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Generalization of cyclic refinements of Jensen's inequality by Fink's identity

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Abstract
We generalize cyclic refinements of Jensen's inequality from a convex function to a higher-order convex function by means of Lagrange–Green's function and Fink's identity. We formulate the monotonicity of the linear functionals obtained from these identities utilizing the theory of inequalities for *n*-convex functions at a point. New Grüss- and Ostrowski-type bounds are found for identities associated with the obtained inequalities. Finally, we investigate the properties of linear functionals regarding exponential convexity and mean value theorems.

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1 Introduction

Arising from the circumstantial analysis of geometrical observations, the theory regarding convex functions serves to aid us in topics such as real analysis and economics. The theory of convex functions has progressed to quite a substantial extent. Several reasons may be attributed to this development: firstly, the application of convex functions is linked, directly or indirectly, to many fields of modern analysis; secondly, convex functions are deeply associated with the philosophy of inequalities and vice versa (see [1]). Systematic study of convex functions started over the period 1905–1906 by thought-provoking ideas and fascinating work of Jensen. However, there also exists some literature about convex functions even before Jensen because one may find the existence of the roots of their definition in the work of Hölder (1889) and Hadamard (1893). The study of convex functions is used as a major tool to solve optimization problems in analysis. However, the impact of inequalities involving convex functions is magical as they solve many problems in different branches of mathematics with a considerably high rate. That is why the study of such inequalities has been given great importance in the literature. Higher-order convexity was introduced by Popoviciu, who defined it under the context of divided differences of a function (see Chap. 1, [1]). Inequalities of higher-order convex functions are very important, and many physicists used them while dealing with higher dimensions. It is interesting to note that the results for convex functions may not be true for convex functions of higher order. There are remarkable changes in the results, which forces to think about the existence of such results. Pečarić et al. in 2015 came up with a powerful idea to generalize suitable inequalities from convex to higher-order convex functions by using approximation theory.

It not only generalizes the results to higher-order convex functions but also extends the domain of interest from non-negative to real value (see [2–4]). The weight functions are also improved from positive to real weights. Butt and Pečarić in their book [5] pay tribute to the Romanian mathematician Professor Tiberiu Popoviciu for his famous Popoviciu's inequality. In the book [5], we consider Popoviciu's inequality for convex function valid for non-negative n-tuples and apply this method to generalize it for higher-order convex functions valid for real n-tuples.

In the current paper, we need the following results of our interest. For $\phi^{(n-1)}$ to be absolutely continuous on $[\delta_1, \delta_2] \subset \mathbb{R}$, Fink in [6] proved the following famous identity:

$$\begin{aligned}\phi(x) &= \frac{n}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\phi(\xi)\,d\xi \\ &\quad + \sum_{\sigma=1}^{n-1}\left(\frac{n-\sigma}{\sigma!}\right)\left(\frac{\phi^{(\sigma-1)}(\delta_2)(x-\delta_2)^{\sigma}-\phi^{(\sigma-1)}(\delta_1)(x-\delta_1)^{\sigma}}{\delta_2-\delta_1}\right) \\ &\quad + \frac{1}{(n-1)!(\delta_2-\delta_1)}\int_{\delta_1}^{\delta_2}(x-\xi)^{n-1}F_{\delta_1}^{\delta_2}(\xi,x)\phi^{(n)}(\xi)\,d\xi, \end{aligned} \tag{1}$$

where

$$F_{\delta_1}^{\delta_2}(\xi,x) = \begin{cases} \xi-\delta_1, & \xi \le x \le \delta_2, \\ \xi-\delta_2, & x < \xi \le \delta_2. \end{cases} \tag{2}$$

Consider the Lagrange–Green function $G_L : [\delta_1,\delta_2]\times[\delta_1,\delta_2] \to \mathbb{R}$ defined as

$$G_L(\xi,\theta) = \begin{cases} \frac{(\delta_2-\xi)(\delta_1-\theta)}{\delta_2-\delta_1}, & \theta \le \xi, \\ \frac{(\delta_2-\theta)(\delta_1-\xi)}{\delta_2-\delta_1}, & \xi \le \theta. \end{cases} \tag{3}$$

For any function $\lambda \in C^2([\delta_1,\delta_2])$ (see [7]), we have

$$\lambda(\xi) = \frac{\delta_2-\xi}{\delta_2-\delta_1}\lambda(\delta_1) + \frac{\xi-\delta_1}{\delta_2-\delta_1}\lambda(\delta_2) + \int_{\delta_1}^{\delta_2} G_L(\xi,\theta)\lambda''(\theta)\,d\theta. \tag{4}$$

The most important inequalities concerning convex functions are the following.

Theorem A (Classical Jensen's inequality, see [8]) *Let g be an integrable function on a probability space $(X,\mathcal{A},\mu)$ taking values in an interval $I \subset \mathbb{R}$. Then $\int_X g\,d\mu$ lies in I. If f is a convex function on I such that $f\circ g$ is integrable, then*

$$f\left(\int_X g\,d\mu\right) \le \int_X f\circ g\,d\mu.$$

Theorem B (Discrete Jensen's inequality, see [8]) *Let C be a convex subset of a real vector space Z, and let $f : C \to \mathbb{R}$ be a convex function. If $p_1,\dots,p_n$ are non-negative numbers with $\sum_{u=1}^{n} p_u = 1$ and $z_1,\dots,z_n \in C$, then*

$$f\left(\sum_{u=1}^{n} p_u z_u\right) \le \sum_{u=1}^{n} p_u f(z_u).$$

Next we consider cyclic refinements of discrete and classical Jensen's inequalities (see [9]). We say that the numbers $p_1,\dots,p_n$ represent a (positive) discrete probability distribution if $(p_u > 0)$ $p_u \geq 0$ $(1 \leq u \leq n)$ and $\sum_{u=1}^{n} p_u = 1$.

To refine the discrete Jensen's inequality, we need the following hypotheses:

(H_1) Let $2 \leq k \leq n$ be integers, and let $p_1,\dots,p_n$ and $\lambda_1,\dots,\lambda_k$ represent positive probability distributions.

(H_2) Let C be a convex subset of a real vector space Z, and let $f : C \to \mathbb{R}$ be a convex function.

Theorem 1 *Assume* (H_1) *and* (H_2). *If* $v_1,\dots,v_n \in C$, *then*

$$\begin{aligned} f\left(\sum_{u=1}^{n} p_u z_u\right) &\leq C_{\text{dis}} = C_{\text{dis}}(f,\mathbf{z},\mathbf{p},\boldsymbol{\lambda}) \\ &:= \sum_{u=1}^{n}\left(\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}\right) f\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}z_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}\right) \\ &\leq \sum_{u=1}^{n} p_u f(z_u), \end{aligned} \tag{5}$$

where $u+v$ *means* $u+v-n$ *in the case of* $u+v>n$.

The previous result can be considered as the weighted form of Theorem 2.1 in [10]. To refine the classical Jensen's inequality, we first introduce some hypotheses and notations.

(H_3) Let $(X,\mathcal{B},\mu)$ be a probability space.

Let $l \geq 2$ be a fixed integer. The σ-algebra in X^l generated by the projection mappings $pr_m : X^l \to X$ $(m = 1,\dots,l)$

$$pr_m(x_1,\dots,x_l) := x_m$$

is denoted by $\mathcal{B}^l$. μ^l means the product measure on $\mathcal{B}^l$: this measure is uniquely (μ is σ-finite) specified by

$$\mu^l(B_1 \times \cdots \times B_l) := \mu(B_1)\cdots\mu(B_l), \quad B_m \in \mathcal{B}, m = 1,\dots,l.$$

(H_4) Let g be a μ-integrable function on X taking values in an interval $I \subset \mathbb{R}$.

(H_5) Let f be a convex function on I such that $f \circ g$ is μ-integrable on X.

Under conditions (H_1) and (H_3)–(H_5), we define

$$\begin{aligned} C_{\text{int}} &= C_{\text{int}}(f,g,\mu,\mathbf{p},\boldsymbol{\lambda}) \\ &:= \sum_{u=1}^{n}\left(\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}\right) \\ &\quad \times \int_{X^n} f\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}g(x_{u+v})}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}\right) d\mu^n(x_1,\dots,x_n), \end{aligned} \tag{6}$$

and for $t \in [0,1]$,

$$\begin{aligned} C_{\mathrm{par}}(t) &= C_{\mathrm{par}}(t, f, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) \\ &:= \sum_{u=1}^{n} \left(\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} \right) \\ &\quad \times \int_{X^n} f\left(t \frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} g(x_{u+v})}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}} + (1-t) \int_X g\, d\mu \right) d\mu^n(x_1, \ldots, x_n), \end{aligned} \tag{7}$$

where $u+v$ means $u+v-n$ in the case of $u+v>n$.

Remark 1 It follows from Lemma 2.1(b) in [11] that the integrals in (6) and (7) exist and are finite.

Theorem 2 *Assume* (H_1) *and* (H_3)–(H_5). *Then*

$$f\left(\int_X g\, d\mu \right) \le C_{\mathrm{par}}(t) \le C_{\mathrm{int}} \le \int_X f \circ g\, d\mu, \quad t \in [0,1].$$

In order to achieve our goals, we consider the following hypotheses for the following sections.

(M_1) Let $I \subset \mathbb{R}$ be an interval, $\mathbf{z} := (z_1, \ldots, z_n) \in I^n$, and let $p_1, \ldots, p_n$ and $\lambda_1, \ldots, \lambda_k$ represent positive probability distributions for $2 \le k \le n$.

