\{r \geq 0 : \mathbb{E}|X|^r < \infty\right\} \quad \textit{a.s.},$$

$$\limsup_{n\to\infty} \hat{\theta}_n = \sup\left\{r \geq 0 : \liminf_{x\to\infty} x^r \mathbb{P}\left(|X| > x\right) = 0\right\} \quad \textit{a.s.},$$

and the following three statements are equivalent:

$$\hat{\theta}_n \xrightarrow{a.s.} \theta, \tag{2.12}$$

$$\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta, \tag{2.13}$$

$$\lim_{x\to\infty} x^r \mathbb{P}\big(|X| > x\big) = \infty \quad \forall r > \theta \text{ if } \theta < \infty. \tag{2.14}$$

If $0 \le \theta < \infty$, *then anyone of* (2.12)–(2.14) *holds if and only if there exists a function* $L(\cdot) : (0,\infty) \to (0,\infty)$ *such that*

$$\mathbb{P}\big(|X| > x\big) \sim \frac{L(x)}{x^\theta} \quad \text{as } x \to \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{\ln L(x)}{\ln x} = 0. \tag{2.15}$$

Remark 2.3 Let $\{X_n; n \ge 1\}$ be a sequence of i.i.d. random variables drawn from the distribution function $F(\cdot)$ of some nonnegative random variable X. For each $n \ge 1$, let $X_{n,1} \le X_{n,2} \le \cdots \le X_{n,n}$ denote the order statistics based on $X_1, X_2, \dots, X_n$. To estimate the tail index of $F(\cdot)$, the well-known Hill estimator, proposed by Hill [1], is defined by

$$\hat{\alpha}_n = \left(\frac{1}{k_n}\sum_{i=1}^{k_n} \ln \frac{X_{n,n-i+1}}{X_{n,n-k_n}}\right)^{-1},$$

where $\{k_n; n \ge 1\}$ is a sequence of positive integers satisfying

$$1 \le k_n < n \quad \text{and} \quad k_n \to \infty \quad \text{and} \quad k_n/n \to 0 \quad \text{as } n \to \infty. \tag{2.16}$$

Mason [2, Theorem 2] showed that, for some constant $\theta \in (0,\infty)$,

$$\hat{\alpha}_n \xrightarrow{\mathbb{P}} \theta \quad \text{for every sequence } \{k_n; n \ge 1\} \text{ satisfying (2.16)}$$

if and only if

$$\mathbb{P}(X > x) \sim \frac{L(x)}{x^\theta}$$
$$\text{as } x \to \infty \text{ where } L(\cdot) : (0,\infty) \to (0,\infty) \text{ is a slowly varying function.} \tag{2.17}$$

Since $L(\cdot)$ defined in (2.17) is a slowly varying function,

$$\lim_{t\to\infty} \frac{\log L(t)}{\log t} = 0$$

is always true and hence (2.15) follows from (2.17). However, the following example shows that (2.15) does not imply (2.17). Thus condtion (2.15) is weaker than (2.17).

Example 2.1 Let $\{X_n; n \ge 1\}$ be a sequence of i.i.d. random variables drawn from the distribution function $F(\cdot)$ of some nonnegative random variable X given by

$$F(x) = 1 - \exp\big(-\theta\big[\ln(x \vee 1)\big]\big), \quad x \ge 0,$$

where $\theta \in (0, \infty)$ is the tail index of the distribution and $[t]$ denotes the integer part of t. Then (2.15) holds but (2.17) is not satisfied. To see this, let

$$L(x) = \exp\bigl(\theta\bigl(\ln x - [\ln x]\bigr)\bigr), \quad x \geq e.$$

Then

$$\mathbb{P}(X > x) = 1 - F(x) = x^{-\theta} L(x), \quad x \geq e.$$

Since, for $x \geq e$, $0 \leq \ln x - [\ln x] \leq 1$, we have

$$1 \leq L(x) \leq \exp(\theta), \quad x \geq 1$$

and hence (2.15) holds. However, for $1 < a < e$ and $x_n = e^n$, $n \geq 1$, we have

$$\ln(ax_n) - \bigl[\ln(ax_n)\bigr] = (n + \ln a) - [n + \ln a] = \ln a \quad \text{and} \quad \ln(x_n) - \bigl[\ln(x_n)\bigr] = n - [n] = 0.$$

Thus, for $\theta \in (0, \infty)$,

$$\frac{L(ax_n)}{L(x_n)} = \frac{\exp(\theta(\ln a))}{\exp(\theta \times 0)} = a^\theta > 1, \quad n \geq 1$$

and hence

$$\lim_{x \to \infty} \frac{L(ax)}{L(x)} = 1 \quad \text{does not hold;}$$

i.e., $L(\cdot)$ is not a slowly varying function. Thus (2.17) is not satisfied and hence, for this example, the well-known Hill estimator cannot be used to estimate the tail index θ.

3 Proofs of the main results

Let $\{A_n; n \geq 1\}$ be a sequence of events. As usual the abbreviation $\{A_n \text{ i.o.}\}$ stands for the case that the events A_n occur infinitely often. That is,

$$\{A_n \text{ i.o.}\} = \{\text{events } A_n \text{ occur infinitely often}\} = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j.$$

For events A and B, we say $A = B$ a.s. if $\mathbb{P}(A \,\Delta\, B) = 0$ where $A \,\Delta\, B = (A \setminus B) \cup (B \setminus A)$. To prove Theorem 2.1, we use the following preliminary result, which can be found in Chandra [3, Example 1.6.25(a), p. 48].

Lemma 3.1 *Let $\{b_n; n \geq 1\}$ be a nondecreasing sequence of positive real numbers such that*

$$\lim_{n \to \infty} b_n = \infty$$

and let $\{V_n; n \geq 1\}$ be a sequence of random variables. Then

$$\Bigl\{\max_{1 \leq k \leq n} V_k \geq b_n \text{ i.o.}\Bigr\} = \{V_n \geq b_n \text{ i.o.}\} \quad \text{a.s.}$$

Proof of Theorem 2.1 *Case I:* $0 < \rho_1 < \infty$. For given $\epsilon > 0$, let $r(\epsilon) = (\frac{1}{\rho_1} + \epsilon)^{-1}$. Then

$$0 < r(\epsilon) < \rho_1 = \sup\Big\{r \geq 0 : \lim_{x\to\infty} x^r \mathbb{P}(X > x) = 0\Big\}$$

and hence

$$\sum_{n=1}^{\infty} \mathbb{P}\big(X > n^{1/r(\epsilon)}\big) < \infty. \tag{3.1}$$

By the Borel–Cantelli lemma, (3.1) implies that

$$\mathbb{P}\big(X_n > n^{1/r(\epsilon)} \text{ i.o.}\big) = 0.$$

By Lemma 3.1, we have

$$\left\{\frac{\log \max_{1\leq k\leq n} X_k}{\log n} > \frac{1}{\rho_1} + \epsilon \text{ i.o.}\right\} = \Big\{\max_{1\leq k\leq n} X_k > n^{1/r(\epsilon)} \text{ i.o.}\Big\} = \big\{X_n > n^{1/r(\epsilon)} \text{ i.o.}\big\} \quad \text{a.s.}$$

and hence

$$\mathbb{P}\left(\frac{\log \max_{1\leq k\leq n} X_k}{\log n} > \frac{1}{\rho_1} + \epsilon \text{ i.o.}\right) = 0.$$

Thus

$$\limsup_{n\to\infty} \frac{\log \max_{1\leq k\leq n} X_k}{\log n} \leq \frac{1}{\rho_1} + \epsilon \quad \text{a.s.}$$

Letting $\epsilon \searrow 0$, we get

$$\limsup_{n\to\infty} \frac{\log \max_{1\leq k\leq n} X_k}{\log n} \leq \frac{1}{\rho_1} \quad \text{a.s.} \tag{3.2}$$

By the definition of ρ_1, we have

$$\limsup_{x\to\infty} x^r \mathbb{P}(X > x) = \infty \quad \forall r > \rho_1,$$

which is equivalent to

$$\limsup_{x\to\infty} x\mathbb{P}\big(X > x^{(1/\rho_1)-\epsilon}\big) = \infty \quad \forall \epsilon > 0.$$