(M_2) Let $f: I \to \mathbb{R}$ be a convex function.

Remark 2 Under conditions (M_1), we define

$$J_1(f) = J_1(\mathbf{z}, \mathbf{p}, \boldsymbol{\lambda}; f) := \sum_{u=1}^{n} p_u f(z_u) - C_{\mathrm{dis}}(f, \mathbf{z}, \mathbf{p}, \boldsymbol{\lambda}),$$

$$J_2(f) = J_1(\mathbf{z}, \mathbf{p}, \boldsymbol{\lambda}; f) := C_{\mathrm{dis}}(f, \mathbf{z}, \mathbf{p}, \boldsymbol{\lambda}) - f\left(\sum_{u=1}^{n} p_u z_u \right),$$

where $f: I \to \mathbb{R}$ is a function. The functionals $f \to J_i(f)$ are linear, $i = 1, 2$, and Theorem 1 implies that

$$J_i(f) \ge 0, \quad i = 1, 2$$

provided that f is a convex function.

Assume (H_1) and (H_3)–(H_5). Then we have the following additional linear functionals:

$$J_3(f) = J_3(f, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) := \int_X f \circ g\, d\mu - C_{\mathrm{int}}(f, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) \ge 0,$$

$$J_4(f) = J_4(t, f, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) := \int_X f \circ g\, d\mu - C_{\mathrm{par}}(t, f, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) \ge 0, \quad t \in [0,1],$$

$$J_5(f) = J_5(t, f, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) := C_{\mathrm{int}}(f, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) - C_{\mathrm{par}}(t, f, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) \ge 0, \quad t \in [0,1],$$

$$J_6(f) = J_6(t, f, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) := C_{\mathrm{par}}(t, f, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) - f\left(\int_X g\, d\mu \right) \ge 0, \quad t \in [0,1].$$

2 Generalization of cyclic refinements of Jensen's inequality by Fink's identity

First, we consider the discrete as well as continuous version of cyclic refinements of Jensen's inequality and construct the following identities having real weights utilizing Fink's identity.

Theorem 3 *Let $n,k \in \mathbb{N}$, $p_1,\dots,p_n$, and $\lambda_1,\dots,\lambda_k$ be real tuples for $3 \leq k \leq n$ such that $\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v} \neq 0$ for $u = 1,\dots,n$ with $\sum_{u=1}^{n} p_u = 1$ and $\sum_{v=1}^{k}\lambda_v = 1$. Also, let $[\alpha_1,\alpha_2] \subset \mathbb{R}$ and $\mathbf{z} \in [\alpha_1,\alpha_2]^n$. Consider the function $\phi : [\alpha_1,\alpha_2] \to \mathbb{R}$ such that $\phi^{(n-1)}$ is absolutely continuous, $\phi(\alpha_1) = \phi(\alpha_2)$ and $F_{\alpha_1}^{\alpha_2}(\xi,\cdot)$, $G_L(\cdot,r)$ are the same as defined in* (2), (3), *respectively.*

(a) *For the linear functionals $J_i(\cdot)$ $(i = 1,2)$, assume further that $\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}z_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}} \in [\alpha_1,\alpha_2]$ for $u = 1,\dots,n$.*

(b) *For the linear functionals $J_i(\cdot)$ $(i = 3,\dots,6)$, assume that* (H_3)–(H_5) *are valid and $\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}g(z_{u+v})}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}} \in [\alpha_1,\alpha_2]$ for $u = 1,\dots,n$. Then, for $i = 1,\dots,6$, we have the following identities*:

(i)

$$J_i(\phi) = \sum_{\sigma=2}^{n-l}\left(\frac{n-\sigma}{\sigma!}\right)\left(\frac{\phi^{(\sigma-1)}(\alpha_2)J_i((z-\alpha_2)^{\sigma}) - \phi^{(\sigma-1)}(\alpha_1)J_i((z-\alpha_1)^{\sigma})}{(\alpha_2-\alpha_1)}\right) + \frac{1}{(n-1)!(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2} J_i\bigl((z-\xi)^{n-1}F_{\alpha_1}^{\alpha_2}(\xi,z)\bigr)\phi^{(n)}(\xi)\,d\xi. \tag{8}$$

(ii)

$$\begin{aligned} J_i(\phi) &= (n-2)\left(\frac{\phi^{(1)}(\alpha_2)-\phi^{(1)}(\alpha_1)}{\alpha_2-\alpha_1}\right)\int_{\alpha_1}^{\alpha_2} J_i\bigl(G_L(z,r)\bigr)\,dr \\ &\quad + \frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2} J_i\bigl(G_L(z,r)\bigr) \\ &\quad \times \left(\sum_{\sigma=1}^{n-3}\left(\frac{n-2-\sigma}{\sigma!}\right)\bigl(\phi^{(\sigma+1)}(\alpha_2)(r-\alpha_2)^{\sigma} - \phi^{(\sigma+1)}(\alpha_1)(r-\alpha_1)^{\sigma}\bigr)\right)dr \\ &\quad + \frac{1}{(n-3)!(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}\phi^{(n)}(\xi)\left(\int_{\alpha_1}^{\alpha_2} J_i\bigl(G_L(z,r)\bigr)(r-\xi)^{n-3}F_{\alpha_1}^{\alpha_2}(\xi,r)\,dr\right)d\xi. \end{aligned} \tag{9}$$

Proof (i) Fix $i = 1,\dots,6$. Using (1) in the Jensen-type functional $J_i(\cdot)$ and practicing the linearity of $J_i(\cdot)$, we have

$$\begin{aligned} J_i(\phi) &= J_i\left(\frac{n}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\phi(\xi)\,d\xi\right) \\ &\quad + \sum_{\sigma=1}^{n-l}\left(\frac{n-\sigma}{\sigma!}\right)\left(\frac{\phi^{(\sigma-1)}(\alpha_2)J_i((z-\alpha_2)^{\sigma})}{\alpha_2-\alpha_1}\right) \\ &\quad - \sum_{\sigma=1}^{n-l}\left(\frac{n-\sigma}{\sigma!}\right)\left(\frac{\phi^{(\sigma-1)}(\alpha_1)J_i((z-\alpha_1)^{\sigma})}{\alpha_2-\alpha_1}\right) \\ &\quad + \frac{1}{(n-1)!(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2} J_i\bigl((z-\xi)^{n-1}F_{\alpha_1}^{\alpha_2}(\xi,z)\bigr)\phi^{(n)}(\xi)\,d\xi. \end{aligned}$$

After simplification, we get (8).

(ii) Fix $i = 1,\dots,6$. Testing (4) in the Jensen-type functional $J_i(\cdot)$ and employing the linearity of $J_i(\cdot)$ along with the assumption $\phi(\alpha_1) = \phi(\alpha_2)$, we have

$$\begin{aligned}
J_i(\phi) &= \phi(\alpha_1)J_i\left(\frac{\alpha_2 - z}{\alpha_2 - \alpha_1}\right) + \phi(\alpha_2)J_i\left(\frac{z - \alpha_1}{\alpha_2 - \alpha_1}\right) + \int_{\alpha_1}^{\alpha_2} J_i\big(G_L(z,r)\big)\phi''(r)\,dr \\
&= \phi(\alpha_1)\frac{J_i(\alpha_2 - z)}{\alpha_2 - \alpha_1} + \phi(\alpha_2)\frac{J_i(z - \alpha_1)}{\alpha_2 - \alpha_1} + \int_{\alpha_1}^{\alpha_2} J_i\big(G_L(z,r)\big)\phi''(r)\,dr \\
&= \frac{1}{\alpha_2 - \alpha_1}\big(\phi(\alpha_1)J_i(\alpha_2) - \phi(\alpha_1)J_i(z) + \phi(\alpha_2)J_i(z) - \phi(\alpha_2)J_i(\alpha_1)\big) \\
&\quad + \int_{\alpha_1}^{\alpha_2} J_i\big(G_L(z,r)\big)\phi''(r)\,dr \\
&= \frac{1}{\alpha_2 - \alpha_1}\big(\phi(\alpha_2)J_i(z) - \phi(\alpha_1)J_i(z)\big) + \int_{\alpha_1}^{\alpha_2} J_i\big(G_L(z,r)\big)\phi''(r)\,dr \\
&= \int_{\alpha_1}^{\alpha_2} J_i\big(G_L(z,r)\big)\phi''(r)\,dr.
\end{aligned} \tag{10}$$

Differentiating (1) twice with respect to variable r and by rearranging the indices, we get

$$\begin{aligned}
\phi''(r) &= \sum_{\sigma=0}^{n-3}\left(\frac{n-2-\sigma}{\sigma!}\right)\left(\frac{\phi^{(\sigma+1)}(\alpha_2)(r-\alpha_2)^{\sigma} - \phi^{(\sigma+1)}(\alpha_1)(r-\alpha_1)^{\sigma}}{\alpha_2 - \alpha_1}\right) \\
&\quad + \frac{1}{(n-3)!(\alpha_2 - \alpha_1)}\int_{\alpha_1}^{\alpha_2} (r-\xi)^{n-3}F_{\alpha_1}^{\alpha_2}(\xi,r)\phi^{(n)}(\xi)\,d\xi.
\end{aligned} \tag{11}$$

Putting (11) in (10) and executing Fubini's theorem in the obtained terms, we get (9).