Then, inductively, we can choose positive integers l_n, $n \geq 1$ such that

$$1 = l_1 < l_2 < \cdots < l_n < \cdots \quad \text{and} \quad l_n \mathbb{P}\big(X > l_n^{(1/\rho_1)-(1/n)}\big) \geq 2\ln n, \quad n \geq 1.$$

Note that, for any $0 \le z \le 1$, $1 - z \le e^{-z}$. Thus, for all sufficiently large n, we have

$$\begin{aligned}
\mathbb{P}\left(\frac{\log \max_{1\le k\le l_n} X_k}{\log l_n} \le \frac{1}{\rho_1} - \frac{1}{n}\right) &= \mathbb{P}\Bigl(\max_{1\le k\le l_n} X_k \le l_n^{(1/\rho_1)-(1/n)}\Bigr) \\
&= \bigl(1 - \mathbb{P}\bigl(X > l_n^{(1/\rho_1)-(1/n)}\bigr)\bigr)^{l_n} \\
&\le \exp\bigl(-l_n\mathbb{P}\bigl(X > l_n^{(1/\rho_1)-(1/n)}\bigr)\bigr) \\
&\le \exp(-2\ln n) \\
&= n^{-2}.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} n^{-2} < \infty$, by the Borel–Cantelli lemma, we get

$$\mathbb{P}\left(\frac{\log \max_{1\le k\le l_n} X_k}{\log l_n} \le \frac{1}{\rho_1} - \frac{1}{n} \text{ i.o.}\right) = 0$$

which ensures that

$$\liminf_{n\to\infty} \frac{\log \max_{1\le k\le l_n} X_k}{\log l_n} \ge \frac{1}{\rho_1} \quad \text{a.s.} \tag{3.3}$$

Clearly, (2.1) and (2.2) follow from (3.2) and (3.3).

Case II: $\rho_1 = \infty$. Using the same argument used in the first half of the proof for Case I, we get

$$\limsup_{n\to\infty} \frac{\log \max_{1\le k\le n} X_k}{\log n} \le \epsilon \quad \text{a.s. } \forall \epsilon > 0$$

and hence

$$\limsup_{n\to\infty} \frac{\log \max_{1\le k\le n} X_k}{\log n} \le 0 \quad \text{a.s.} \tag{3.4}$$

Note that

$$0 \le \frac{\log \max_{1\le k\le n} X_k}{\log n} \quad \forall n \ge 1.$$

We thus have

$$\liminf_{n\to\infty} \frac{\log \max_{1\le k\le n} X_k}{\log n} \ge 0 \quad \text{a.s.} \tag{3.5}$$

It thus follows from (3.4) and (3.5) that

$$\lim_{n\to\infty} \frac{\log \max_{1\le k\le n} X_k}{\log n} = 0 \quad \text{a.s.}$$

proving (2.1) and (2.2) (with $l_n = n$, $n \ge 1$).

Case III: $\rho_1 = 0$. By the definition of ρ_1, we have

$$\limsup_{x\to\infty} x^r\mathbb{P}(X > x) = \infty \quad \forall r > 0,$$

which is equivalent to

$$\limsup_{x\to\infty} x\mathbb{P}\big(X > x^r\big) = \infty \quad \forall r > 0.$$

Then, inductively, we can choose positive integers l_n, $n \geq 1$ such that

$$1 = l_1 < l_2 < \cdots < l_n < \cdots \quad \text{and} \quad l_n\mathbb{P}\big(X > l_n^n\big) \geq 2\ln n, \quad n \geq 1.$$

Thus, for all sufficiently large n, we have by the same argument as in Case I

$$\mathbb{P}\left(\frac{\log \max_{1\leq k\leq l_n} X_k}{\log l_n} \leq n\right) \leq n^{-2}$$

and hence by the Borel–Cantelli lemma

$$\mathbb{P}\left(\frac{\log \max_{1\leq k\leq l_n} X_k}{\log l_n} \leq n \text{ i.o.}\right) = 0$$

which ensures that

$$\lim_{n\to\infty} \frac{\log \max_{1\leq k\leq l_n} X_k}{\log l_n} = \infty \quad \text{a.s.}$$