More willingly, we adopt formula (1) with function ϕ'' by substituting $n \to n-2$ ($n \geq 3$) to get

$$\begin{aligned}
\phi''(r) &= (n-2)\left(\frac{\phi^{(1)}(\alpha_2) - \phi^{(1)}(\alpha_1)}{\alpha_2 - \alpha_1}\right) \\
&\quad + \sum_{\sigma=1}^{n-3}\left(\frac{n-2-\sigma}{\sigma!}\right)\left(\frac{\phi^{(\sigma+1)}(\alpha_2)(r-\alpha_2)^{\sigma} - \phi^{(\sigma+1)}(\alpha_1)(r-\alpha_1)^{\sigma}}{\alpha_2 - \alpha_1}\right) \\
&\quad + \frac{1}{(n-3)!(\alpha_2 - \alpha_1)}\int_{\alpha_1}^{\alpha_2} (r-\xi)^{n-3}F_{\alpha_1}^{\alpha_2}(\xi,r)\phi^{(n)}(\xi)\,d\xi.
\end{aligned} \tag{12}$$

Employing (12) in (10) and utilizing Fubini's theorem in the obtained term, we get (9). □

Now we obtain generalizations of discrete and integral Jensen-type linear functionals with real weights.

Theorem 4 *Under the suppositions of Theorem* 3, *let ϕ be an n-convex function. Then we conclude the following two results*:

(i) *If*

$$J_i\big((z-\xi)^{n-1}F_{\alpha_1}^{\alpha_2}(\xi,z)\big) \geq 0, \quad \xi \in [\alpha_1,\alpha_2] \tag{13}$$

holds, then we have

$$J_i(\phi) \geq \sum_{\sigma=2}^{n-l} \left(\frac{n-\sigma}{\sigma!} \right) \left(\frac{\phi^{(\sigma-1)}(\alpha_2) J_i((z-\alpha_2)^\sigma) - \phi^{(\sigma-1)}(\alpha_1) J_i((z-\alpha_1)^\sigma)}{(\alpha_2-\alpha_1)} \right) \tag{14}$$

for $i = 1, \ldots, 6$.

(ii) *If*

$$\int_{\alpha_1}^{\alpha_2} J_i\big(G_L(z,r)\big)(r-\xi)^{n-3} F_{\alpha_1}^{\alpha_2}(\xi,r)\,dr \geq 0, \quad \xi \in [\alpha_1,\alpha_2] \tag{15}$$

holds, then we have

$$\begin{aligned} J_i(\phi) &\geq (n-2)\left(\frac{\phi^{(1)}(\alpha_2) - \phi^{(1)}(\alpha_1)}{\alpha_2-\alpha_1} \right) \int_{\alpha_1}^{\alpha_2} J_i\big(G_L(z,r)\big)\,dr \\ &\quad + \frac{1}{(\alpha_2-\alpha_1)} \int_{\alpha_1}^{\alpha_2} J_i\big(G_L(z,r)\big) \\ &\quad \times \left(\sum_{\sigma=1}^{n-3} \left(\frac{n-2-\sigma}{\sigma!} \right) \big(\phi^{(\sigma+1)}(\alpha_2)(r-\alpha_2)^\sigma - \phi^{(\sigma+1)}(\alpha_1)(r-\alpha_1)^\sigma \big) \right) dr \end{aligned} \tag{16}$$

for $i = 1, \ldots, 6$.

Proof (i) Fix $i = 1, \ldots, 6$.

As $\phi^{(n-1)}$ is absolutely continuous on $[\alpha_1,\alpha_2]$, $\phi^{(n)}$ exists almost everywhere. Since ϕ is n-convex, so $\phi^{(n)}(x) \geq 0$ for almost everywhere on $[\alpha_1,\alpha_2]$ (see [1], p. 16). Therefore, by applying Theorem 3, we get (14).

(ii) Follow similar steps as in (i). □

We will finish the present section by stating the following theorem.

Theorem 5 *If the assumptions of Theorem* 3 *are fulfilled with additional conditions that* $p_1,\ldots,p_n$ *and* $\lambda_1,\ldots,\lambda_k$ *are non-negative tuples for* $3 \leq k \leq n$, *such that* $\sum_{i=1}^n p_i = 1$, $\sum_{i=1}^k \lambda_i = 1$. *Then, for* $\phi : [\alpha_1,\alpha_2] \to \mathbb{R}$ *being n-convex function, we conclude the following results*:

(i) *For even* $n > 3$, (14) *holds.*

(ii) *If inequality* (14) *is valid and the function*

$$F(z) := \sum_{\sigma=2}^{n-1} \left(\frac{n-\sigma}{\sigma!(\alpha_2-\alpha_1)} \right) \big(\phi^{(\sigma-1)}(\alpha_2)(z-\alpha_2)^\sigma - \phi^{(\sigma-1)}(\alpha_1)(z-\alpha_1)^\sigma \big) \tag{17}$$

is convex, then the inequality

$$J_i(\phi) \geq 0, \quad i = 1, \ldots, 6. \tag{18}$$

(iii) *For even* $n > 3$, (16) *holds.*

(iv) *If inequality* (16) *is true and*

$$\sum_{\sigma=0}^{n-3}\left(\frac{n-2-\sigma}{\sigma!}\right)\left(\phi^{(\sigma+1)}(\alpha_2)(s-\alpha_2)^{\sigma}-\phi^{(\sigma+1)}(\alpha_1)(s-\alpha_1)^{\sigma}\right)\geq 0, \tag{19}$$

then we have (18) *for* $i = 1,\ldots,6$.

Proof (i) Fix $i = 1,\ldots,6$.

For

$$\vartheta(z) := (z-\xi)^{n-1}F_{\alpha_1}^{\alpha_2}(\xi,z) = \begin{cases}(z-\xi)^{n-1}(\xi-\alpha_1), & \xi\leq z\leq\alpha_2,\\ (z-\xi)^{n-1}(\xi-\alpha_2), & z<\xi\leq\alpha_2.\end{cases}$$

Applying the second derivative test on ϑ, it can be seen easily that it is convex for even $n > 3$. Since the weights are non-negative, so by advantage of Remark 2, (13) holds. Pursuing Theorem 4(i), (14) is evident.

(ii) Since $J_i(\cdot)$ is a linear functional for each $i = 1,\ldots,6$, so we can rewrite the R.H.S. of (14) in the form $J_i(F)$ where F is defined in (17). Since F is (assumed to be) convex, therefore the R.H.S. of (14) is non-negative as a consequence of Remark 2 and (18) is evident.

(iii) Fix $i = 1,\ldots,6$.

Since Green's function $G_L(z,r)$ is convex and the weights are assumed to be positive, thus, by practicing Remark 2, $J_iG_L(z,r)\geq 0$. Moreover, if we analyze the function $(r-\xi)^{n-3}F_{\alpha_1}^{\alpha_2}(\xi,r)$, then by using small calculus it can be observed that it is positive for even $n > 3$. Hence, (15) holds. By virtue of Theorem 4(ii), we can obtain (16).

(iv) Utilizing (19) in (16), one can obtain (18). □

3 Related inequalities for *n*-convex functions at a point

In the present section we formulate related results for the class of n-convex functions at a point introduced by Pečarić et al. in [12].

Definition 1 Let I be an interval in $\mathbb{R}$, $d\in I^{o}$ (interior of I) and $n\in\mathbb{N}$. A function $\phi: I\to\mathbb{R}$ is said to be n-convex at point d if there exists a constant C such that the function

$$\Phi(z)=\phi(z)-\frac{\mathrm{C}}{(n-1)!}z^{n-1} \tag{20}$$

is $(n-1)$-concave on $I\cap(-\infty,d]$ and $(n-1)$-convex on $I\cap[d,\infty)$. A function ϕ is said to be n-concave at point d if the function $-\phi$ is n-convex at point d.

Pečarić et al. in [12] study necessary and sufficient conditions on two linear functionals $\varOmega: C([\alpha_1,d])\to\mathbb{R}$ and $\varDelta: C([d,\alpha_2])\to\mathbb{R}$ so that the inequality $\varOmega(\phi)\leq\varDelta(\phi)$ holds for every function ϕ that is n-convex at point d. In the present section we give inequalities of such type for the particular linear functionals obtained from the inequalities in the previous section. Let ζ^{i} denote the monomials $\zeta^{i}(x)=x^{i}$, $i\in\mathbb{N}_0$. For the rest of this section, $\varOmega_i(\phi_{[\alpha_1,d]})$ and $\varDelta_i(\phi_{[d,\alpha_2]})$ will denote the linear functionals obtained as the difference of the L.H.S. and R.H.S. of inequality (14), where $i = 1,\ldots,6$, applied to the intervals $[\alpha_1,d]$ and

$[d,\alpha_2]$, respectively, i.e., for $\mathbf{z} \in [\alpha_1, d]^n$, $\mathbf{p} \in \mathbb{R}^n$, $\boldsymbol{\lambda} \in \mathbb{R}^k$, $\mathbf{y} \in [d,\alpha_2]^m$, $\mathbf{q} \in \mathbb{R}^m$, and $\tilde{\boldsymbol{\lambda}} \in \mathbb{R}^{\tilde{k}}$, let

$$\begin{aligned}\Omega_i(\phi_{[\alpha_1,d]}) &:= J_i(\phi) \\ &\quad - \sum_{w=2}^{n-l} \left(\frac{n-\sigma}{\sigma!}\right)\left(\frac{\phi^{(\sigma-1)}(d)J_i((z-d)^\sigma) - \phi^{(\sigma-1)}(\alpha_1)J_i((z-\alpha_1)^\sigma)}{(d-\alpha_1)}\right),\end{aligned} \tag{21}$$

$$\begin{aligned}\Delta_i(\phi_{[d,\alpha_2]}) &:= J_i(\phi) \\ &\quad - \sum_{\sigma=2}^{n-l} \left(\frac{n-\sigma}{\sigma!}\right)\left(\frac{\phi^{(\sigma-1)}(\alpha_2)J_i((y-\alpha_2)^\sigma) - \phi^{(\sigma-1)}(d)J_i((y-d)^\sigma)}{(\alpha_2-d)}\right).\end{aligned} \tag{22}$$