Thus (2.1) and (2.2) hold. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2 *Case I:* $0 < \rho_2 < \infty$. For given $\rho_2 < r < \infty$, let $r_1 = (r + \rho_2)/2$ and $\tau = 1 - (r_1/r)$. Then $\rho_2 < r_1 < r < \infty$ and $\tau > 0$. By the definition of ρ_2, we have

$$\lim_{x\to\infty} x^{r_1}\mathbb{P}(X > x) = \infty$$

and hence, for all sufficiently large x,

$$\mathbb{P}(X > x) \geq x^{-r_1}.$$

Thus, for all sufficiently large n,

$$n\mathbb{P}\big(X > n^{1/r}\big) \geq n\big(n^{1/r}\big)^{-r_1} = n^{1-(r_1/r)} = n^\tau$$

and hence

$$\mathbb{P}\Big(\max_{1\leq k\leq n} X_k \leq n^{1/r}\Big) = \big(1 - \mathbb{P}\big(X > n^{1/r}\big)\big)^n \leq e^{-n\mathbb{P}(X>n^{1/r})} \leq e^{-n^\tau}.$$

Since

$$\sum_{n=1}^{\infty} e^{-n^\tau} < \infty,$$

by the Borel–Cantelli lemma, we have

$$\mathbb{P}\Big(\max_{1\leq k\leq n} X_k \leq n^{1/r} \text{ i.o.}\Big) = 0,$$

which implies that

$$\liminf_{n\to\infty} \frac{\log \max_{1\leq k\leq n} X_k}{\log n} \geq 1/r \quad \text{a.s.}$$

Letting $r \searrow \rho_2$, we get

$$\liminf_{n\to\infty} \frac{\log \max_{1\leq k\leq n} X_k}{\log n} \geq \frac{1}{\rho_2} \quad \text{a.s.} \tag{3.6}$$

Again, by the definition of ρ_2, we have

$$\liminf_{x\to\infty} x^r \mathbb{P}(X > x) = 0 \quad \forall r < \rho_2,$$

which is equivalent to

$$\liminf_{x\to\infty} x\mathbb{P}\bigl(X > x^{(1/\rho_2)+\epsilon}\bigr) = 0 \quad \forall \epsilon > 0.$$

Then, inductively, we can choose positive integers m_n, $n \geq 1$ such that

$$1 = m_1 < m_2 < \cdots < m_n < \cdots \quad \text{and} \quad m_n\mathbb{P}\bigl(X > m_n^{(1/\rho_2)+(1/n)}\bigr) \leq n^{-2}, \quad n \geq 1.$$

Then we have

$$\sum_{n=1}^{\infty} \mathbb{P}\Bigl(\max_{1\leq k\leq m_n} X_k > m_n^{(1/\rho_2)+(1/n)}\Bigr) \leq \sum_{n=1}^{\infty} m_n\mathbb{P}\bigl(X > m_n^{(1/\rho_2)+(1/n)}\bigr) \leq \sum_{n=1}^{\infty} n^{-2} < \infty.$$

Thus, by the Borel–Cantelli lemma, we get

$$\mathbb{P}\left(\frac{\log \max_{1\leq k\leq m_n} X_k}{\log m_n} > \frac{1}{\rho_2} + \frac{1}{n} \text{ i.o.}\right) = 0$$

which ensures that

$$\limsup_{n\to\infty} \frac{\log \max_{1\leq k\leq m_n} X_k}{\log m_n} \leq \frac{1}{\rho_2} \quad \text{a.s.} \tag{3.7}$$

Clearly, (2.3) and (2.4) follow from (3.6) and (3.7).