In a similar fashion, taking into account inequality (16), we can define linear functionals for $i = 1,\ldots,6$ as follows:

$$\begin{aligned}\hat{\Omega}_i(\phi_{[\alpha_1,d]}) &:= J_i(\phi) - \int_{\alpha_1}^{d} J_i\big(G_L(z,r)\big) \\ &\quad \times \sum_{\sigma=0}^{n-3} \left(\frac{n-2-\sigma}{\sigma!}\right)\left(\frac{\phi^{(\sigma+1)}(d)(r-d)^\sigma - \phi^{(\sigma+1)}(\alpha_1)(r-\alpha_1)^\sigma}{d-\alpha_1}\right) dr,\end{aligned} \tag{23}$$

$$\begin{aligned}\hat{\Delta}_i(\phi_{[d,\alpha_2]}) &:= J_i(\phi) - \int_{d}^{\alpha_2} J_i\big(G_L(y,r)\big) \\ &\quad \times \sum_{\sigma=0}^{n-3} \left(\frac{n-2-\sigma}{\sigma!}\right)\left(\frac{\phi^{(\sigma+1)}(\alpha_2)(r-\alpha_2)^\sigma - \phi^{(\sigma+1)}(d)(r-d)^\sigma}{\alpha_2-d}\right) dr.\end{aligned} \tag{24}$$

It is important to notify that by introducing new linear functionals $\Omega_i(\phi_{[\alpha_1,d]})$ and $\Delta_i(\phi_{[d,\alpha_2]})$, identity (8) for $i = 1,\ldots,6$, applied to the respective intervals $[\alpha_1,d]$ and $[d,\alpha_2]$, takes the following shape:

$$\Omega_i(\phi_{[\alpha_1,d]}) = \frac{1}{(n-1)!(d-\alpha_1)} \int_{\alpha_1}^{d} J_i\big((z-\xi)^{n-1}F_{\alpha_1}^{d}(\xi,z)\big)\phi^{(n)}(\xi)\,d\xi, \tag{25}$$

$$\Delta_i(\phi_{[d,\alpha_2]}) = \frac{1}{(n-1)!(\alpha_2-d)} \int_{d}^{\alpha_2} J_i\big((y-\xi)^{n-1}F_{d}^{\alpha_2}(\xi,y)\big)\phi^{(n)}(\xi)\,d\xi. \tag{26}$$

Moreover, identity (9) for $i = 1,\ldots,6$, applied to the respective intervals $[\alpha_1,d]$ and $[d,\alpha_2]$, takes the following shape:

$$\begin{aligned}\hat{\Omega}_i(\phi_{[\alpha_1,d]}) &= \frac{1}{(n-3)!(d-\alpha_1)} \\ &\quad \times \int_{\alpha_1}^{d} \phi^{(n)}(\xi)\left(\int_{\alpha_1}^{d} J_i\big(G_L(z,r)\big)(r-\xi)^{n-3}F_{\alpha_1}^{d}(\xi,r)\,dr\right) d\xi,\end{aligned} \tag{27}$$

$$\begin{aligned}\hat{\Delta}_i(\phi_{[d,\alpha_2]}) &= \frac{1}{(n-3)!(\alpha_2-d)} \\ &\quad \times \int_{d}^{\alpha_2} \phi^{(n)}(\xi)\left(\int_{d}^{\alpha_2} J_i\big(G_L(y,r)\big)(r-\xi)^{n-3}F_{d}^{\alpha_2}(\xi,r)\,dr\right) d\xi.\end{aligned} \tag{28}$$

Now we are ready to state the following theorem for inequalities involving n-convex function at a point.

Theorem 6 *Let $z \in [\alpha_1, d]^n$, $p \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^k$, $y \in [d, \alpha_2]^m$, $q \in \mathbb{R}^m$, and $\tilde{\lambda} \in \mathbb{R}^{\tilde{k}}$ be such that*

(i) *For $i = 1, \ldots, 6$, assume*

$$J_i\big((z-\xi)^{n-1} F_{\alpha_1}^{d}(\xi, z)\big) \geq 0, \quad \xi \in [\alpha_1, d], \tag{29}$$

$$J_i\big((y-\xi)^{n-1} F_{d}^{\alpha_2}(\xi, y)\big) \geq 0, \quad \xi \in [d, \alpha_2], \tag{30}$$

$$\int_{\alpha_1}^{d} J_i\big((z-\xi)^{n-1} F_{\alpha_1}^{d}(\xi, z)\big)\, d\xi = \left(\frac{d-\alpha_1}{\alpha_2 - d}\right) \int_{d}^{\alpha_2} J_i\big((y-\xi)^{n-1} F_{d}^{\alpha_2}(\xi, y)\big)\, d\xi, \tag{31}$$

where $F_{\alpha_1}^{\alpha_2}(\xi, \cdot)$ is given by (2), *and let $\Omega_i(\phi_{[\alpha_1,d]})$ and $\Delta_i(\phi_{[d,\alpha_2]})$ be the linear functionals given by* (21) *and* (22). *If $\phi : [\alpha_1, \alpha_2] \to \mathbb{R}$ is $(n+1)$-convex at point d, then*

$$\Omega_i(\phi_{[\alpha_1,d]}) \leq \Delta_i(\phi_{[d,\alpha_2]}) \quad \text{for } i = 1, \ldots, 6. \tag{32}$$

(ii) *Analogously, for $i = 1, \ldots, 6$, assume*

$$\int_{\alpha_1}^{d} J_i\big(G_L(z, r)\big)(r-\xi)^{n-3} F_{\alpha_1}^{d}(\xi, r)\, dr \geq 0, \quad \xi \in [\alpha_1, d], \tag{33}$$

$$\int_{d}^{\alpha_2} J_i\big(G_L(y, r)\big)(r-\xi)^{n-3} F_{d}^{\alpha_2}(\xi, r)\, dr \geq 0, \quad \xi \in [d, \alpha_2], \tag{34}$$

$$\begin{aligned} &\int_{\alpha_1}^{d} \left(\int_{\alpha_1}^{d} J_i\big(G_L(z, r)\big)(r-\xi)^{n-3} F_{\alpha_1}^{d}(\xi, r)\, dr \right) d\xi \\ &\quad = \left(\frac{d-\alpha_1}{\alpha_2 - d}\right) \int_{d}^{\alpha_2} \left(\int_{d}^{\alpha_2} J_i\big(G_L(y, r)\big)(r-\xi)^{n-3} F_{d}^{\alpha_2}(\xi, r)\, dr \right) d\xi, \end{aligned} \tag{35}$$

where $F_{\alpha_1}^{\alpha_2}(\xi, \cdot)$, $G_L(\cdot, r)$ are the same as defined in (2), (3), *respectively, and let $\hat{\Omega}_i(\phi_{[\alpha_1,d]})$ and $\hat{\Delta}_i(\phi_{[d,\alpha_2]})$ be the linear functionals given by* (23) *and* (24). *If $\phi : [\alpha_1, \alpha_2] \to \mathbb{R}$ is $(n+1)$-convex at point d, then*

$$\hat{\Omega}_i(\phi_{[\alpha_1,d]}) \leq \hat{\Delta}_i(\phi_{[d,\alpha_2]}) \quad \text{for } i = 1, \ldots, 6. \tag{36}$$

Proof (i) Fix $i = 1, \ldots, 6$. Using Definition 1, construct function $\Phi = \phi - \frac{C}{n!}\zeta^n$ in such a way that the function Φ is n-concave on $[\alpha_1, d]$ and n-convex on $[d, \alpha_2]$. Applying Theorem 4 to Φ on the interval $[\alpha_1, d]$, we have

$$0 \geq \Omega_i(\Phi) = \Omega_i(\phi_{[\alpha_1,d]}) - \frac{C}{(n)!} \Omega_i\big(\zeta_{[\alpha_1,d]}^{n}\big). \tag{37}$$

Analogously, applying Theorem 4 to Φ on the interval $[c, \alpha_2]$, we get

$$0 \leq \Delta_i(\Phi) = \Delta_i(\phi_{[d,\alpha_2]}) - \frac{C}{(n)!} \Delta_i\big(\zeta_{[d,\alpha_2]}^{n}\big). \tag{38}$$

Moreover, identities (25) and (26) applied to the function x^n give

$$\Omega_i\big(\zeta_{[\alpha_1,d]}^{n}\big) = \frac{n}{(d-\alpha_1)} \int_{\alpha_1}^{d} J_i\big((z-\xi)^{n-1} F_{\alpha_1}^{d}(\xi, z)\big)\, d\xi, \tag{39}$$

$$\Delta_i\big(\zeta^n_{[d,\alpha_2]}\big) = \frac{n}{(\alpha_2 - d)} \int_d^{\alpha_2} J_i\big((y-\xi)^{n-1} F_d^{\alpha_2}(\xi, y)\big)\, d\xi. \tag{40}$$

Therefore assumption (31) is equivalent to

$$\Omega_i\big(\zeta^n_{[\alpha_1,d]}\big) = \Delta_i\big(\zeta^n_{[d,\alpha_2]}\big).$$

So, from (37) and (38), one can get

$$\Omega_i(\phi_{[\alpha_1,d]}) \le \frac{\mathrm{C}}{(n)!}\Omega_i\big(\zeta^n_{[\alpha_1,d]}\big) = \frac{\mathrm{C}}{(n)!}\Delta_i\big(\zeta^n_{[d,\alpha_2]}\big) \le \Delta_i(\phi_{[d,\alpha_2]}). \tag{41}$$

So (32) is established for $i = 1, \dots, 6$.

(ii) We can employ the similar method as above using identities (27) and (28). So, by deducing assumption (35), we get (36) for $i = 1, \dots, 6$. □

Remark 3 In fact, inequalities (32) and (36) still hold if we replace assumptions (31) and (35) with the weaker assumptions that $\mathrm{C}(\Delta_i(\zeta^n_{[d,\alpha_2]}) - \Omega_i(\zeta^n_{[\alpha_1,d]})) \ge 0$ and $\mathrm{C}(\hat{\Delta}_i(\zeta^n_{[d,\alpha_2]}) - \hat{\Omega}_i(\zeta^n_{[\alpha_1,d]})) \ge 0$ for $i = 1, \dots, 6$, respectively.

4 New upper bounds for generalized cyclic refinements of Jensen's inequalities by Fink's identity

In the present section we use Čebyšev functional defined for Lebesgue integrable functions $\mho_1, \mho_2 : [\alpha_1, \alpha_2] \to \mathbb{R}$ as

$$\mathbb{C}(\mho_1, \mho_2) = \frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} \mho_1(\xi)\mho_2(\xi)\, d\xi - \frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} \mho_1(\xi)\, d\xi . \frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} \mho_2(\xi)\, d\xi$$

to construct some new upper bounds.

The following inequalities of Grüss type were given in [13].

Theorem 7 *Let $\mho_1 \in L[\alpha_1, \alpha_2]$ and $\mho_2 : [\alpha_1, \alpha_2] \to \mathbb{R}$ be an absolutely-continuous function along with $(\cdot - \alpha_1)(\alpha_2 - \cdot)[\mho_2']^2 \in L[\alpha_1, \alpha_2]$. Then the inequality*

$$\big|\mathbb{C}(\mho_1, \mho_2)\big| \le \frac{1}{\sqrt{2}}\left[\frac{\mathbb{C}(\mho_1, \mho_1)}{(\alpha_2 - \alpha_1)}\right]^{\frac{1}{2}} \left(\int_{\alpha_1}^{\alpha_2} (z - \alpha_1)(\alpha_2 - z)\big[\mho_2'(z)\big]^2\, dz\right)^{\frac{1}{2}} \tag{42}$$

holds with $\frac{1}{\sqrt{2}}$ being the best possible constant.

Theorem 8 *Let $\mho_1 : [\alpha_1, \alpha_2] \to \mathbb{R}$ be absolutely continuous with $\mho_1' \in L_\infty[\alpha_1, \alpha_2]$ and $\mho_2 : [\alpha_1, \alpha_2] \to \mathbb{R}$ be a monotonic nondecreasing function. Then the inequality*

$$\big|\mathbb{C}(\mho_1, \mho_2)\big| \le \frac{\|\mho_1'\|_\infty}{2(\alpha_2 - \alpha_1)} \int_{\alpha_1}^{\alpha_2} (z - \alpha_1)(\alpha_2 - z)\, d\mho_2(z) \tag{43}$$

holds with best possible constant $\frac{1}{2}$.

For the current section, let us denote, for $i = 1, \dots, 6$,

$$\Re_i(\xi) = J_i\big((z - \xi)^{n-1} F_{\alpha_1}^{\alpha_2}(\xi, z)\big) \ge 0, \quad \xi \in [\alpha_1, \alpha_2], \tag{44}$$

and

$$\mathfrak{O}_i(\xi) = \int_{\alpha_1}^{\alpha_2} J_i\big(G_L(z,r)\big)(r-\xi)^{n-3} F_{\alpha_1}^{\alpha_2}(\xi,r)\,dr \geq 0, \quad \xi \in [\alpha_1,\alpha_2]. \tag{45}$$

Now we are in a position to formulate our new upper bounds both for discrete and integral Jensen-type inequalities with real weights with the help of the above theorems.

Theorem 9 *Consider the suppositions of Theorem* 3 *be fulfilled, let* $\phi^{(n)}$ *be absolutely continuous with* $(\cdot-\alpha_1)(\alpha_2-\cdot)[\phi^{(n+1)}]^2 \in L[\alpha_1,\alpha_2]$ *such that* $\mathfrak{R}_i$ *and* $\mathfrak{O}_i$ *are defined in* (44) *and* (45), *respectively.*

(i) *Then*

$$\begin{aligned} J_i(\phi) = {}& \sum_{\sigma=2}^{n-l}\left(\frac{n-\sigma}{\sigma!}\right)\left(\frac{\phi^{(\sigma-1)}(\alpha_2)J_i((z-\alpha_2)^\sigma)-\phi^{(\sigma-1)}(\alpha_1)J_i((z-\alpha_1)^\sigma)}{(\alpha_2-\alpha_1)}\right) \\ & + \frac{\phi^{(n-1)}(\alpha_2)-\phi^{(n-1)}(\alpha_1)}{(\alpha_2-\alpha_1)^2(n-1)!}\int_{\alpha_1}^{\alpha_2}\mathfrak{R}_i(\xi)\,d\xi + \mathfrak{K}_{n,i}(\alpha_1,\alpha_2;\phi), \end{aligned} \tag{46}$$

where the remainder $\mathfrak{K}_{n,i}(\alpha_1,\alpha_2;\phi)$ *satisfies the bound*

$$\begin{aligned} \big|\mathfrak{K}_{n,i}(\alpha_1,\alpha_2;\phi)\big| \leq {}& \frac{1}{\sqrt{2}(n-1)!}\big[\mathbb{C}(\mathfrak{R}_i,\mathfrak{R}_i)\big]^{\frac{1}{2}} \\ & \times \frac{1}{\sqrt{\alpha_2-\alpha_1}}\left|\int_{\alpha_1}^{\alpha_2}(\xi-\alpha_1)(\alpha_2-\xi)\big[\phi^{(n+1)}(\xi)\big]^2\,d\xi\right|^{\frac{1}{2}} \end{aligned} \tag{47}$$

for $i = 1,\dots,6$.

(ii)

$$\begin{aligned} J_i(\phi) = {}& \frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2} J_i\big(G_L(z,r)\big) \\ & \times\left(\sum_{\sigma=0}^{n-3}\left(\frac{n-2-\sigma}{\sigma!}\right)\big(\phi^{(\sigma+1)}(\alpha_2)(r-\alpha_2)^\sigma-\phi^{(\sigma+1)}(\alpha_1)(r-\alpha_1)^\sigma\big)\right)dr \\ & + \frac{\phi^{(n-1)}(\alpha_2)-\phi^{(n-1)}(\alpha_1)}{(\alpha_2-\alpha_1)^2(n-3)!}\int_{\alpha_1}^{\alpha_2}\mathfrak{O}_i(\xi)\,d\xi + \mathfrak{K}_{n,i}(\alpha_1,\alpha_2;\phi), \end{aligned} \tag{48}$$

where the remainder $\mathfrak{K}_{n,i}(\alpha_1,\alpha_2;\phi)$ *satisfies the bound*

$$\begin{aligned} \big|\mathfrak{K}_{n,i}(\alpha_1,\alpha_2;\phi)\big| \leq {}& \frac{1}{\sqrt{2}(n-3)!}\big[\mathbb{C}(\mathfrak{O}_i,\mathfrak{O}_i)\big]^{\frac{1}{2}} \\ & \times \frac{1}{\sqrt{\alpha_2-\alpha_1}}\left|\int_{\alpha_1}^{\alpha_2}(\xi-\alpha_1)(\alpha_2-\xi)\big[\phi^{(n+1)}(\xi)\big]^2\,d\xi\right|^{\frac{1}{2}} \end{aligned} \tag{49}$$

for $i = 1,\dots,6$.

Proof (i) Fix $i = 1,\dots,6$.