Case II: $\rho_2 = \infty$. By the definition of ρ_2, we have

$$\liminf_{x\to\infty} x^r \mathbb{P}(X > x) = 0 \quad \forall r > 0,$$

which is equivalent to

$$\liminf_{x\to\infty} x\mathbb{P}\bigl(X > x^r\bigr) = 0 \quad \forall r > 0.$$

Then, inductively, we can choose positive integers m_n, $n \geq 1$ such that

$$1 = m_1 < m_2 < \cdots < m_n < \cdots \quad \text{and} \quad m_n\mathbb{P}\bigl(X > m_n^{1/n}\bigr) \leq n^{-2}, \quad n \geq 1.$$

Thus

$$\sum_{n=1}^{\infty} \mathbb{P}\Bigl(\max_{1\le k\le m_n} X_k > m_n^{1/n}\Bigr) \le \sum_{n=1}^{\infty} m_n \mathbb{P}\bigl(X > m_n^{1/n}\bigr) \le \sum_{n=1}^{\infty} n^{-2} < \infty$$

and hence by the Borel–Cantelli lemma

$$\mathbb{P}\Bigl(\max_{1\le k\le m_n} X_k > m_n^{1/n} \text{ i.o.}\Bigr) = 0,$$

which ensures that

$$\limsup_{n\to\infty} \frac{\log \max_{1\le k\le m_n} X_k}{\log m_n} \le 0 \quad \text{a.s.} \tag{3.8}$$

It is clear that

$$\liminf_{n\to\infty} \frac{\log \max_{1\le k\le n} X_k}{\log n} \ge 0 \quad \text{a.s.} \tag{3.9}$$

It thus follows from (3.8) and (3.9) that

$$\liminf_{n\to\infty} \frac{\log \max_{1\le k\le n} X_k}{\log n} = 0 \quad \text{a.s.} \quad \text{and} \quad \lim_{n\to\infty} \frac{\log \max_{1\le k\le m_n} X_k}{\log m_n} = 0 \quad \text{a.s.};$$

i.e., (2.3) and (2.4) hold.

Case III: $\rho_2 = 0$. Using the same argument used in the first half of the proof for Case I, we get

$$\liminf_{n\to\infty} \frac{\log \max_{1\le k\le n} X_k}{\log n} \ge \frac{1}{r} \quad \text{a.s. } \forall r > 0.$$

Letting $r \searrow 0$, we get

$$\liminf_{n\to\infty} \frac{\log \max_{1\le k\le n} X_k}{\log n} = \infty \quad \text{a.s.}$$

Thus

$$\lim_{n\to\infty} \frac{\log \max_{1\le k\le n} X_k}{\log n} = \infty \quad \text{a.s.}$$

and hence (2.3) and (2.4) hold with $m_n = n$, $n \ge 1$. □

Proof of Theorem 2.3 It follows from Theorems 2.1 and 2.2 that

$$(2.5) \Longleftrightarrow (2.7) \Longleftrightarrow (2.8).$$

Since (2.6) follows from (2.5), we only need to show that (2.6) implies (2.8). It follows from (2.6) that

$$\lim_{n\to\infty} \mathbb{P}\biggl(\frac{\log \max_{1\le k\le n} X_k}{\log n} \le \frac{1}{r}\biggr) = \begin{cases} 1 & \forall r < \rho \text{ if } \rho > 0, \\ 0 & \forall r > \rho \text{ if } \rho < \infty. \end{cases} \tag{3.10}$$

Since, for $n \geq 3$,

$$\mathbb{P}\left(\frac{\log \max_{1\leq k\leq n} X_k}{\log n} \leq \frac{1}{r}\right) = \mathbb{P}\left(\max_{1\leq k\leq n} X_k \leq n^{1/r}\right) = \left(1 - \mathbb{P}\left(X > n^{1/r}\right)\right)^n = e^{n\ln(1-\mathbb{P}(X>n^{1/r}))}$$

and

$$n\ln\left(1 - \mathbb{P}\left(X > n^{1/r}\right)\right) \sim -n\mathbb{P}\left(X > n^{1/r}\right) \quad \text{as } n \to \infty,$$

it follows from (3.10) that

$$\lim_{n\to\infty} n\mathbb{P}\left(X > n^{1/r}\right) = \begin{cases} 0 & \forall r < \rho \text{ if } \rho > 0, \\ \infty & \forall r > \rho \text{ if } \rho < \infty, \end{cases}$$

which is equivalent to (2.8).