Executing Theorem 7 for $\mho_1 \to \mathfrak{R}_i$ and $\mho_2 \to \phi^{(n)}$, we get

$$\left|\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathfrak{R}_i(\xi)\phi^{(n)}(\xi)\,d\xi - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathfrak{R}_i(\xi)\,d\xi\cdot\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\phi^{(n)}(\xi)\,d\xi\right|$$
$$\leq \frac{1}{\sqrt{2}}\big[\mathbb{C}(\mathfrak{R}_i,\mathfrak{R}_i)\big]^{\frac{1}{2}}\frac{1}{\sqrt{\alpha_2-\alpha_1}}\left|\int_{\alpha_1}^{\alpha_2}(\xi-\alpha_1)(\alpha_2-\xi)\big[\phi^{(n+1)}(\xi)\big]^2\,d\xi\right|^{\frac{1}{2}}. \tag{50}$$

Dividing both sides of (50) by $(n-1)!$, we have

$$\left|\frac{1}{(n-1)!(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}\mathfrak{R}_i(\xi)\phi^{(n)}(\xi)\,d\xi\right.$$
$$\left. - \frac{1}{(n-1)!(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}\mathfrak{R}_i(\xi)\,d\xi\cdot\frac{\phi^{(n-1)}(\alpha_2)-\phi^{(n-1)}(\alpha_1)}{(\alpha_2-\alpha_1)}\right|$$
$$\leq \frac{1}{\sqrt{2}(n-1)!}\big[\mathbb{C}(\mathfrak{R}_i,\mathfrak{R}_i)\big]^{\frac{1}{2}}\frac{1}{\sqrt{\alpha_2-\alpha_1}}\left|\int_{\alpha_1}^{\alpha_2}(\xi-\alpha_1)(\alpha_2-\xi)\big[\phi^{(n+1)}(\xi)\big]^2\,d\xi\right|^{\frac{1}{2}}. \tag{51}$$

By denoting

$$\mathfrak{K}_{n,i}(\alpha_1,\alpha_2;\phi) = \frac{1}{(n-1)!(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}\mathfrak{R}_i(\xi)\phi^{(n)}(\xi)\,d\xi$$
$$- \frac{1}{(n-1)!(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}\mathfrak{R}_i(\xi)\,d\xi\cdot\frac{\phi^{(n-1)}(\alpha_2)-\phi^{(n-1)}(\alpha_1)}{(\alpha_2-\alpha_1)} \tag{52}$$

in (51), we have (47). Hence, we have

$$\frac{1}{(n-1)!(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}\mathfrak{R}_i(\xi)\phi^{(n)}(\xi)\,d\xi$$
$$= \frac{\phi^{(n-1)}(\alpha_2)-\phi^{(n-1)}(\alpha_1)}{(\alpha_2-\alpha_1)^2(n-1)!}\int_{\alpha_1}^{\alpha_2}\mathfrak{R}_i(\xi)\,d\xi + \mathfrak{K}_{n,i}(\alpha_1,\alpha_2;\phi),$$

where the remainder $\mathfrak{K}_{n,i}(\alpha_1,\alpha_2;\phi)$ satisfies the bound (47). Utilizing identity (8), we get (46).

(ii) Fix $i = 1,\ldots,6$.

Now applying similar method opted in (i) and then using identity (9) , we obtain (48), where the remainder $\mathfrak{K}_{n,i}(\alpha_1,\alpha_2;\phi)$ satisfies estimation (49). □

Adopting Theorem 8, we deduced the following Grüss-type inequalities.

Theorem 10 *Consider the suppositions of Theorem* 3 *be fulfilled. Let* $\phi^{(n)}$ *be absolutely continuous such that* $\phi^{(n+1)} \geq 0$ *on* $[\alpha_1,\alpha_2]$ *with* $\mathfrak{R}_i$ *and* $\mathfrak{O}_i$ *defined in* (44) *and* (45), *respectively.*

(i) *Then illustration* (46) *along with remainder* $\mathfrak{K}_{n,i}(\alpha_1,\alpha_2;\phi)$ *satisfies the estimation*

$$\big|\mathfrak{K}_{n,i}(\alpha_1,\alpha_2;\phi)\big|$$
$$\leq \frac{\|\mathfrak{R}_i'\|_\infty}{(n-1)!}\left[\frac{\phi^{(n-1)}(\alpha_2)+\phi^{(n-1)}(\alpha_1)}{2} - \frac{\phi^{(n-2)}(\alpha_2)-\phi^{(n-2)}(\alpha_1)}{\alpha_2-\alpha_1}\right] \tag{53}$$

for $i = 1,\ldots,6$.

(ii) *Illustration* (48) *along with remainder* $\mathfrak{K}_{n,i}(\alpha_1,\alpha_2;\phi)$ *satisfies the estimation*

$$\begin{aligned}&\left|\mathfrak{K}_{n,i}(\alpha_1,\alpha_2;\phi)\right|\\&\quad\le\frac{\|\mho_i'\|_\infty}{(n-3)!}\left[\frac{\phi^{(n-1)}(\alpha_2)+\phi^{(n-1)}(\alpha_1)}{2}-\frac{\phi^{(n-2)}(\alpha_2)-\phi^{(n-2)}(\alpha_1)}{\alpha_2-\alpha_1}\right],\end{aligned}\tag{54}$$

for $i=1,\dots,6$.

Proof (i) Fix $i=1,\dots,6$. Applying Theorem 8 for $\mho_1\to\mathfrak{R}_i$ and $\mho_2\to\phi^{(n)}$, we get

$$\begin{aligned}&\left|\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathfrak{R}_i(\xi)\phi^{(n)}(\xi)\,d\xi-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathfrak{R}_i(\xi)\,d\xi\cdot\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\phi^{(n)}(\xi)\,d\xi\right|\\&\quad\le\frac{1}{2(\alpha_2-\alpha_1)}\left\|\mathfrak{R}_i'\right\|_\infty\int_{\alpha_1}^{\alpha_2}(\xi-\alpha_1)(\alpha_2-\xi)\phi^{(n+1)}(\xi)\,d\xi.\end{aligned}\tag{55}$$

Now, solving the integral part of R.H.S. of inequality (55) and taking into account identity (8), we formulate (53).

(ii) Fix $i=1,\dots,6$. Now, applying similar method opted in (i) and then using identity (9), we obtain (54). □

Now we express some Ostrowski-type inequalities affiliated with our generalized Jensen-type inequalities.

Theorem 11 *Consider the suppositions of Theorem* 3 *be fulfilled. Let* $|\phi^{(n)}|^q:[\alpha_1,\alpha_2]\to\mathbb{R}$ *be an R-integrable function with* $q,q'\in[1,\infty]$ *such that* $\frac{1}{q}+\frac{1}{q'}=1$. *Then, for* $i=1,\dots,6$, *we have*

(i)

$$\begin{aligned}&\left|J_i(\phi)-\sum_{\sigma=2}^{n-l}\left(\frac{n-\sigma}{\sigma!}\right)\left(\frac{\phi^{(\sigma-1)}(\alpha_2)J_i((z-\alpha_2)^\sigma)-\phi^{(\sigma-1)}(\alpha_1)J_i((z-\alpha_1)^\sigma)}{(\alpha_2-\alpha_1)}\right)\right|\\&\quad\le\frac{1}{(n-1)!(\alpha_2-\alpha_1)}\left\|\phi^{(n)}\right\|_q\left(\int_{\alpha_1}^{\alpha_2}\left|J_i\big((z-\xi)^{n-1}F_{\alpha_1}^{\alpha_2}(\xi,z)\big)\right|^{q'}d\xi\right)^{1/q'},\end{aligned}\tag{56}$$

(ii)

$$\begin{aligned}&\Bigg|J_i(\phi)-\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}J_i\big(G_L(z,r)\big)\\&\qquad\times\left(\sum_{\sigma=0}^{n-3}\left(\frac{n-2-\sigma}{\sigma!}\right)\big(\phi^{(\sigma+1)}(\alpha_2)(r-\alpha_2)^\sigma-\phi^{(\sigma+1)}(\alpha_1)(r-\alpha_1)^\sigma\big)\right)dr\Bigg|\\&\quad\le\frac{1}{(n-3)!(\alpha_2-\alpha_1)}\left\|\phi^{(n)}\right\|_q\\&\qquad\times\left(\int_{\alpha_1}^{\alpha_2}\left|\int_{\alpha_1}^{\alpha_2}J_i\big(G_L(z,r)\big)(r-\xi)^{n-3}F_{\alpha_1}^{\alpha_2}(\xi,r)\,dr\right|^{q'}d\xi\right)^{1/q'}.\end{aligned}\tag{57}$$

The constants on the R.H.S. of (56) *and* (57) *are sharp for* $q\in(1,\infty]$ *and best possible for* $q=1$.

Proof (i) Fix $i = 1,\dots,6$. Let us denote

$$h = \frac{1}{(n-1)!(\alpha_2-\alpha_1)}\bigl(J_i\bigl((z-\xi)^{n-1}F_{\alpha_1}^{\alpha_2}(\xi,z)\bigr)\bigr), \quad \xi \in [\alpha_1,\alpha_2].$$

Applying identity (8), we get

$$\begin{aligned}&\left|J_i(\phi) - \sum_{\sigma=2}^{n-l}\left(\frac{n-\sigma}{\sigma!}\right)\left(\frac{\phi^{(\sigma-1)}(\alpha_2)J_i((z-\alpha_2)^\sigma) - \phi^{(\sigma-1)}(\alpha_1)J_i((z-\alpha_1)^\sigma)}{(\alpha_2-\alpha_1)}\right)\right|\\ &\quad = \left|\int_{\alpha_1}^{\alpha_2} h(\xi)\phi^{(n)}(\xi)\,d\xi\right|. \end{aligned} \tag{58}$$

Employing Hölder's inequality on the R.H.S. of (58) gives

$$\left|\int_{\alpha_1}^{\alpha_2} h(\xi)\phi^{(n)}(\xi)\,d\xi\right| \le \left(\int_{\alpha_1}^{\alpha_2}\bigl|\phi^{(n)}(\xi)\bigr|^q\,d\xi\right)^{\frac{1}{q}}\left(\int_{\alpha_1}^{\alpha_2}\bigl|h(\xi)\bigr|^{q'}\,d\xi\right)^{\frac{1}{q'}},$$

which combined together with (58) leads to (56).

In order to prove the sharpness of the constant $(\int_{\alpha_1}^{\alpha_2}|h(\xi)|^{q'}\,d\xi)^{1/q'}$, we need to choose an appropriate function ϕ so that we can obtain the equality case in (56).

For $1 < q \le \infty$, let us define ϕ in such a way that

$$\phi^{(n)}(\xi) = \operatorname{sgn} h(\xi)\bigl|h(\xi)\bigr|^{\frac{1}{q-1}}.$$

For $q = \infty$, take $\phi^{(n)}(\xi) = \operatorname{sgn} h(\xi)$.

For $q = 1$, we prove that

$$\left|\int_{\alpha_1}^{\alpha_2} h(\xi)\phi^{(n)}(\xi)\,d\xi\right| \le \max_{\xi\in[\alpha_1,\alpha_2]}\bigl|h(\xi)\bigr|\left(\int_{\alpha_1}^{\alpha_2}\phi^{(n)}(\xi)\,d\xi\right) \tag{59}$$

is the best possible inequality. Suppose $|h(\xi)|$ acquires its maximum at $\xi_0 \in [\alpha_1,\alpha_2]$. To move further, first assume $h(\xi_0) > 0$ and define $\phi_\lambda(\xi)$ by

$$\phi_\lambda(\xi) = \begin{cases} 0, & \alpha_1 \le \xi \le \xi_0,\\ \frac{1}{\lambda n!}(\xi-\xi_0)^n, & \xi_o \le \xi \le \xi_0+\lambda,\\ \frac{1}{n!}(\xi-\xi_0)^{n-1}, & \xi_0+\lambda \le \xi \le \alpha_2. \end{cases}$$

Then, for λ small enough,

$$\left|\int_{\alpha_1}^{\alpha_2} h(\xi)\phi^{(n)}(\xi)\,dt\right| = \left|\int_{\xi_0}^{\xi_0+\lambda} h(\xi)\frac{1}{\lambda}\,d\xi\right| = \frac{1}{\lambda}\int_{\xi_0}^{\xi_0+\lambda} h(\xi)\,d\xi.$$

Now, from inequality (59), we have

$$\frac{1}{\lambda}\int_{\xi_0}^{\xi_0+\lambda} h(\xi)\,d\xi \le h(\xi_0)\int_{\xi_0}^{\xi_0+\lambda}\frac{1}{\lambda}\,d\xi = h(\xi_0).$$

Since

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\int_{\xi_0}^{\xi_0+\lambda} h(\xi)\,d\xi = h(\xi_0),$$

the statement follows. The case when $h(\xi_0) < 0$, we define $\phi_\lambda(\xi)$ by

$$\phi_\lambda(\xi) = \begin{cases} \frac{1}{n!}(\xi - \xi_0 - \lambda)^{n-1}, & \alpha_1 \le \xi \le \xi_0, \\ \frac{-1}{\lambda n!}(\xi - \xi_0 - \lambda)^n, & \xi_o \le \xi \le \xi_0 + \lambda, \\ 0, & \xi_0 + \lambda \le \xi \le \alpha_2, \end{cases}$$

and follow the same steps.

(ii) Fix $i = 1, \ldots, 6$. Now, applying a similar method to that opted in (i) to identity (9), we obtain the desired results. □

5 *n*-Exponential convexity and related results

We start this section by presenting some important results from [14, 15], and [16].

Definition 2 A function $\phi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{j,l=1}^{n} \mu_j \mu_l \phi\left(\frac{z_l + z_j}{2}\right) \ge 0$$

holds for all choices $\mu_1, \ldots, \mu_n \in \mathbb{R}$ and all choices $z_1, \ldots, z_n \in I$. If ϕ is additionally continuous on I, then it is n-exponentially convex.

Remark 4 If the function ϕ is n-exponentially convex in the Jensen sense (exponential convex) for all $n \in \mathbb{N}$, then it is exponentially convex in the Jensen sense (exponential convex) on I.

Proposition 1 *If $\phi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense, then the matrix $[\phi(\frac{z_j+z_l}{2})]_{j,l=1}^{k}$ is a positive semi-definite matrix for all $k \in \mathbb{N}, k \le n$. Particularly,*

$$\det\left[\phi\left(\frac{z_j + z_l}{2}\right)\right]_{j,l=1}^{k} \ge 0$$

for all $k \in \mathbb{N}$, $k = 1, 2, \ldots, n$.

Remark 5 It is known that $\phi : I \to \mathbb{R}$ is log-convex in the Jensen sense if and only if

$$\alpha_1^2\phi(z_1) + 2\alpha_1\alpha_2\phi\left(\frac{z_1 + z_2}{2}\right) + \alpha_2^2\phi(z_2) \ge 0$$

holds for every $\alpha_1, \alpha_2 \in \mathbb{R}$ and $z_1, z_2 \in I$. This shows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

Remark 6 As a consequence of Theorem 4, we define the positive linear functionals with respect to n-convex function ϕ in the following way:

(i) From (14), for $i = 1,\ldots,6$, we construct

$$\begin{aligned}\varOmega_i(\phi) &:= J_i(\phi) - \sum_{\sigma=2}^{n-l} \\ &\quad \times \left(\frac{n-\sigma}{\sigma!}\right)\left(\frac{\phi^{(\sigma-1)}(\alpha_2)J_i((z-\alpha_2)^{\sigma}) - \phi^{(\sigma-1)}(\alpha_1)J_i((z-\alpha_1)^{\sigma})}{(\alpha_2-\alpha_1)}\right) \\ &\geq 0. \end{aligned} \tag{60}$$

(ii) From (16), for $i = 1,\ldots,6$, we construct

$$\begin{aligned}\varXi_i(\phi) &:= J_i(\phi) - \frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2} J_i\bigl(G_L(z,r)\bigr) \\ &\quad \times \left(\sum_{\sigma=0}^{n-3}\left(\frac{n-2-\sigma}{\sigma!}\right)\bigl(\phi^{(\sigma+1)}(\alpha_2)(r-\alpha_2)^{\sigma} - \phi^{(\sigma+1)}(\alpha_1)(r-\alpha_1)^{\sigma}\bigr)\right)dr \\ &\geq 0. \end{aligned} \tag{61}$$

In the later part of the paper, we would like to establish our further results by considering the linear functionals $\varOmega_i(\cdot)$ defined in (60). However, we can also formulate the same results for the functionals $\varXi_i(\cdot)$ defined in (61) for $i = 1,\ldots,6$.

First of all, we will formulate mean value theorems of Lagrange and Cauchy type related to defined functionals in the form of the following theorems.

Theorem 12 *Let $\phi : [\alpha_1,\alpha_2] \to \mathbb{R}$ be such that $\phi \in C^n[\alpha_1,\alpha_2]$. If the inequalities in* (13) *for $i = 1,\ldots,6$ are valid, then there exist $\mu_i \in [\alpha_1,\alpha_2]$ such that*

$$\varOmega_i(\phi) = \phi^{(n)}(\mu_i) \cdot \frac{\varOmega_i(z^n)}{n!}, \quad i = 1,\ldots,6,$$

where $\varOmega_i(\cdot)$ are defined in Remark 6.

Proof Similar to the proof of Theorem 4.1 in [17] (see also [18]). □

Theorem 13 *Let $\phi,\psi : [\alpha_1,\alpha_2] \to \mathbb{R}$ be such that $\phi,\psi \in C^n[\alpha_1,\alpha_2]$. If the inequalities in* (13) *for $i = 1,\ldots,6$ are valid, then there exist $\mu_i \in [\alpha_1,\alpha_2]$ such that*

$$\frac{\varOmega_i(\phi)}{\varOmega_i(\psi)} = \frac{\phi^{(n)}(\mu_i)}{\psi^{(n)}(\mu_i)} \quad i = 1,\ldots,6,$$

with $\varOmega_i(\cdot)$ as defined in Remark 6.

Proof Similar to the proof of Corollary 4.2 in [17] (see also [18]). □

As an application of Theorem 13, we can define Cauchy means of α_1,α_2 for given functions ϕ and ψ in the following way:

$$\mu_i = \left(\frac{\phi^{(n)}}{\psi^{(n)}}\right)^{-1}\left(\frac{\varOmega_i(\phi)}{\varOmega_i(\psi)}\right).$$

Next we construct the nontrivial examples of n-exponentially and exponentially convex functions from positive linear functionals $\Omega_i(\cdot)$ ($i = 1,\dots,6$). We use the idea given in [16]. In the sequel I and J are intervals in $\mathbb{R}$.

Theorem 14 *Let $\Theta = \{\phi_h : h \in J\}$ be a family of functions defined on I in such a way that for every one of $(n + 1)$ mutually different points $z_0,\dots,z_n \in I$ the function $h \mapsto [z_0,\dots,z_n;\phi_h]$ is n-exponentially convex in the Jensen sense on J. Then the following statements are valid for each $i = 1,\dots,6$ for the linear functionals $\Omega_i(\phi_h)$ as defined in Remark* 6:

(i) *The function $h \to \Omega_i(\phi_h)$ is n-exponentially convex in the Jensen sense on J and the matrix $[\Omega_i(\phi_{\frac{h_j+h_l}{2}})]_{j,l=1}^k$ is positive semi-definite for all $k \in \mathbb{N}$, $k \le n$, $h_1,\dots,h_k \in J$. Particularly,*

$$\det\bigl[\Omega_i(\phi_{\frac{h_j+h_l}{2}})\bigr]_{j,l=1}^k \ge 0$$

for all $k \in \mathbb{N}$, $k = 1,2,\dots,n$.

(ii) *If the function $h \to \Omega_i(\phi_h)$ is continuous on J, then it is n-exponentially convex on J.*

Proof The proof is similar to Theorem 4.6 in [4]. □

Remark 7 The results of Theorem 14 still hold, if we replace the family of functions $\Theta = \{\phi_h : h \in J\}$ from n-exponentially convex to exponentially convex functions.