For $0 \leq \rho < \infty$, note that

$$\mathbb{P}(X > x) = x^{-\rho}\left(x^{\rho}\mathbb{P}(X > x)\right) = e^{-\rho\ln x + \ln(x^{\rho}\mathbb{P}(X>x))} \quad \forall x > 0.$$

We thus see that, if $0 \leq \rho < \infty$, then (2.8) is equivalent to

$$\lim_{x\to\infty} \frac{\ln(x^{\rho}\mathbb{P}(X > x))}{\log x} = 0.$$

(We leave it to the reader to work out the details of the proof.) We thus see that (2.8) implies (2.9) with $L(x) = x^{\rho}\mathbb{P}(X > x)$, $x > 0$. It is easy to verify that (2.8) follows from (2.9). This completes the proof of Theorem 2.3. □

Proof of Theorem 2.4 For fixed $x \in (-\infty, \infty)$, write

$$a_n(x) = \frac{\ln n + \tau \ln\ln n + \ln h(n) - \tau \ln \rho + x}{\rho} \quad \text{and} \quad b_n(x) = e^{a_n(x)}, n \geq 2.$$

Then

$$b_n(x) = n^{1/\rho}(\ln n)^{\tau/\rho}\left(h(n)\right)^{1/\rho}\rho^{-\tau/\rho}e^{x/\rho}, n \geq 2.$$

Since $h(\cdot) : [0,\infty) \to (0,\infty)$ is a monotone function with $\lim_{x\to\infty} h(x^2)/h(x) = 1$, $h(\cdot)$ is a slowly varying function such that $\lim_{x\to\infty} h(x^r)/h(x) = 1$ $\forall r > 0$ and hence

$$h\left(b_n(x)\right) \sim h(n) \quad \text{as } n \to \infty.$$

Clearly,

$$\left(\ln b_n(x)\right)^{\tau} \sim \rho^{-\tau}(\ln n)^{\tau} \quad \text{as } n \to \infty.$$

It thus follows from (2.10) that, as $n \to \infty$,

$$\begin{aligned} n\ln\bigl(1-\mathbb{P}\bigl(X>b_n(x)\bigr)\bigr) &\sim -n\mathbb{P}\bigl(X>b_n(x)\bigr) \\ &\sim -n \times \frac{(\ln(b_n(x)))^{\tau} h(b_n(x))}{(b_n(x))^{\rho}} \\ &\sim -n \times \frac{\rho^{-\tau}(\ln n)^{\tau} h(n)}{n(\ln n)^{\tau} h(n)\rho^{-\tau} e^{x}} \\ &= -e^{-x} \end{aligned}$$

so that

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}\Bigl(\log \max_{1\le k\le n} X_k \le a_n(x)\Bigr) &= \lim_{n\to\infty} \mathbb{P}\Bigl(\max_{1\le k\le n} X_k \le b_n(x)\Bigr) \\ &= \lim_{n\to\infty} \bigl(1-\mathbb{P}\bigl(X>b_n(x)\bigr)\bigr)^{n} \\ &= \lim_{n\to\infty} e^{n\ln(1-\mathbb{P}(X>b_n(x)))} \\ &= \exp\bigl(-e^{-x}\bigr); \end{aligned}$$

i.e., (2.11) holds. □

Proof of Theorem 2.5 Since $\hat{\theta}_n = \frac{\log n}{\log \max_{1\le k\le n}|X_k|}$, $n \ge 1$, Theorem 2.5 follows immediately from Theorems 2.1–2.3. □

4 Conclusions

In this paper we propose the following simple point estimator of θ, the power of moments of the random variable X, and investigate its asymptotic properties:

$$\hat{\theta}_n = \frac{\log n}{\log \max_{1\le k\le n}|X_k|}.$$

In particular, we show that

$$\hat{\theta}_n \to_{\mathbb{P}} \theta \quad \text{if and only if} \quad \lim_{x\to\infty} x^{r}\mathbb{P}\bigl(|X|>x\bigr) = \infty \quad \forall r>\theta.$$

This means that, under very reasonable conditions on $F(\cdot)$, $\hat{\theta}_n$ is actually a consistent estimator of θ. From Remark 2.3 and Example 2.1, we see that, for a nonnegative random variable X, $\hat{\theta}_n$ is a consistent estimator of θ whenever the well-known Hill estimator $\hat{\alpha}_n$ is a consistent estimator of θ. However, the converse is not true.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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