The following corollary is an immediate consequence of the above theorem.

Corollary 1 *Let $\Theta = \{\phi_h : h \in J\}$ be a family of functions defined on I in such a way that for every one of $(n+1)$ mutually different points $z_0,\dots,z_n \in I$ the function $h \mapsto [z_0,\dots,z_n;\phi_h]$ is 2-exponentially convex in the Jensen sense on J. Let $\Omega_i(\cdot)$ ($i = 1,\dots,6$) be linear functionals, then the following statements are valid*:

(i) *If the function $h \mapsto \Omega_i(\phi_h)$ is continuous on J, then it is a 2-exponentially convex function on J. If $h \mapsto \Omega_i(\phi_h)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true*:

$$\bigl[\Omega_i(\phi_s)\bigr]^{h-r} \le \bigl[\Omega_i(\phi_r)\bigr]^{h-s}\bigl[\Omega_i(\phi_h)\bigr]^{s-r}$$

for every choice $r,s,h \in J$ such that $r < s < h$.

(ii) *If the function $h \mapsto \Omega_i(\phi_h)$ is strictly positive and differentiable on J, then for every $h,s,u,v \in J$, such that $h \le u$ and $s \le v$, we have*

$$\mathfrak{M}(h,s,\Omega_i,\Theta) \le \mathfrak{M}(u,v,\Omega_i,\Theta), \tag{62}$$

where

$$\mathfrak{M}(h,s,\Omega_i,\Theta) = \begin{cases} \left(\frac{\Omega_i(\phi_h)}{\Omega_i(\phi_s)}\right)^{\frac{1}{h-s}}, & h \neq s, \\ \exp\left(\frac{\frac{d}{dh}\Omega_i(\phi_h)}{\Omega_i(\phi_h)}\right), & h = s, \end{cases} \tag{63}$$

for $\phi_h,\phi_s \in \Theta$.

Proof The proof is similar to Corollary 4.8 in [4]. □

6 Cauchy means

Finally, we choose special families of functions which empower us to establish new exponentially convex functions and related results.

Example 1 For real $h \in \mathbb{R}$, consider a family of functions $\Theta = \{\phi_h : (0,\infty) \to \mathbb{R}\}$ defined by

$$\phi_h(z) = \begin{cases} \frac{z^h}{h(h-1)\cdots(h-n+1)}, & h \notin \{0,1,\ldots,n-1\}, \\ \frac{z^w \log z}{(-1)^{n-1-w} w!(n-1-w)!}, & h = w \in \{0,1,\ldots,n-1\}. \end{cases}$$

As $\phi_h^{(n)}(z) = z^{h-n} > 0$, therefore the function ϕ_h is n-convex for $z > 0$ and $h \mapsto \frac{d^n \phi_h}{dz^n}(z)$ is exponentially convex by definition. Now, employing Theorem 14, it is straightforward that $h \mapsto [z_0,\ldots,z_n;\phi_h]$ is exponentially convex. Therefore, by Remark 7, we deduce that $h \mapsto \Omega_i(\phi_h)$ $(i = 1,\ldots,6)$ are exponentially convex. Thus, taking into account the present family of functions $\mathfrak{M}(h,s,\Omega_i,\Theta)$ $(i = 1,\ldots,6)$, from (63), are equal to

$$\mathfrak{M}(h,s,\Omega_i,\Theta) = \begin{cases} \left(\frac{\Omega_i(\phi_h)}{\Omega_i(\phi_s)}\right)^{\frac{1}{h-s}}, & h \neq s, \\ \exp\left(\frac{\Omega_i(\phi_0\phi_h)}{\Omega_i(\phi_h)} \cdot (-1)^{n-1}(n-1)! + \sum_{t=0}^{n-1} \frac{1}{t-h}\right), & h = s \notin \{0,1,\ldots,n-1\}, \\ \exp\left(\frac{\Omega_i(\phi_0\phi_h)}{2\Omega_i(\phi_h)} \cdot (-1)^{n-1}(n-1)! + \sum_{\substack{t=0 \\ t\neq h}}^{n-1} \frac{1}{t-h}\right), & h = s \in \{0,1,\ldots,n-1\}. \end{cases}$$

For the case $i = 1$, the explicit expressions are as follows:

$$\begin{aligned} &\mathfrak{M}(h,s,\Omega_1,\Theta) \\ &\quad = \Biggl(\frac{s(s-1)\cdots(s-n+1)}{h(h-1)\cdots(h-n+1)} \\ &\qquad \times \frac{J_1(z^h) - \sum_{\sigma=2}^{n-l}(\frac{n-\sigma}{\sigma!(\alpha_2-\alpha_1)})(\prod_{r=0}^{\sigma-2}(h-r)\alpha_2^{h-(\sigma-1)}J_1((z-\alpha_2)^\sigma) - \prod_{r=0}^{\sigma-2}(h-r)\alpha_1^{h-(\sigma-1)}J_1((z-\alpha_1)^\sigma))}{J_1(z^s) - \sum_{\sigma=2}^{n-l}(\frac{n-\sigma}{\sigma!(\alpha_2-\alpha_1)})(\prod_{r=0}^{\sigma-2}(s-r)\alpha_2^{s-(\sigma-1)}J_1((z-\alpha_2)^\sigma) - \prod_{r=0}^{\sigma-2}(s-r)\alpha_1^{s-(\sigma-1)}J_1((z-\alpha_1)^\sigma))}\Biggr)^{\frac{1}{h-s}}, \\ &\quad \text{for } h \neq s \notin \{0,1,\ldots,n-1\}; \end{aligned}$$

whereas

$$J_1(z^h) = J_1(\mathbf{z},\mathbf{p},\boldsymbol{\lambda};z^h) := \sum_{i=1}^{n} p_u {z_u}^h - C_{\text{dis}}(z^h,\mathbf{z},\mathbf{p},\boldsymbol{\lambda})$$

and

$$C_{\text{dis}}(z^h,\mathbf{z},\mathbf{p},\boldsymbol{\lambda}) = \sum_{u=1}^{n}\left(\sum_{\nu=0}^{k-1}\lambda_{\nu+1}p_{u+\nu}\right)\left(\frac{\sum_{\nu=0}^{k-1}\lambda_{\nu+1}p_{u+\nu}z_{u+\nu}}{\sum_{j=0}^{k-1}\lambda_{\nu+1}p_{u+\nu}}\right)^h.$$

$$\begin{aligned} &\mathfrak{M}(h,h,\Omega_1,\Theta) \\ &\quad = \exp\Biggl(\frac{J_1(z^h\log z) - \sum_{\sigma=2}^{n-l}(\frac{n-\sigma}{\sigma!(\alpha_2-\alpha_1)})(\frac{d^{\sigma-1}}{dz^{\sigma-1}}(z^h\log z)|_{z=\alpha_2}J_1((z-\alpha_2)^\sigma) - \frac{d^{\sigma-1}}{dz^{\sigma-1}}(z^h\log z)|_{z=\alpha_1}J_1((z-\alpha_1)^\sigma))}{J_1(z^h) - \sum_{\sigma=2}^{n-l}(\frac{n-\sigma}{\sigma!(\alpha_2-\alpha_1)})(\prod_{r=0}^{\sigma-2}(h-r)\alpha_2^{h-(\sigma-1)}J_1((z-\alpha_2)^\sigma) - \prod_{r=0}^{\sigma-2}(h-r)\alpha_1^{h-(\sigma-1)}J_1((z-\alpha_1)^\sigma))} \\ &\qquad + \sum_{t=0}^{n-1}\frac{1}{t-h}\Biggr), \\ &\quad \text{for } h = s \notin \{0,1,\ldots,n-1\}. \end{aligned}$$

$$\begin{aligned}
&\mathfrak{M}(h,h,\Omega_1,\Theta)\\
&\quad = \exp\left(\frac{J_1(z^h\log^2 z) - \sum_{\sigma=2}^{n-l}(\frac{n-\sigma}{\sigma!(\alpha_2-\alpha_1)})(\frac{d^{\sigma-1}}{dz^{\sigma-1}}(z^h\log^2 z)|_{z=\alpha_2}J_1((z-\alpha_2)^\sigma) - \frac{d^{\sigma-1}}{dz^{\sigma-1}}(z^h\log^2 z)|_{z=\alpha_1}J_1((z-\alpha_1)^\sigma))}{2(J_1(z^h\log z) - \sum_{\sigma=2}^{n-l}(\frac{n-\sigma}{\sigma!(\alpha_2-\alpha_1)})(\frac{d^{\sigma-1}}{dz^{\sigma-1}}(z^h\log z)|_{x=\alpha_2}J_1((z-\alpha_2)^\sigma) - \frac{d^{\sigma-1}}{dz^{\sigma-1}}(z^h\log z)|_{z=\alpha_1}J_1((z-\alpha_1)^\sigma)))}\right.\\
&\qquad \left. + \sum_{\substack{t=0\\ t\neq h}}^{n-1}\frac{1}{t-h}\right),\\
&\qquad \text{for } h = s \in \{0,1,\ldots,n-1\}.
\end{aligned}$$

Now, using Theorem 13, we conclude that

$$\alpha_1 \le \left(\frac{\Omega_i(\phi_h)}{\Omega_i(\phi_s)}\right)^{\frac{1}{h-s}} \le \alpha_2, \quad i = 1,\ldots,6.$$

Hence $\mathfrak{M}(h,s,\Omega_i,\Theta)$ $(i = 1,\ldots,6)$ are means, and their monotonicity is followed by (62).

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